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The Splitting Problem and the Direct Factor Problem in Modular Abelian Group Algebras

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Let K be an algebraic closed field of prime characteristic p and G be a torsion abelian group. In this paper we demonstrate that the normalized unit group $V(KG)$ of a group ring KG is a direct sum of countable groups, respectively is divisible or is algebraically compact if and only if the p-component G_p of G is identity. Besides, $V(KG)$ is a splitting group and the quotient $V(KG)/tV(KG)$ is being described, where $tV(KG)$ is the torsion part in $V(KG)$. A necessary condition is also obtained, the complement $V(KG)/tV(KG)$ to be divisible, i.e. $V(KG)$ to be divisible modulo torsion.

Moreover, it is shown that if G is a direct sum of countables such that G/G_p is divisible, then G is a direct factor of $V(KG)$ with a direct sum of countables complementary factor. But if G/G_p is not divisible, then G is not a direct factor of $V(KG)$.

Finally, suppose that the arbitrary group G belongs to $\mathcal{K}$, any class of abelian groups such that G is a direct factor of $V(KG)$. Then, each coproduct $A = \coprod_{i\in I} G_i$ so that $G_i \in \mathcal{K}$ for all $i \in I$, is a direct factor of $V(KA)$. Thus in particular, it is proved that if G is a coproduct of torsion-complete p-groups or if G is a coproduct of p-mixed algebraically compact groups, G is a direct factor of $V(KG)$.

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1. Introduction

Throughout the rest in this paper, let G be an abelian group with torsion subgroup tG and p-primary component G_p, R be an unitary commutative ring of char$R = p$, F be a field of char$F = p > 0$ and K be an algebraic closed field with the same char$K = p$. Thus RG, FG and KG are abelian group rings with 1, with normed unit groups $V(RG)$, $V(FG)$ and $V(KG)$, and p-components $S(RG)$, $S(FG)$ and $S(KG)$, respectively. All other notations from the abelian group theory are in agreement with [6, 11], and these from the group algebra theory with [9, 10].

1.1. Direct Factor

In the theory of the group algebras, there exist *some open problems* (see [10], p.178 and p.184) for the direct factor, namely:

(1) *If G is p-torsion, then whether G is a direct factor of $V(FG)$?*

(2) *If G is p-mixed (i.e. $tG = G_p$), then whether G is a direct factor of $V(FG)$?*

(3) *If G is torsion, then G is not however a direct factor of $V(FG)$. But of some interest is the question when the last is true, for some restrictions on G and F?*

In this paper, we study these phenomena.

1.2. Splitting

(1) If G is torsion-free, then by a classical result of Higman [10], $V(FG) = G$.

(2) If tG has only p-torsion, i.e. in other words G is p-mixed, then probably $V(FG)$ splits if and only if G splits.

($*$) G splitting yields $V(FG)$ splitting (see May [14], p.307), which is however trivial (see Proposition 1 below).

($**$) As we see, there exists a conjecture that G is a direct factor of $V(FG)$. So, if this assertion is true and $V(FG)$ splits, then G splits too.

(3) If G is torsion, then when $V(FG)$ splits? In this article, we investigate this question.

2. Preliminary and main results

We start with the following necessary proposition.

Proposition 1. *Let $G = G_p \times M$. Then $V(FG) \cong V(FM) \times S(FG)$.*

Proof. Using the proposition from [1], we deduce $V(FG) \cong V(FM) \times V((FM)G_p)$ and so, $S(FG) \cong S(FM) \times S((FM)G_p)$. But $M_p = 1$ and hence FM is semisimple by the well-known classical Passman criterion for semisimplicity, [9]. Therefore FM has a trivial nilradical, immediately we detect $S(FM) = 1$, and $S(FG) \cong S((FM)G_p) = V((FM)G_p)$ since $\mathrm{char} FM = p$. Finally $V(FG) \cong V(FM) \times S(FG)$, which proves the formula. ∎

Let ε_n be a primitive n-th root of unity in the algebraic closure $\overline{F}$ of F and let $U(F)$ be the multiplicative group of F. The following proposition holds.

Proposition 2. *Suppose that G is torsion and $U(F(\varepsilon_n))$ is divisible modulo torsion, for each n which is an order of any element in G. Then $V(FG)$ is divisible modulo torsion.*

Proof. By the above proposition, $V(FG)/tV(FG) \cong V(FM)/tV(FM)$. Now we need only to apply [17], and we are done. ■

Corollary 1. *Let G be torsion. Then $V(KG)$ is divisible modulo torsion.*

Proof. The field K is algebraically closed. Thus, $U(K(\varepsilon_n)) = U(K)$ is divisible hence divisible modulo torsion [6], and from the above statement, $V(KG)$ is divisible modulo torsion. ■

Lemma 1. *Suppose A is abelian so that $A = B \times E$. Then A splits if and only if B and E both split.*

Proof. Let A split. Furthermore $A = tA \times C$. Moreover $tA = tB \times tE$, consequently $A = tB \times T$ and tB is a direct factor of $B \subseteq A$. Similarly for E.

Conversely, if tB is a direct factor of B and tE is a direct factor of E, then $tB \times tE = tA$ is a direct factor of A. The lemma is proved. ■

The next statement was announced in [2].

Theorem 1. (Splitting) *Let G be torsion. Then $V(KG)$ is a splitting group, i.e. $V(KG) = tV(KG) \times D$, where $D = 1$ if K is an algebraic cover of a simply field or D is divisible torsion-free of rank* $\max(|K|, |G/G_p|)$, *otherwise. Moreover, if G/G_p is infinite, then $tV(KG)/S(KG) \cong \prod_{q\neq p} \prod_{|G/G_p|} Z(q^\infty)$ or if G/G_p is finite, then $tV(KG)/S(KG) \cong \prod_{q\neq p} \prod_{|G/G_p|-1} Z(q^\infty)$.*

Proof. Write down, $G = G_p \times M$. Therefore Proposition 1 does imply $V(KG) \cong V(KM) \times S(KG)$. By virtue of [16] (see also [2]), $V(KM)$ is divisible, whence it is splitting. That is why, Lemma 1 yields, $V(KG)$ splits. Besides, by virtue of Corollary 1, we obtain that $V(KG)/tV(KG)$ is divisible. Further, the proof is based on [15]. ■

Now, we give a characterization of $V(KG)$ for some important classes of abelian groups. More specially, the following statement, announced in [2], holds.

Theorem 2. (Structure) *Suppose G is torsion. Then:*

(∘) *$V(KG)$ is divisible if and only if G_p is divisible.*

(∘∘) *$V(KG)$ is algebraically compact if and only if G_p is algebraically compact.*

(ooo) *$V(KG)$ is a direct sum of countables if and only if G_p is a direct sum of countables.*

Proof. By application of Proposition 1, $V(KG) = V(KM) \times S(KG)$, since $G = G_p \times M$, where M is p-divisible. Following [16] (see also [2]), $V(KM)$ is divisible.

(o) By the above observations, $V(KG)$ is divisible if and only if $S(KG)$ is divisible. This is equivalent to $S^p(KG) = S(KG^p) = S(KG)$, i.e. to $G^p = G$, i.e. to G_p is divisible.

(oo) $V(KG)$ is algebraic compact if and only if $S(KG)$ is algebraic compact, because the divisible groups are algebraically compact (see [6]). As we have seen in the proof of Proposition 1, it is valid that $S(KG) \cong S((KM)G_p)$. The field K is closed, hence perfect. So, KM is a perfect ring with no nilpotents and therefore by [1], $S(KG)$ is algebraically compact if and only if G_p is the same.

(ooo) $V(KG)$ is a direct sum of countables if and only if $S(KG)$ is one also, which fact follows directly or by a result of Kaplansky-Walker ([6], p.63, Prop. 9.10). But it is well-known that $S(KG) \cong S((KM)G_p)$, where KM is perfect without nilpotents. By application of [12], $S(KG)$ is a direct sum of countables if and only if G_p is. This completes the proof. ■

Now, we can state the following theorem.

Theorem 3. (Direct Factor) *Let G be a torsion abelian group so that G_p is a direct sum of countables (in particular, let G be a torsion direct sum of countable groups). Then:*

($\diamond$) *If G/G_p is divisible, G is a direct factor of $V(KG)$ with complement, which is a direct sum of countables.*

($\diamond\diamond$) *If G/G_p is not divisible, G is not a direct factor of $V(KG)$.*

Proof. Write $G = G_p \times M$. Further, Proposition 1 implies $V(KG) \cong V(KM) \times S(KG)$. Obviously, M is a direct factor of $V(KM)$, since it is divisible. From [12], G_p is a direct factor of $S(KG_p)$, hence of $S(KG)$, because clearly $S(KG_p)$ is a direct factor of $S(KG)$. Finally, $G_p \times M = G$ is a direct factor of $V(KG)$. Moreover Theorem 2 implies that $V(KG)/G$ is a direct sum of countables.

Assuming G/G_p no divisible, G is not a direct factor of $V(KG)$, since $M \cong G/G_p$ is not a direct factor of the divisible $V(KM)$. The assertion is fulfilled. ■

We generalize now the above theorem to the following one.

Theorem 4. (Direct Factor) *Let G be torsion whose G_p is a direct sum of groups of cardinality $\aleph_1$ or is simply presented. Then G is a direct factor* *of $V(KG)$ provided G/G_p is divisible. Otherwise G is not a direct factor of $V(KG)$. The complementary factor is a direct product of a divisible group and of a simply presented p-group.*

Proof. It follows by the same scheme as the proof of the above theorem, but according to the main results in [7] and [13], respectively. ■

We continue with considering the action of some special conditions on R and G that guarantee that G itself is a direct factor of $V(RG)$. But first and foremost, we summarize some known results; the best results for the direct factor problem of p-groups, by this moment, are the following stated as below:

Theorem. *In each of the following cases, the p-torsion group G is a direct factor of $V(FG)$, namely:*

(a) ([12, 13]) *G is totally projective (simply presented) and in particular, G is a direct sum of countables. The complementary factor belongs to the same group class as G, provided F is perfect.*

(a') ([18]) *G is an A-group (which generalizes (a)). The complement is still not fully known in general.*

(b) ([7, 14]) *G is a direct sum (= coproduct) of p-groups with the cardinality of each factor not exceeding $\aleph_1$. The complementary factor is totally projective (simply presented) assuming F is perfect.*

(c) ([4]) *G is summable with countable length. The complement is totally projective presuming F is perfect.*

(d) ([5]) *G is a C_λ-group with length$G = \lambda < \Omega$. The complement is totally projective provided F is perfect.*

We need some preliminary results before stating and proving the central theorems. So we start with the following major lemma.

Lemma 5. *Let $M \leq G$ and $C \leq G$, where C is p-torsion, $1 \in P \leq R$. Then:*

$(*)$ $V(PM) \cap V(RG;C) = V(PM; M \cap C)$.

$(**)$ $V(RG) \cap RM = V(RM)$.

Proof. $(*)$ Take x in the left hand-side. Therefore $x = \sum_{m \in M} \alpha_m m$, $\alpha_m \in P$ and $\sum_{m \in \overline{m}C} \alpha_m = \begin{cases} 1, & \overline{m} \in C \\ 0, & \overline{m} \notin C \end{cases}$ for any $\overline{m} \in M$. But since $\overline{m}C \cap M = \overline{m}(C \cap M)$, we conclude that $\sum_{m \in \overline{m}(M \cap C)} \alpha_m = \begin{cases} 1, & \overline{m} \in M \cap C \\ 0, & \overline{m} \notin M \cap C \end{cases}$. Furthermore, it is a simple matter to see that $x \in V(PM; M \cap C)$, whence the left relation "$\subseteq$" is fulfilled. On the other hand, because $M \cap C$ is p-primary, we derive,

$V(PM; M \cap C) \subseteq V(PM)$ as a p-group. Thus and the right relation "$\supseteq$" is valid, to finish the proof.

(**) Given $x \in V(RG) \cap RM$. Hence $x = \sum_i r_i m_i$ ($r_i \in R$, $m_i \in M$) and there exists an element $y \in RG$, say $y = \sum_i \alpha_i g_i$ ($\alpha_i \in R$, $g_i \in G$), such that $\sum_i r_i m_i . \sum_i \alpha_i g_i = \sum_{i,j} r_i \alpha_j m_i g_j = 1$. We will show that $y \in RM$. In fact, without loss of generality we may assume that $m_1 g_1 = m_2 g_2 = \cdots = m_k g_k = 1$ for any fixed $k \in \mathrm{N}$, such that $\alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_k r_k = 1$. Let now $l > k (l \in \mathrm{N})$; then if $g_l \in g_j M = M$ for some $j = 1, \ldots, k$, we have $g_l \in M$. Otherwise, i.e. if $g_l \notin g_j M = M$, it is not difficult to verify that $\alpha_l = 0$. Finally, $y \in RM$, as claimed. Thus the right hand-side contains the left hand-side. The converse is trivial. The lemma is true. ■

Remark 6. The point (**) was proved also in [9], but when R is a field. Moreover the technique used there is different to that given as above.

We begin with the formulation of the following key matter of a technical character.

Proposition 7. *Suppose $M, C \leq G$, where C is p-torsion. Then $V(RG) = V(RM) \times V(RG; C)$ if and only if $G = M \times C$, where R is a field if the necessity is fulfilled.*

Proof. *"Sufficiency"*. Because $G = M \times C$, we deduce that $RG = (RM)C$. Therefore for each $x \in V(RG)$ we establish $x = \sum_{c \in C} x_c c$, where $x_c \in RM$. Choose $\overline{x} = \sum_{c \in C} x_c \in RM$. Evidently, $x = \overline{x} + \sum_{c \in C \setminus \{1\}} x_c (c-1)$. But C is p-primary and thus obviously, $x^{p^k} = \overline{x}^{p^k}$ for some natural k. Thus it is a routine matter to see that $\overline{x} \in V(RG)$ and consequently Lemma 5 yields $\overline{x} \in V(RG) \cap RM = V(RM)$. Moreover, select $v = 1 + \overline{x}^{-1} \sum_{c \in C \setminus \{1\}} x_c (c-1)$. Apparently $v \in V(RG; C)$, and on the other hand, $x = \overline{x} v$. So $V(RG) \subseteq V(RM).V(RG; C)$. In this light, Lemma 5 leads us to $V(RM) \cap V(RG; C) = V(RM; M \cap C) = 1$, since $M \cap C = 1$ by hypothesis. As a final, $V(RG) = V(RM) \times V(RG; C)$ as desired. The first part is completely proved.

"Necessity". Indeed, $M \cap C \subseteq V(RM) \cap V(RG; C) = 1$ and so $M \cap C = 1$. Now, for given $x \in G \subseteq V(RG)$ we can write $x = (\sum_i r_i m_i)(1 + \sum_{i,j} \alpha_{i,j} g_{i,j}(1 -$

$c_i)) = \sum_k r_k m_k + \sum_{i,j,k} r_k \alpha_{i,j} m_k g_{i,j}(1 - c_i)$, where $r_k, \alpha_{i,j} \in R$; $m_k \in M$; $g_{i,j} \in G$, $c_i \in C$. Hence $x = mgc$ or eventually $x = m'g'$ for some fixed $m \in M$, $m' \in M$; $g \in G$, $g' \in G$; $c \in C$. Moreover, observing that $r_k\alpha_{i,j} \neq 0$ for all k, i, j, we have that $g \in MC$ or eventually $g' \in MC$. Finally, we derive $x \in MC = M \times C$ and this finishes the proof. The proposition is verified. ∎

The next statement is important.

Proposition 8. *Assume that* $A = \coprod_{i \in I} G_i$ *is a* p*-group. Then* $V(RA) = \coprod_{i \in I} V(R(\coprod_{j \in J \cup \{i\}} G_j); G_i)$, *where* $J \subseteq I$ *is a subset.*

Proof. Put $I = \lambda$ for some fixed ordinal λ. Hence by our assumption $A = \coprod_{\mu<\lambda} G_\mu$. Choose arbitrary $\alpha < \lambda$, and in this direction take $B_\alpha = \coprod_{\mu<\alpha} G_\mu$. It is clear that $B_{\alpha+1} = B_\alpha \times G_\alpha$ and so owing to Proposition 7 we derive $V(RB_{\alpha+1}) = V(RB_\alpha) \times V(RB_{\alpha+1}; G_\alpha)$. In this light, it is easily seen that $V(RA) = \coprod_{\alpha<\lambda} V(RB_{\alpha+1}; G_\alpha)$. In fact, $A = \coprod_{\alpha<\lambda} B_\alpha$ and so, again Proposition 7 is transfinite inductively applicable to obtain the claim. The proposition is shown. ∎

Here we consider the direct factor problem for coproducts of abelian p-groups. Now we are in position to state the following theorem.

Theorem 9. *Let* G *be an abelian* p*-group belonging to any class* $\mathcal{K}$ *of abelian* p*-groups such that* G *is a direct factor of* $V(RG)$. *Then,* $A = \coprod_{i \in I} G_i$ *with* $G_i \in \mathcal{K}$ *for all* $i \in I$, *is a direct factor of* $V(RA)$.

Proof. According to Proposition 8, we can write $A = \coprod_{\mu<\lambda} G_\mu$ and $V(RA) = \coprod_{\alpha<\lambda} V(R(\coprod_{\mu<\alpha} G_\mu \times G_\alpha); G_\alpha) = \coprod_{\alpha<\lambda} V(R \coprod_{\mu\leq\alpha} G_\mu; G_\alpha)$, where $\lambda = I$. By the hypothesis, G_α is a direct factor of $V(RG_\alpha)$. But on the other hand, G_α is a direct factor of $\coprod_{\mu\leq\alpha} G_\mu$ and so, Proposition 8 guarantees that $V(RG_\alpha)$ is a direct factor of $V(R \coprod_{\mu\leq\alpha} G_\mu) = V(R \coprod_{\mu\leq\alpha} G_\mu; \coprod_{\mu\leq\alpha} G_\mu) \supseteq V(R \coprod_{\mu\leq\alpha} G_\mu; G_\alpha) \supseteq G_\alpha$. Thus G_α is a direct factor of $V(R \coprod_{\mu\leq\alpha} G_\mu)$, whence and of $V(R \coprod_{\mu\leq\alpha} G_\mu; G_\alpha)$.

Consequently, $V(RA) = (\coprod_{\alpha<\lambda} G_\alpha) \times M = A \times M$ for some group M, since $A = \coprod_{\mu<\lambda} G_\mu = \coprod_{\alpha<\lambda} \coprod_{\mu<\alpha} G_\mu = \coprod_{\alpha<\lambda} G_\alpha$. Finally A is a direct factor of $V(RA)$, as stated. The proof is finished. ■

The next assertion is only announced in [2].

Corollary 10. *Suppose G is a direct sum of torsion-complete p-groups. Then G is a direct factor of $V(RG)$.*

Proof. Indeed, we can write $G = \coprod_{i\in I} G_i$, where G_i are torsion-complete for all $i \in I$. But every G_i as a pure subgroup is a direct factor of $V(RG_i)$ owing to the well-known Kulikov-Papp theorem [6], and thus, Theorem 9 is applicable to obtain that G is a direct factor of $V(RG)$, as claimed. ■

Remark 11. The structure of the complement is unknown yet. Probably it is a direct sum of cycles.

Theorem 12. *Let G be a torsion abelian group whose G_p is a direct sum of torsion-complete groups. Then:*

(♣) *If G/G_p is divisible, G is a direct factor of $V(KG)$.*

(♣♣) *If G/G_p is not divisible, G is not a direct factor of $V(KG)$.*

Proof. It follows by the same method as in Theorem 3 owing to Corollary 10. ■

In the sequel we examine the direct factor problem for p-mixed groups. The significant known facts on this theme may be found in [3, 8, 14]. Foremost, we shall obtain a generalization of Theorem 9 for arbitrary commutative groups and rings by means of another technique. So, we can formulate the next theorem.

Theorem 13. *Assume that A is an abelian group and L is an abelian ring. Then $A = \coprod_{i\in I} A_i$ is a direct factor of $V(LA)$ if and only if A_i is a direct factor of $V(LA_i)$ for each $i \in I$.*

Proof. The necessity is trivial. For the sufficiency observe that the projection maps $A \to A_i$ induce projections $V(LA) \to V(LA_i)$. Since the elements of group algebras have finite supports, these projections induce a homomorphism $V(LA) \to \coprod_{i\in I} V(LA_i)$, which is clearly the identity map on the inner coproduct $\coprod_{i\in I} V(LA_i)$ in $V(LA)$. Thus, $\coprod_{i\in I} V(LA_i)$ is a direct factor of $V(LA)$. That is why, if we presume that A_i is a direct factor of $V(LA_i)$ for every $i \in I$, then A is a direct factor of $V(LA)$, proving the more general theorem. ■

Next, we close the study with an extension, for p-mixed algebraically compact groups, of a similar fact for algebraically compact p-groups that we have given in [1].

Theorem 14. *Let G be a coproduct of p-mixed algebraically compact abelian groups. Then G is a direct factor of $V(FG)$.*

Proof. Because G is p-mixed and hence $V(FG) = GS(FG)$ [13, 14, 3, 4], it is a routine matter to be seen that G is pure in $V(FG)$. Consequently the result follows in view of the definition for an algebraically compact group [6] and the latter theorem. The proof is over. ■

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