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The Formation of a Differential Equation Involving the Ratio of the Θ-theta Functions

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In this work, a non-linear differential equation which contains the squares of the theta functions and is also satisfied by $w(z) = \frac{\theta_2(z)}{\theta_1(z)}$, is established by using the known properties of the theta functions θ_1, θ_2, θ_3 and θ_4.

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1. Introduction

Definition 1.1. A lattice Ω of complex numbers in an aggregate of complex numbers with the two properties:

i) Ω is a group with respect to addition,

ii) The absolute magnitudes of the non-zero elements are bounded below, i.e. there is a real number $k > 0$ such that $|w| \geq k$ for all $w \neq 0$ in Ω, [2].

The set

$$\Omega = \{mw_1 + nw_2 : m, n \in Z\}$$

is a 2-dimensional lattice, where w_1 and w_2 are linearly independent complex numbers. The pair (w_1, w_2) which generates the lattice is called a basis of the lattice. The periodic function $f(z)$ is called doubly-periodic, if its period lattice has dimension 2.

Definition 1.2. A uniform function which has no essential singularity in a given region is said to be meromorphic function.

Definition 1.3. A doubly-periodic function which is meromorphic in the open z-plane is called an elliptic function.

Any fundamental region of period lattice is called a period-parallelogram of an elliptic function. The number of zeros or poles (multiplicities are taken into account) in a period-parallelogram of an elliptic function is known as the order of the elliptic function.

Theorem 1.1. ([1]) *A non-constant elliptic function of order one does not exist.*

2. Θ-theta functions and some properties

Definition 2.1. We define a *theta characteristic*, usually abbreviated to *characteristic*, to be a two by one matrix of integers, writen as $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$. Given a complex number z, and another complex number $\tau = \dfrac{w_1}{w_2}$, satisfying $\mathrm{Im}\tau > 0$, i.e., the upper halfplane, and a characteristic $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$, we define the first order general theta function with characteristic $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$, argument z and theta period τ, usually abbreviated to theta function with characteristic $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$, by:

$$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z,\tau) = \sum_n \exp \pi i \left\{\tau(n+\frac{\varepsilon}{2})^2 + 2(n+\frac{\varepsilon}{2})(z+\frac{\varepsilon'}{2})\right\}, \tag{1}$$

where n ranges over all the integers ($-\infty$ to ∞) and exp is the usual exponential function [5].

The first order general theta function can usually be thought as a function of z only by assuming that q and τ are constants. Hence, it is also denoted by $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z)$. Moreover, the basis of period lattice of this function is taken as $(\pi, \pi\tau)$, unless otherwise stated.

If we put $q = e^{i\pi\tau}$ (where $|q| = e^{-\pi s} < 1$ if $\tau = \dfrac{w_1}{w_2} = r + is$ and $s > 0$) and $\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ for the characteristic $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ in (1), we get the following four theta functions:

$$\theta_1(z,q) = \theta\begin{bmatrix}1\\1\end{bmatrix}(z,\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{iz(2n+1)}, \tag{2}$$

$$\theta_2(z,q) = \theta\begin{bmatrix}1\\0\end{bmatrix}(z,\tau) = \sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2} e^{iz(2n+1)}, \tag{3}$$

(4) $$\theta_3(z,q) = \theta\begin{bmatrix}0\\0\end{bmatrix}(z,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2\pi i z},$$

and

(5) $$\theta_4(z,q) = \theta\begin{bmatrix}0\\1\end{bmatrix}(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2\pi i z},$$

respectively. It can easily seen from the above equalities that these four theta functions are periodic and also θ_1, θ_2 have 2π as a fundamental period and θ_3, θ_4 have π as a fundamental period. Moreover, the zeros of the functions θ_1 , θ_2 , θ_3 and θ_4 are the points congruent to 0, $\frac{\pi}{2}$, $\frac{\pi}{2}+\frac{\pi\tau}{2}$ and $\frac{\pi\tau}{2}$, respectively.

The transformations between the functions θ_1 , θ_2, θ_3 and θ_4 are given by the following table, [1] (where $N = q^{-1}e^{-2iz}$ and $M = q^{\frac{1}{4}}e^{iz}$).

z	$z+\pi$	$z+\pi\tau$	$z+\frac{\pi}{2}$	$z+\frac{\pi}{2}+\frac{\pi\tau}{2}$	$z+\frac{\pi\tau}{2}$
$\theta_1(z)$	$-\theta_1(z)$	$-N\theta_1(z)$	$\theta_2(z)$	$M^{-1}\theta_3(z)$	$iM^{-1}\theta_4(z)$
$\theta_2(z)$	$-\theta_2(z)$	$N\theta_2(z)$	$-\theta_1(z)$	$-iM^{-1}\theta_4(z)$	$M^{-1}\theta_3(z)$
$\theta_3(z)$	$\theta_3(z)$	$N\theta_3(z)$	$\theta_4(z)$	$iM^{-1}\theta_1(z)$	$M^{-1}\theta_2(z)$
$\theta_4(z)$	$\theta_4(z)$	$-N\theta_4(z)$	$\theta_3(z)$	$M^{-1}\theta_2(z)$	$iM^{-1}\theta_1(z)$

Table I.

Theorem 2.1. ([1, 4]) *We have the well-known identity:*

$$\theta_1'(0) = \theta_2(0)\theta_3(0)\theta_4(0).$$

Theorem 2.2. ([1, 3]) *The squares of the functions $\theta_1(z)$, $\theta_2(z)$, $\theta_3(z)$ and $\theta_4(z)$ satisfy the following functional relations:*

(6) $$\theta_2^2(z)\theta_3^2(0) = \theta_3^2(z)\theta_2^2(0) - \theta_1^2(z)\theta_4^2(0),$$

(7) $$\theta_1^2(z)\theta_3^2(0) = \theta_4^2(z)\theta_2^2(0) - \theta_2^2(z)\theta_4^2(0).$$

After this preface, we shall give the following main theorem.

Theorem 2.3. *The quotient function* $w(z) = \dfrac{\theta_2(z)}{\theta_1(z)}$ *satisfies the non-linear differential equation*

$$(\frac{dw}{dz})^2 = [\theta_4^2(0) + w^2(z)\theta_3^2(0)][\theta_3^2(0) + w^2(z)\theta_4^2(0)].$$

P r o o f. From Table I, we see that the function $w(z) = \dfrac{\theta_2(z)}{\theta_1(z)}$ has periodicity factors ∓ 1, associated with the periods π and $\pi\tau$ respectively. That is,

$$w(z+\pi) = \frac{\theta_2(z+\pi)}{\theta_1(z+\pi)} = \frac{-\theta_2(z)}{-\theta_1(z)} = w(z),$$

$$w(z+\pi\tau) = \frac{\theta_2(z+\pi\tau)}{\theta_1(z+\pi\tau)} = \frac{N\theta_2(z)}{-N\theta_1(z)} = -w(z), \quad (N = q^{-1}e^{-2iz}).$$

The derivative function

$$\frac{d}{dz}w(z) = \frac{\theta_2'(z)\theta_1(z) - \theta_1'(z)\theta_2(z)}{\theta_1^2(z)}$$

of $w(z)$, has also the same periodicity factors ∓ 1, due to within the same periods. Then we have

$$\frac{d}{dz}w(z+\pi) = \frac{d}{dz}w(z), \quad \frac{d}{dz}w(z+\pi\tau) = -\frac{d}{dz}w(z).$$

Again with the aid of Table 1, it can be easily shown that $w_1(z) = \dfrac{\theta_3(z)\theta_4(z)}{\theta_1^2(z)}$ has also the periodicity factors ∓ 1, associated with the periods π and $\pi\tau$ respectively.

Now consider the function

$$\psi(z) = \frac{\theta_2'(z)\theta_1(z) - \theta_1'(z)\theta_2(z)}{\theta_3(z)\theta_4(z)} \tag{8}$$

which is indeed the ratio of $\dfrac{d}{dz}w(z)$ to $w_1(z)$. The function $\psi(z)$ is doubly-periodic with periods π and $\pi\tau$, having simple poles at the zeros of $\theta_3(z)$ and $\theta_4(z)$. Therefore, the simple poles of $\psi(z)$ are at the points congruent to $\dfrac{\pi}{2} + \dfrac{\pi\tau}{2}$ and $\dfrac{\pi\tau}{2}$.

Using Table I, we notice

$$\psi(z+\frac{\pi}{2}) = \frac{d}{dz}\left[\frac{\theta_2(z+\frac{\pi}{2})}{\theta_1(z+\frac{\pi}{2})}\right] \frac{\theta_1^2(z+\frac{\pi}{2})}{\theta_3(z+\frac{\pi}{2})\theta_4(z+\frac{\pi}{2})}$$

$$= \frac{d}{dz}\left[\frac{-\theta_1(z)}{\theta_2(z)}\right] \frac{\theta_2^2(z)}{\theta_4(z)\theta_3(z)}$$

$$= \frac{-\theta_1'(z)\theta_2(z) + \theta_2'(z)\theta_1(z)}{\theta_4(z)\theta_3(z)} = \psi(z).$$

This implies that $\psi(z)$ is doubly-periodic with periods π and $\frac{\pi}{2}$. On account of the fact that $\psi(\frac{\pi}{2}) = \psi(0) \neq \infty$ and $z = 0$ is not a pole of $\psi(z)$, it follows that $\psi(z)$ has no pole at $z = \frac{\pi}{2}$. So, the only pole of $\psi(z)$ is the point $z = \frac{\pi\tau}{2}$. Hence, $\psi(z)$ is a first order elliptic function with the periods π and $\frac{\pi}{2}$ and according to Theorem 1.1 it must be a constant. Thus,

$$\frac{\theta_2'(z)\theta_1(z) - \theta_1'(z)\theta_2(z)}{\theta_3(z)\theta_4(z)} = C, \quad (C \text{ constant}). \tag{9}$$

One can find out that

$$C = -\theta_2^2(0)$$

by using Theorem 2.1 and the fact that $\theta_1(0) = 0$ making $z \to \infty$. Writing C in (9) we obtain

$$\frac{\theta_2'(z)\theta_1(z) - \theta_1'(z)\theta_2(z)}{\theta_3(z)\theta_4(z)} = -\theta_2^2(0). \tag{10}$$

From (10) we deduce

$$\frac{d}{dz}\left[\frac{\theta_2(z)}{\theta_1(z)}\right] \frac{\theta_1^2(z)}{\theta_3(z)\theta_4(z)} = -\theta_2^2(0),$$

$$\frac{d}{dz}\left[\frac{\theta_2(z)}{\theta_1(z)}\right] = -\theta_2^2(0)\frac{\theta_3(z)\theta_4(z)}{\theta_1^2(z)}.$$

We raise the squares of both sides to obtain

$$\left(\frac{dw}{dz}\right)^2 = \frac{\theta_2^2(0)\theta_3^2(z)}{\theta_1^2(z)} \frac{\theta_2^2(0)\theta_4^2(z)}{\theta_1^2(z)}.$$

We consider the functional relations of Theorem 2.2 to get

$$\left(\frac{dw}{dz}\right)^2 = \frac{\theta_1^2(z)\theta_1^2(0) + \theta_2^2(z)\theta_3^2(0)}{\theta_1^2(z)} \frac{\theta_2^2(z)\theta_4^2(0) + \theta_1^2(z)\theta_3^2(0)}{\theta_1^2(z)}. \tag{11}$$

The above differential equation reduces to the form

$$(12) \qquad \left(\frac{dw}{dz}\right)^2 = \left[\theta_4^2(0) + w^2(z)\theta_3^2(0)\right]\left[\theta_3^2(0) + w^2(z)\theta_4^2(0)\right],$$

by writing $w(z) = \dfrac{\theta_2(z)}{\theta_1(z)}$ in (11). ■

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