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Regular Filter Convergence Spaces

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There are various generalizations of the usual topological T_3-axiom to topological categories defined in [1] and [7]. In this paper, a characterization of each of them is given in the categories of filter and local filter convergence spaces. Furthermore, the relationships among these various forms of T_3 structures as well as some invariance properties of them are investigated in these categories.

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1. Introduction

Let $\mathcal{E}$ be a category and SET be the category of sets. The functor $U : \mathcal{E} \to$ SET is said to be topological, or $\mathcal{E}$ is said to be a topological category over SET, if U is concrete (i.e. faithful and amnestic, i.e. if $U(f) = \text{id}$ and f is an isomorphism, then $f = \text{id}$), has small (i.e. sets) fibers, and for which every U-source has an initial lift or, equivalently, for which each U-link has a final lift, [8] or [10].

Observe that for a T_1 topological space X, X is T_3 iff: (a) X/F is T_2 if it is T_1, where F is any nonempty subset of X; (b) X/F is $PreT_2$ (for any two distinct points, if there is a neighbourhood of one missing the other, then the points have disjoint neighbourhoods) if it is T_1, where $\emptyset \neq F$ in X; (c) X/F is $PreT_2$ for closed $\emptyset \neq F$ in X. The equivalence of (a), (b) and (c) follows from the facts that for T_1 topological spaces, T_2 is equivalent to $PreT_2$ and F is closed iff X/F is T_1.

In view of (c) and (d) from [1], there are four ways of generalizing the usual T_3 separation axiom to arbitrary set based topological categories. In view of (a) and (b) from [7], there are four more ways of generalizing the usual T_3 separation axiom to arbitrary set based topological categories.

In this paper, we give a characterization of all T_3-objects in the categories of filter and local filter convergence spaces. Furthermore, we investigate the relationships among various forms of the T_3-objects in these categories and give some invariance properties of them.

1.1. Let A be a set and K be a function which assigns to each point x of A a set of filters (proper or not, where a filter α is proper iff α does not contain the empty set $\emptyset$, i.e. $\alpha \neq [\emptyset]$ (the "filters converging to x") is called a convergence structure on A ((A, K) a filter convergence space), iff it satisfies the following two conditions:

1. $[x] = [\{x\}] \in K(x)$ for each $x \in A$ (where $[F] = \{B \subset A : F \subset B\}$.
2. $\beta \supset \alpha \in K(x)$ implies $\beta \in K(x)$ for any filter β on A.

A map $f : (A, K) \to (B, L)$ between filter convergence spaces is called continuous, iff $\alpha \in K(x)$ implies $f(\alpha) \in L(f(x))$ (where $f(\alpha)$ denotes the filter generated by $\{f(D) : D \in \alpha\}$. The category of filter convergence spaces and continuous maps is denoted by FCO [12]. A filter convergence space (A, K) is said to be a local filter convergence space if $\alpha \cap [x] \in K(x)$ whenever $\alpha \in K(x)$, see [11], p.1374. The category LFCO of local filter convergence spaces is the full subcategory of FCO, [11], p.1374.

For filters α and β, we denote by $\alpha \cup \beta$ the smallest filter (proper or not) containing both α and β, i.e. $\alpha \cup \beta = \{M \subset A : U \cap V \subset M$ for some $U \in \alpha$ and $V \in \beta\}$.

Let $U : \mathcal{E} \to$ SET be topological and X an object in $\mathcal{E}$ with $UX = B$. Let F be a nonempty subset of B. We denote by X/F the final lift of the epi U-sink $q : U(X) = B \to B/F = (B \setminus F) \cup \{*\}$, where q is the epi map that is identity on $B \setminus F$ and identifying F with a point $*$. Let p be a point in B.

Lemma 1.2. (cf. [7], Lemma 1.4) *Let α and β be proper filters on B. Then $q(\alpha) \cup q(\beta)$ is proper iff either $\alpha \cup \beta$ is proper or both $\alpha \cup [F]$ and $\beta \cup [F]$ are proper.*

Lemma 1.3. (cf. [2], Lemma 3.16)
(1) For $a \in B$ with $a \notin F$, $q(\alpha) \subset [a]$ iff $\alpha \subset [a]$,
(2) $q(\alpha) \subset [*]$ iff $\alpha \cup [F]$ is proper.

Lemma 1.4. (cf. [2], Lemma 3.19)
(1) If $\alpha \cup [F]$ is improper, then $q(\sigma) \subset q(\alpha)$ iff $\sigma \subset \alpha$,
(2) If $\alpha \cup [F]$ is proper, then $q(\sigma) \subset q(\alpha)$ iff $\sigma \cap [F] \subset \alpha$ and $\sigma \cup [F]$ is proper.

Lemma 1.5 *Let $X = (B, K)$ be in FCO or LFCO and $\emptyset \neq F \subset B$.*
(1) X is ΔT_2 iff for all $x \neq y$ in B, $[x] \notin K(y)$ iff X is T_1, [3].

2. X is ST_3' iff X is T_1 and X/F is $PreT_2'$ for all strongly closed $F \neq \emptyset$ in $U(X)$, [1].

3. X is $\bar{T}_3$ iff X is T_1 and X/F is $Pre\bar{T}_2$ for all closed $F \neq \emptyset$ in $U(X)$, [1].

4. X is T_3' iff X is T_1 and X/F is $PreT_2'$ for all closed $F \neq \emptyset$ in $U(X)$, [1].

5. X is KT_3 iff X is T_1 and X/F is $Pre\bar{T}_2$ if it is T_1, where $F \neq \emptyset$ in $U(X)$, [7].

6. X is LT_3 iff X is T_1 and X/F is $PreT_2'$ if it is T_1, where $F \neq \emptyset$ in $U(X)$, [7].

7. X is ST_3 iff X is T_1 and X/F is ST_2 if it is T_1, where $F \neq \emptyset$ in $U(X)$, [7].

8. X is ΔT_3 iff X is T_1 and X/F is ΔT_2 if it is T_1, where $F \neq \emptyset$ in $U(X)$, [7].

Remark 2.2. (1) For the category TOP of topological spaces, all of T_3's reduce to the usual T_3 separation axiom by above (see Introduction) and [7].

(2) If $U : \mathcal{E} \to \mathcal{B}$, where $\mathcal{B}$ is a topos [9], then Parts (1), (2), and (5)–(7) of Definitions 2.1 do make sense since each of these notations requires only finite products and finite colimits in their definitions. Furthermore, if $\mathcal{B}$ has infinite products and infinite wedge products, then Definitions 2.1, (3), (4) and (8) also make sense.

Theorem 2.3. *$X = (B, K)$ in FCO or LFCO is $S\bar{T}_3$ iff conditions (1), (2), and (3) hold, where the conditions are:*

(1) for all $a \neq b$ in B, $K(a) \cap K(b) = \{[\emptyset]\}$;

(2) for any nonempty strongly closed subset F of X, $a \in B$, and any proper filters $\alpha, \delta \in K(a)$:

(i) if $a \notin F$ and $\alpha \cup \delta$ is proper, then there exists a filter $\beta \in K(a)$ such that $\beta \subset \alpha \cup \delta$;

(ii) if $a \in F$ and either $\alpha \cup \delta$ is proper or both $\alpha \cup [F]$ and $\delta \cup [F]$ are proper, then $\exists d \in F$ and a filter $\beta \in K(d)$ such that either $\beta \subset \alpha \cap \delta$ or $\beta \cap [F] \subset \alpha \cap \delta$ and $\beta \cup [F]$ is proper;

(3) for any nonempty strongly closed subset F of X and any proper filters $\alpha \in K(c)$ and $\delta \in K(d)$ with $c, d \in F$, if both $\alpha \cup [F]$ and $\delta \cup [F]$ are proper, then there exists $e \in F$ and a filter $\beta \in K(e)$ such that $\beta \cap [F] \subset \alpha \cap \delta$ and $\beta \cup [F]$ is proper.

Proof. Suppose X is $S\bar{T}_3$. Let $\alpha \in (K(a) \cap K(b))$ with $a \neq b$ in B. Then $q(\alpha) \in K'(q(a)) \cap K'(q(b))$, where k' is the quotient structure on B/F induced by the map $q : B \to B/F$ that identifies F to a point $*$. X is $S\bar{T}_3$, in particular X/F is $\bar{T}_2$ and so, by 1.5, $q(\alpha) = [\emptyset]$. Hence $\alpha = [\emptyset]$. This shows that condition (1) holds.

Suppose that for any nonempty strongly closed subset F of X, $a \in B$, and any proper filters $\alpha, \delta \in K(a)$.

Suppose $a \notin F$ and $\alpha \cup \delta$ is proper. Then, by 1.2, $q(\alpha) \cup q(\delta)$ is proper. Note that $q(\alpha), q(\delta) \in K'(q(a))$. Since X/F is $Pre\bar{T}_2$, by 1.5, $q(\alpha \cap \delta) = q(\alpha) \cap q(\delta) \in K'(q(a))$. It follows from definition of K' that there exists $\beta \in K(a)$ such that $q(\beta) \subset q(\alpha \cap \delta)$. If $(\alpha \cap \delta) \cup [F]$ is proper, then, by Lemma 1.3 (2), $q(\alpha \cap \delta) \subset [*]$ and thus, $[*] \in K'(q(a))$, a contradiction. Therefore, $(\alpha \cap \delta) \cup [F]$ must be improper and by Lemma 1.4 (1), $\beta \subset \alpha \cap \delta$.

Suppose $a \in F$ and either $\alpha \cup \delta$ is proper or both $\alpha \cup [F]$ and $\delta \cup [F]$ are proper. It follows from definition of K' that there exists $d \in F$ and there exists $\beta \in K(d)$ such that $q(\beta) \subset q(\alpha \cap \delta)$ and $q(d) = * = q(q)$. If $(\alpha \cap \delta) \cup [F]$ is improper, then, by Lemma 1.4 (1), $\beta \subset \alpha \cap \delta$. If $(\alpha \cap \delta) \cup [F]$ is proper, then, by Lemma 1.4 (2), $\beta \cap [F] \subset \alpha \cap \delta$ and $\beta \cup [F]$ is proper. Thus, condition (2) also holds.

Suppose that for any nonempty strongly closed subset F of X, any proper filters $\alpha \in K(c)$ and $\delta \in K(d)$ with $c, d \in F$, $\alpha \cup [F]$ and $\delta \cup [F]$ are proper. Then, by 1.2, $q(\alpha) \cup q(\delta)$ is proper. Note that $q(\alpha), q(\delta) \in K'(*)$. Since X/F is $Pre\bar{T}_2$, by 1.5, $q(\alpha \cap \delta) = q(\alpha) \cap q(\delta) \in K'(*)$. It follows that there exists $e \in F$ and a filter $\beta \in K(e)$ such that $q(\beta) \subset q(\alpha \cap \delta)$ and $q(e) = *$. Since $(\alpha \cap \delta) \cup [F]$ is proper, by Lemma 1.4 (2), $\beta \cap [F] \subset \alpha \cap \delta$ and $\beta \cup [F]$ is proper.

Conversely, suppose that the conditions hold. By (1) and 1.5, X is T_1. Suppose F is strongly closed subset of X. Note, by 1.6, that X/F is T_1. Hence, it is sufficient to show that X/F is $\bar{T}_2$ for any nonempty strongly closed subset F of X. Let $x \neq y$ in B/F and $\sigma \in K'(x) \cap K'(y)$. If $\sigma = [\emptyset]$, then we are done. Suppose $\sigma \neq [\emptyset]$. It follows that there exist $\alpha \in K(a)$ and $\delta \in K(b)$ such that $q(\alpha) \subset \sigma$, $q(\delta) \subset \sigma$ and $q(a) = x$, $q(b) = y$. Notice that $q(\alpha) \cup q(\delta)$ is proper, and so, by 1.2, either $\alpha \cup \delta$ is proper or both $\alpha \cup [F]$ and $\delta \cup [F]$ are proper. By assumption (1), the first case can not occur. The second case can not happen either, since F is strongly closed subset of X (by 1.5, we may assume that $a \notin F$). Hence, we must have $\sigma = [\emptyset]$.

It remains to show that for any proper filters $\sigma, \gamma \in K'(x)$ with $\sigma \cup \gamma$ proper, $\sigma \cap \gamma \in K'(x)$. Let $x \neq *$. If $\sigma, \gamma \in K'(x)$, then there exist $\alpha, \delta \in K(a)$ such that $q(\alpha) \subset \sigma$, $q(\delta) \subset \gamma$ and $q(a) = a = x$. It follows that $q(\alpha) \cup q(\delta)$ is proper, and so, by 1.2, either $\alpha \cup \delta$ is proper or both $\alpha \cup [F]$ and $\delta \cup [F]$ are proper. The second case can not occur since F is strongly closed subset of X (by 1.5). Hence, we must have $\alpha \cup \delta$ is proper. By the assumption (2), there exists $\beta \in K(a)$ such that $\beta \subset \alpha \cap \delta$. Note that $q(\beta) \subset q(\alpha) \cap q(\delta) \subset \sigma \cap \gamma$ and consequently, $\sigma \cap \gamma$ is in $K'(x)$.

Suppose $x = *$ and $\sigma, \gamma \in K'(*)$, then there exist $c, d \in F$ and $\alpha \in K(c)$, $\delta \in K(d)$ such that $q(\alpha) \subset \sigma$, $q(\delta) \subset \gamma$ and $q(c) = * = q(d)$. It follows that $q(\alpha) \cup q(\delta)$ is proper, and so, by 1.2 either $\alpha \cup \delta$ is proper or both $\alpha \cup [F]$ and $\delta \cup [F]$ are proper.

If $c \neq d$, then the first case can not hold since $\alpha \cup \delta \in K(c) \cap K(d)$. Thus, the second case must hold. By the assumption (3), there exist $e \in F$ and $\beta \in K(e)$ such that $\beta \cap [F] \subset \alpha \cap \delta$ and $\beta \cup [F]$ is proper. Hence, $q(\beta) = q(\beta \cap [F]) = q(\beta) \cap [*] \subset \sigma \cap \gamma$ and consequently $\sigma \cap \gamma \in K'(*)$, since by Lemma 1.3 (2), $\beta \cup [F]$ is proper iff $q(\beta) \subset [*]$.

Suppose $c = d$ and either $\alpha \cup \delta$ is proper or both $\alpha \cup [F]$ and $\delta \cup [F]$ are proper. Then, by assumption (2), there exist $e \in F$ and $\beta \in K(e)$ such that $\beta \subset \alpha \cap \delta$ or $\beta \cap [F] \subset \alpha \cap \delta$ and $\beta \cup [F]$ is proper. If the first case holds, then $q(\beta) \subset q(\alpha) \cap q(\delta) \subset \sigma \cap \gamma$ and consequently, $\sigma \cap \gamma \in K'(*)$. If the second case holds, then $q(\beta) = q(\beta \cap [F]) = q(\beta) \cap [*] \subset \sigma \cap \gamma$ and consequently, $\sigma \cap \gamma \in K'(*)$, since by Lemma 1.3 (2), $\beta \cup [F]$ is proper iff $q(\beta) \subset [*]$. Hence, by 1.5, X/F is $\bar{T}_2$ and thus, X is $S\bar{T}_3$. ■

Theorem 2.4. *Let $X = (B, K)$ be in FCO or LFCO and F be any nonemty subset of X such that $\forall x \in B$ if $x \notin F$ and $\alpha \in K(x)$, then $\alpha \cup [F]$ is improper.*

Then X is $\bar{T}_3$ iff X is $S\bar{T}_3$.

Proof. It follows from 1.6, 2.1, and 2.3. ■

Theorem 2.5. (Theorem 2.8 in [7]) *Let $X = (B, K)$ be in FCO or LFCO.*

(1) X is ΔT_3 iff X is T_1.

(2) X is ST_3 iff X is ST_2.

(3) X is LT_3 iff ST_3' iff for all $x \neq y$ in B, $[x] \notin K(x)$, for any nonempty strongly closed subset F of B, $x \in B$, and any proper filter $\alpha \in K(x)$ either $\alpha = [x]$ or $F \in \alpha$.

(4) X is T_3' iff for all $x \neq y$ in B, $[x] \notin K(y)$, for any nonempty subset F of B, $x \in B$, and any proper filter $\alpha \in K(x)$ either $\alpha = [x]$ or $F \in \alpha$.

Remark 2.6. (1) For the Category FCO or LFCO, by 2.3, 2.4 and 2.5, $ST_3' = LT_3 \Rightarrow T_3' \Rightarrow ST_3 \Rightarrow \Delta T_3$ and $ST_3' = LT_3 \Rightarrow T_3' \Rightarrow S\bar{T}_3' = KT_3 \Rightarrow \bar{T}_3'$ but the converse of each implication is not true, in general.

(2) By 1.5 and 2.5, if X is ST_3', LT_3 or T_3', then all subsets of X are both closed and strongly closed.

(3) By 1.5 and 2.5, if X is ΔT_3, then F is always closed and F is strongly closed iff $\forall x \in B$ if $x \notin F$ and $\alpha \in K(x)$, then $\alpha \cup [F]$ is improper.

(4) By 1.5 and 2.5, if X is ST_3', LT_3 or T_3', then X is T_2' and LT_2, [5].
(5) Let $U : \mathcal{E} \to \text{SET}$ be topological and X an object in $\mathcal{E}$.
In [7], we have showed the following implications:
$T_3' \Rightarrow \bar{T}_3$, $ST_3' \Rightarrow S\bar{T}_3$, and $LT_3 \Rightarrow KT_3$.

Now we give some invariance properties of regular filter convergence spaces.

Let $X = (B, K)$ be in FCO or LFCO and F be a nonempty subset of X. Let $q : X \to X/F$ be the identification map defined in introduction.

Theorem 2.7. *If X is ST_3', LT_3 or T_3', then X/F is ST_3', LT_3 or T_3'.*

Proof. Suppose X is ST_3'. If $[x] \in K'(y)$ for some $x \neq y$ in B/F, where K' is the quotient structure on B/F then there exists $\alpha \in K(a)$ such that $[x] \supset q(\alpha)$ and $q(a) = y$. Since X is ST_3', by 2.5, $\alpha = [a]$ or $F \in \alpha$. The first case can not hold since $[q(a) = y] = q(\alpha) \subset [x]$ and $x \neq y$. If the second case holds, then by 1.3, $q(\alpha) = [*]$ and consequently, $y = [*]$, i.e., $q(a) = y = * = x$, a contradiction, since X is ST_3'.

Suppose $x \in B/F$ and α is any proper filter in $K'(x)$. We must show that $\alpha = [x]$ or $F' \in \alpha$ for all nonempty strongly closed subset F' of B/F. It follows that there exists a proper filter $\beta \in K(a)$ such that $\alpha \supset q(\beta)$ and $q(a) = x$. Since X is ST_3', by 2.5, $\beta = [a]$ or $F \in \beta$. If the first case holds, then $[q(a) = x] = q(\beta) \subset \alpha$ and consequently, $\alpha = [x]$. If the second case holds, then by 1.3, $q(\beta) = [*]$ and so, $\alpha = [*]$. Hence, by 2.5, X/F is ST_3'. The proof for T_3' is similar. ■

Theorem 2.8. *If X is ΔT_3 and F is strongly closed, then X/F is ΔT_3.*

Proof. Suppose that X is ΔT_3 and F is strongly closed. By 2.5, we have to show that for all $x \neq y$ in B/F, $[x] \notin K'(y)$. Suppose $[x] \in K'(y)$ for some $x \neq y$ in B/F. It follows that there exists $\alpha \in K(a)$ such that $[x] \supset q(\alpha)$ and $q(a) = y$. If $x \neq *$, i.e. $x \notin F$, then, by 1.3, $\alpha \subset [x]$ and consequently, $[x] \in K(a)$, a contradiction since X is ΔT_3. If $x = *$, then by 1.3, $\alpha \cup [F]$ is proper, and consequently, $a \in F$ (since F is strongly closed). It follows that $x = * = q(a) = y$, a contradiction. Hence X/F is ΔT_3. ■

Theorem 2.9. *If X is ST_3 and F is strongly closed, then X/F is ST_3.*

Proof. Suppose that X is ΔT_3 and F is strongly closed. By 2.5, we need to show that for all $x \neq y$ in B/F, $K'(x) \cap K'(y) = [\emptyset]$. Suppose that $\alpha \in K'(x) \cap K'(y)$. It follows that there exists $\delta \in K(a)$ and $\beta \in K(b)$ such that $q(\beta) \subset \alpha$, $q(\delta) \subset \alpha$, and $qa = x$, $qb = y$. If α is improper, then we are done.

Suppose α is proper. It follows that $q(\beta) \cup q(\delta)$ is proper and so, by 1.2, $\beta \cup \delta$ is proper or both $\beta \cup [F]$ and $\delta \cup [F]$ are proper. The first case can not occur since X is ST_3 and $a \neq b$. The second case can not occur, either, since F is strongly closed (we may assume $a \notin F$ since $x \neq y$). This completes the proof. ■

Theorem 2.10. *Suppose X is $S\bar{T}_3$. If F is strongly closed and $\forall x \in F$, $K(x) = \{[\emptyset], [x]\}$, then X/F is $S\bar{T}_3$.*

Proof. Let $A = B/F$ and $Y = (A, K')$. If $x \neq y$ in B/F, then, by 2.9, $K'(x) \cap K'(y) = [\emptyset]$. Suppose that for any nonempty strongly closed subset F' of A, $x \in A$, and any proper filters $\sigma, \gamma \in K'(x)$. Suppose $x \notin F'$ and $\sigma \cap \gamma$ is proper. It follows that there exist $\delta \in K(a)$ and $\alpha \in K(b)$ such that $q(\beta) \subset \alpha$, $q(\delta) \subset \alpha$, and $qa = x = qb$.

Suppose $x \neq *$. If $\sigma \cup \gamma$ is proper, then $q(\delta) \cup q(\alpha)$ is proper, and by 1.2, either $\delta \cup \alpha$ is proper or both $\delta \cup [F]$ and $\alpha \cup [F]$ are proper, where $q : B \to B/F$ is the epi map that identifies F to $*$. The second case can not occur since F is strongly closed. Since X is $S\bar{T}_3$, by 2.3, there exists $\beta \in K(x = a = b)$ such that $\beta \subset \alpha \cap \delta$. It follows that $q(\beta) \in K'(x)$ and $q(\beta) \subset q(\alpha) \cap q(\delta) \subset \sigma \cap \gamma$. Suppose $x \notin F'$ and $x = *$. It follows that $a, b \in F$, and by assumption $\delta = [b]$ and $\alpha = [a]$. Hence, $q(\sigma) = [*] = q(\gamma)$. Let $\beta = [a]$ and note that $q(\beta) = [*] \subset \sigma \cap \gamma$.

Suppose $x \in F'$ and either $\sigma \cup \gamma$ is proper or both $\sigma \cup [F']$ and $\gamma \cup [F']$.

Suppose also that $x \neq *$. If $\sigma \cup \gamma$ is proper, then $q(\delta) \cup q(\alpha)$ is proper, and by 1.2 either $\delta \cup \alpha$ is proper or both $\delta \cup [F]$ and $\alpha \cup [F]$ are proper. The second case can not occur since F is strongly closed and $x \notin F$. Since X is $S\bar{T}_3$, by 2.3, there exists $\beta \in K(x)$ such that $\beta \subset \alpha \cap \delta$. Note that $q(\beta) \in K'(x)$ and $q(\beta) \subset q(\alpha) \cap q(\delta) \subset \sigma \cap \gamma$.

Suppose that $\sigma \cup [F']$ and $\gamma \cup [F']$ are proper. If follows that $q(\delta) \cup [F']$ and $q(\alpha) \cup [F']$ are proper. Since $q^{-1}F'$ is strongly closed subset of B (by 1.6(5)) and F is strongly closed, it follows that $\alpha \cup [q^{-1}F']$ and $\delta \cup [q^{-1}F']$ are proper. Since X is $S\bar{T}_3$, by 2.3(2), there exists $d \in q^{-1}F'$ and $\beta \in K(d)$ such that either $\beta \subset \alpha \cap \delta$ or both $\beta \cap [q^{-1}F'] \subset \alpha \cap \delta$ and $\beta \cup [q^{-1}F']$ is proper. Note that $q(d) \in F'$ and $q(\beta) \in K'(q(d))$. Hence, $q(\beta) \subset \sigma \cap \gamma$ or $q(\beta \cap [q^{-1}F']) = q(\beta) \cap [F'] \subset q(\alpha) \cap q(\delta) \subset \sigma \cap \gamma$ and $q(\beta) \cup [qq^{-1}F' = F'])$ is proper.

Suppose $x \in F'$ and $x = *$. It follows that $a, b \in F$, and by assumption $\delta = [b]$ and $\alpha = [a]$. Hence, $q(\sigma) = [*] = q(\gamma)$. Let $\beta = [\alpha]$ and note that $q(\beta) = [*] \subset \sigma \cap \gamma$.

Suppose that for any nonempty strongly closed subset F' of Y and any proper filters $\sigma \in K'(x)$ and $\gamma \in K'(y)$ with $x, y \in F'$. It follows that there exist $\delta \in K(a)$ and $\alpha \in K(b)$ such that $q(\beta) \subset \sigma$, $q(\delta) \subset \gamma$, and $qa = x$, $qb = y$.

Suppose that $\sigma \cup [F']$ and $\gamma \cup [F']$ are proper. Then $q(\delta) \cup [F']$ and $q(\alpha) \cup [F']$ are proper. Let $x \neq * \neq y$. Since $q^{-1}F'$ is strongly closed subset of B (by 1.6) and F is strongly closed, it follows that $\alpha \cup [q^{-1}F']$ and $\delta \cup [q^{-1}F']$ are proper. Since X is $S\bar{T}_3$, by 2.3, there exists $d \in q^{-1}F'$ and $\beta \in K(d)$ such that either $\beta \subset \alpha \cap \delta$ or $\beta \cap [q^{-1}F'] \subset \alpha \cap \delta$ and $\beta \cup [q^{-1}F']$ is proper. It follows that $q(d) \in F'$, $q(\beta) \in K'(q(d))$ and $q(\beta) \subset q(\alpha) \cap q(\delta) \subset \sigma \cap \gamma$ or $q(\beta \cap [q^{-1}F'] = q(\beta) \cap [F'] \subset q(\alpha) \cap q(\delta) \subset \sigma \cap \gamma$ and $q(\beta) \cup [F']$ is proper. Let $x = * \neq y$. Then $a \in F$ and, by assumption, $\alpha = [a]$. Note that $\alpha \cup [q^{-1}[F']$ and $\delta \cup [q^{-1}F']$ are proper. Since X is $S\bar{T}_3$, by 2.3, there exists $d \in q^{-1}F'$ and $\beta \in K(d)$ such that $\beta \cap [q^{-1}F'] \subset \alpha \cap \delta$ and $\beta \cup [q^{-1}F']$ is proper. Note that $q(d) \in F'$, $q(\beta) \in K'(x)$. It follows that $q(\beta \cap [q^{-1}F']) = q(\beta) \cap [F'] \subset \sigma \cap \gamma$ and $q(\beta) \cup [F']$ is proper.

If $x \neq * = y$, then interchange the role of x and y in the above.

Suppose $x = * = y$. Then $a, b \in F$, and by assumption $\delta = [b]$ and $\alpha = [a]$.

Hence, $q(\sigma) = [*] = q(\gamma)$. Let $\beta = [a]$ and note that $q(\beta) = [*] \subset \sigma \cap \gamma$. ■

Remark 2.11. Let $X = (B, K)$ be in FCO or LFCO.

Let I be finite set and for each $i \in I$, $\emptyset \neq F_i \subset B$ such that for all $i \neq j$ in I, $F_i \cap F_j = \emptyset$. Let R be the equivalence relation $\cup_{i \in I}(F_i \times F_i) \cup \{(x, x) : x \in B\}$. Note that the quotient map $q : X \to X/R$ is the composition of the quotient maps $q_1 : X \to X/F_1 = Y_1$, $q_2 : Y_1 \to Y_1/F_2 = Y_2$, ... and $q_n : Y_{n-1} \to Y_{n-1}/F_n = X/R$. Note that if $I = \{1\}$, a one-point set, then $X/R = X/F$, the quotient space defined in the introduction. Then, we have:

(1) By induction, Theorem 2.7 holds for this quotient space X/R.

(2) By induction, Theorem 2.8 holds for this quotient space X/R provided that each F_i is strongly closed, $i \in I$.

(3) By induction, Theorem 2.8 holds for this quotient space X/R provided that each F_i is strongly closed, $i \in I$, and $\forall x \in F_i$, $K(x) = \{[\emptyset], [x]\}$.

Lemma 2.12. *Let $X = (B, K)$ be an object in FCO or LFCO and $Y = (A, L)$ be a subspace of X.*

If X is ΔT_3, ST_3 or T_3', then Y is ΔT_3, ST_3 or T_3', respectively.

Proof. It follows easily from 2.5. ■

Lemma 2.13. *If $A \subset B$ and α is a filter on B, then the following are equivalent:*

(1) There exists a filter β on A such that $\alpha = \{U \subset B : V \subset U$ for some $V \in \beta\}$,

(2) $A \in \alpha$.

Proof. If $A \in \alpha$, then let $\beta = \{V : A \supset V\}$. For the converse note that note that $A \in \beta \subset \{U \subset B : V \subset U \text{for some } V \in \beta\} = \alpha$. ■

Lemma 2.14. *Let $X = (B, K)$ be in FCO or LFCO, $Y = (A, L)$ be a subspace of X. Suppose that if $\alpha \in K(x)$, $x \in B$, then $A \in \alpha$. Then if X is $S\bar{T}_3$, KT_3, ST_3' or LT_3, then Y is $S\bar{T}_3$, KT_3, ST_3' or LT_3, respectively.*

Proof. Suppose X is $S\bar{T}_3$ By 2.12, if $a \neq b$ in A, then $K(a) \cap K(b) = \{[\emptyset]\}$. It remains to show that conditions (2) and (3) of Theorem 2.3 hold. Let F be any nonempty strongly closed subset F of Y, $a \in A$, and α, δ be any proper filters in$L(a)$. Suppose $a \notin F$. It follows that $i(\alpha), i(\delta) \in K(a)$, where i is the inclusion map $Y \subset X$. We show that F is strongly closed subset of B. By 1.6 (3), we need to show that $\forall a \in B$ if $a \notin F$ and$\beta \in K(a)$, then $\beta \cup [F]$ is improper. By assumption, $A \in \beta$ and by 2.13, there exists a filter σ on A such that $\beta = i(\sigma)$. Since Y is a subspace of X, $\sigma \in L(a)$. Since F is strongly closed subset of Y, $\sigma \cup [F]$ is improper. It follows that $\beta \cup [F]$ is improper. Note that $i(\alpha) \cup i(\beta)$ is proper and so, there exists a filter $\beta \in K(a)$ such that $\beta \subset i(\alpha) \cap i(\delta)$ since X is $S\bar{T}_3$. By assumption, $A \in \beta$ and by 2.13, there exists a filter σ on A such that $\beta = i(\sigma)$. Since Y is a subspace of X, $\sigma \in L(a)$ and $\sigma \subset \alpha \cap \delta$.

Suppose that $a \in F$ and either $\alpha \cup \delta$ is proper or both $\alpha \cup [F]$ and $\delta \cup [F]$ are proper. It follows that either $i(\alpha) \cup i(\delta)$ is proper or both $i(\alpha) \cup [i(F) = F]$ and $i(\delta) \cup [F]$ are proper. Since X is $S\bar{T}_3$, by 2.3, $\exists d \in F$ and a filter $\beta \in K(d)$ such that either $\beta \subset i(\alpha \cap \delta)$ or $\beta \cap [F] \subset i(\alpha \cap \delta)$ and $\beta \cup [F]$ is proper. By assumption, $A \in \beta$ and by 2.13, there exists a filter σ on A such that $\beta = i(\sigma)$. Since Y is a subspace of X, $\sigma \in L(a)$ and $\sigma \subset \alpha \cap \delta$ or $\sigma \cap [F] \subset \alpha \cap \delta$ and $\sigma \cup [F]$ is proper.

The condition (3) of 2.3 can be proved similarly.

Hence, Y is $S\bar{T}_3$.

The proof for ST_3' follows easily from 2.5 and 2.13. ■

Let $\mathcal{E}$ be a set based topological category and $f : X \to Y$ be a morphism in $\mathcal{E}$. Recall from [6], p. 225, that f is strongly closed iff the image of each strongly closed subset of X is a strongly closed subset of Y.

Theorem 2.15. *Let $X_i = (B_i, L_i)$ be objects in FCO or LFCO and $B = \prod_{i \in I} B_i$.*

(1) The cartesian product $X = (B, L)$ is ΔT_3 or ST_3 iff each X_i is ΔT_3 or ST_3, respectively.

(2) If each X_i is $S\bar{T}_3$ and the projections $\pi_i : B \to B_i$ are strongly closed, then $X = (B, L)$ is $S\bar{T}_3$.

(3) If each X_i, I is finite, is ST_3' and the projections $\pi_i : B \to B_i$ are strongly closed, then $X = (B, L)$ is ST_3'.

(4) If each X_i, I is finite, is T_3', then $X = (B, L)$ is T_3'.

Proof. (1) Suppose $X = (B, L)$ is ΔT_3 or ST_3. Then it is easy to see that each X_i is isomorphic to some slice in X and by 2.14, all X_i are ΔT_3 or ST_3, respectively.

Suppose each X_i is ΔT_3. We show that $X = (B, L)$, where L is the product structure on B, is ΔT_3. Suppose there exist $x \neq y$ in B such that $[x] \in L(y)$. It follows that there exists $m \in I$ such that $x_m \neq y_m$ in B_m and $\pi_m([x]) = [x_m] \in L_m(\pi_m(y) = y_m)$, a contradiction. Hence, for any $x \neq y$ in B, $[x] \notin L(y)$, i.e., by 2.5, X is ΔT_3.

Suppose X_i is ST_3 and for any $x \neq y$ in B, $\alpha \in L(x) \cap L(y)$. It follows that there exists $m \in I$ such that $x_m \neq y_m$ in B_m and $\pi_m(\alpha) \in L_m(\pi_m(x) = x_m) \cap L_m(\pi_m(y) = y_m)$. Since X_m is ST_3, it follows that $\pi_m(\alpha) = [\emptyset]$ and consequently $\alpha = [\emptyset]$. Hence, X is ST_3.

Suppose that each X_i is $S\bar{T}_3$. By above and 2.3, we need to show that the conditions (2) and (3) of 2.3 hold.

Let F be nonempty strongly closed subset of X , $x \in B$, and α, δ be any proper filters in $L(x)$. Let $\sigma = \cup_{m\in I}(\pi_m^{-1}\pi_m\alpha)$, $\gamma = \cup_{m\in I}(\pi_m^{-1}\pi_m\delta)$, and $F' = \cup_{m\in I}(\pi_m^{-1}\pi_m(F)) = \prod_{m\in I}\pi_m F$. Note that $\sigma \subset \alpha$ (since $\pi_m^{-1}\pi_m\alpha \subset \alpha$), $\gamma \subset \delta$, $F \subset F'$, $\pi_m\sigma = \pi_m\alpha$, $\pi_m\gamma = \pi_m\delta$, $\pi_m F = \pi_m F'$, and consequently, $\sigma, \gamma \in L(x)$ and, by 1.6, F' is strongly closed subset of X. So, we may work with σ, γ and F'.

Suppose that $x \notin F'$ and $\sigma \cup \gamma$ is proper. It follows that $\pi_m(\sigma), \pi_m(\gamma) \in L_m(\pi_m(x) = x)$, and $\pi_m(\sigma) \cup \pi_m(\gamma)$ is proper. Since $\pi_m F')$is strongly closed subset of B_m and X_m is $S\bar{T}_3$, it follows that there exists a filter $\beta_m \in L_m(x_m)$ such that $\beta_m \subset \pi_m(\sigma) \cap \pi_m(\gamma) = \pi_m(\sigma \cap \gamma)$. Let $\beta = \cup_{m\in I}(\pi_m^{-1}\beta_m)$. Then, it follows easily that $\pi_m(\beta) \supset \beta_m \in L_m(x_m)$ (since $\pi_m\beta \supset \pi_m^{-1}\pi_m\beta_m \supset \beta_m$), $\beta \in L(x)$ and $\beta \subset \sigma \cap \gamma$ (since for each $m \in I$, $\pi_m^{-1}\beta \subset \pi_m^{-1}\pi_m(\sigma \cap \gamma) \subset \sigma \cap \gamma$). Suppose that $x \in F'$ and either $\sigma \cup \gamma$ is proper or both $\sigma \cup [F']$ and $\gamma \cup [F']$ are proper. It follows that $\pi_m(\sigma), \pi_m(\gamma) \in L_m(\pi_m(x) = x_m)$ and $\pi_m(\sigma) \cup \pi_m(\gamma)$ is proper or both $\pi_m(\sigma) \cup [\pi_m F']$ and $\pi_m(\gamma) \cup [\pi_m F']$ are proper. Since $\pi_m F'$ is strongly closed subset of B_m and X_m is $S\bar{T}_3$, it follows that there exists $d_m \in \pi_m F'$ and a filter $\beta_m \in L_m(d_m)$ such that $\beta_m \subset \pi_m(\sigma) \cap \pi_m(\gamma)$ or $\beta_m \cap [\pi_m F'] \subset \pi_m(\sigma) \cap \pi_m(\gamma)$ and $\beta_m \cup [\pi_m F']$ is proper.

Let $d = (d_m)$ and $\beta = \cup_{m\in I}(\pi^{-1}\beta_m)$. Note that $\pi_m(\beta) \in L_m(d_m)$ and consequently, $\beta \in L(d)$. If $\beta_m \subset \pi_m(\sigma) \cap \pi_m(\gamma)$, then, by the above argument, $\beta \subset \sigma \cap \gamma$. Hence, $\sigma \cap \gamma \in L(d)$ and cosequently, $\sigma \in L(d) \cap L(x)$. Since σ is proper and condition (1) of 2.3 holds, we must have $d = x$ and so, $d \in F'$. If

$\beta_m \cap [\pi_m F'] \subset \pi_m(\sigma) \cap \pi_m(\gamma)$ and $\beta_m \cup [\pi_m F']$ is proper, then it follows easily that $\beta \cap [F'] \subset \sigma \cap \gamma$ and $\beta \cup [F']$ is proper. Since $\beta \cup [F']$ is proper and F' is strongly closed subset of B, then $d \in F'$.

Let F' be nonempty strongly closed subset of X and $\sigma \in L(x)$, $\gamma \in L(y)$, $x, y \in F'$ with $\sigma \cup [F']$ and $\gamma \cup [F']$ proper. It follows that $\pi_m(\sigma) \in L_m(x_m)$, $\pi_m(\gamma) \in L_m(\pi_m(y) = y_m)$ and both $\pi_m(\sigma) \cup [\pi_m F']$ and $\pi_m(\gamma) \cup [\pi_m F']$ are proper. Since $\pi_m F'$ is strongly closed subset of B_m and X_m is $S\bar{T}_3$, it follows that there exists $d_m \in \pi_m F'$ and a filter $\beta_m \in L_m(d_m)$ such that $\beta_m \cap [\pi_m F'] \subset \pi_m(\sigma) \cap \pi_m(\gamma)$ and $\beta_m \cup [\pi_m F']$ is proper. Let $d = (d_m)$ and $\beta = \cup_{m \in I}(\pi^{-1}\beta_m)$. Note that $\pi_m(\beta) \in L_m(d_m)$ and consequently, $\beta \in L(d)$ and $\beta \cap [F'] \subset \sigma \cap \gamma$ and $\beta \cup [F']$ is proper. Since $\beta \cup [F']$ is proper and F is a strongly closed subset of B, then $d \in F'$.

(3) Suppose X is LT_3. By (1), for all $x \neq y$ in B, $[x] \notin L(y)$. Let F be any nonempty strongly closed subset F of B, $x \in B$, and α be any proper filter in $L(x)$. Let $\gamma = \cup_{m \in I}(\pi_m^{-1}\pi_m \alpha)$ and $F' = \cup_{m \in I}(\pi_m^{-1}\pi_m F)$, where I is finite. Note that $\pi_m \gamma = \pi_m \alpha$, $\gamma \in L(x)$, and $\pi_m F' = \pi_m F$. So, we may work with γ and F'. Since $\pi_m F'$ is strongly closed subset of B_m and X_m is LT_3, by 2.5, $\pi_m \gamma = [x_m]$ or $\pi_m F' \in \pi_m \gamma$. If $\pi_m \gamma = [x_m]$, then $\gamma = [x]$. If $\pi_m F' \in \pi_m \gamma$, then $F' \in \gamma$.

(4) The proof for T_3' is similar to the proof of (3). ■

Let $T_3\mathcal{E}$ denote the class (full subcategory) of regular objects in a topological category $\mathcal{E}$, where T_3 is one of ΔT_3 or ST_3, respectively.

Theorem 2.16. *Let $\mathcal{E}$ = FCO or LFCO. The subcategories $\Delta T_3\mathcal{E}$ and $ST_3\mathcal{E}$ are quotient-reflective in $\mathcal{E}$.*

Proof. It is easy to see that each of these subcategories are full, isomorphism-closed, and closed under finer structures (i.e., if $X \in T_3\mathcal{E}$, $Y \in \mathcal{E}$, $UX = UY$, and $id : Y \to X$ is a $\mathcal{E}$-morphism, then $Y \in T_3\mathcal{E}$). By 2.12 and 2.15, each of these subcategories are closed under formation of subspaces and products. Hence, the subcategories $\Delta T_3\mathcal{E}$ and $ST_3\mathcal{E}$ are quotient-reflective in $\mathcal{E}$.

■

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