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Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica Balkanica

New Series Vol. 14, 2000, Fasc. 3-4

More Cardinal Inequalities on Urysohn Spaces

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Presented by P. Kenderov

Some results concerning cardinality of Urysohn spaces are proven.

AMS Subj. Classification: 54A25

Key Words: Urysohn space, cardinality, cardinal inequalities

1. Introduction

Throughout the paper X will denote an infinite Urysohn space (i.e. a space such that if $x, y \in X$, $x \neq y$, x and y have disjoint closed neighborhoods).

We use the standard notation and terminology, following [3], [4]. For definitions of θ-versions of some cardinal invariants, see [5], [6]. α, β, γ are ordinal numbers, while κ, λ denote infinite cardinals and κ^+ is the successor cardinal of κ. As As usual, cardinals are assumed to be initial ordinals. If X is a set, then $[X]^{\leq\kappa}$ denotes the collection of all subsets of X whose cardinality is $\leq \kappa$.

Recall that the *θ-closure* $Cl_\theta A$ of a subset A of a space X is the set $\{x \in X : \overline{U} \cap A \neq \emptyset$ for every neighborhood U of $x\}$.

In [2], the cardinal function $sL(X)$ was introduced as being the smallest cardinal κ such that for every $A \subset X$ and every open collection $\mathcal{U}$, with $\overline{A} \subset \cup\mathcal{U}$, there exists $\mathcal{V} \subset \mathcal{U}$ satisfying $|\mathcal{V}| \leq \kappa$ and $A \subset \overline{\cup\mathcal{V}}$. It was also shown that for a Hausdorff space X, $|X| \leq 2^{sL(X)\chi(X)}$.

We introduce the following definition.

Definition 1. For a space X, $sL_\theta(X)$ is the smallest cardinal κ such that if $A \subset X$, $\mathcal{U}$ is an open collection and $cl_\theta(A) \subset \cup\mathcal{U}$, there is $\mathcal{V} \subset \mathcal{U}$ with $|\mathcal{V}| \leq \kappa$ and $A \subset \overline{\cup\mathcal{V}}$.

† The author thanks FARESP and the University of Niš for its hospitality;
‡ Supported by the Serbian Scientific Foundation, Grant N^0 04M01.

It is immediate that $sL_\theta(X) \leq sL(X)$ for every space X.

Definition 2. For a Hausdorff space X let $\kappa(X)$ be the smallest cardinal number κ such that for each point $x \in X$, there is a collection $\mathcal{V}_\S$ of *closed* neighborhoods of x so that $|\mathcal{V}_\S| \leq \kappa$ and if W is a *closed* neighborhood of x, then W contains a member of $\mathcal{V}_\S$.

We need the following lemma which can be easily shown.

Lemma. *For a subset A of a Urysohn space X, $|cl_\theta(A)| \leq |A|^{\kappa(X)}$.*

It is immediate that $\kappa(X) \leq \chi(X)$ for every Hausdorff space X; as a matter of fact, $\kappa(X)$ is equal to the character of the semiregularization of X. The following example shows that $sL_\theta(X)$ may be strictly smaller than $sL(X)$ and $\kappa(X)$ strictly smaller than $\chi(X)$.

Example. Let $X = \omega_2 \times \{0,1\}^{\omega_2} \cup \{a\}$, where $a \notin \omega_2 \times \{0,1\}^{\omega_2}$, and consider the following topology τ on X:

(i) $V \subset X$ is a neighborhood of a iff there is a $\lambda < \omega_2$ such that

$$V \supset \{a\} \cup (\omega_2 \setminus \lambda) \times (\{0,1\}^{\omega_2} \setminus \{(0,0,\ldots)\});$$

(ii) $W \subset X$ is a neighborhood of a point $(\beta, t) \in \omega_2 \times \{0,1\}^{\omega_2}$ iff

$$W \supset \{\beta\} \times (U \setminus S),$$

where U is a neighborhood of t in the usual product topology and S is a subset of $\{0,1\}^{\omega_2}$ of cardinality $\leq \omega_2$ to which t does not belong.

It is easy to prove that:

1) (X, τ) is Urysohn;

2) $\chi(X) \geq \omega_3$ and $\kappa(X) = \omega_2$;

3) $sL(X) \geq \omega_2$, because $F = \omega_2 \times \{(0,0,\ldots)\}$ is closed and $\mathcal{U} = \{\{\alpha\} \times \{\prime, \infty\}^{\omega_\in} : \alpha \in \omega_\in\}$ is an open cover of F such that if $\mathcal{V} \subset \mathcal{U}$ and $|\mathcal{V}| < \omega_\in$, then $F \not\subseteq \overline{\cup\mathcal{V}}$;

4) $sL_\theta(X) = \omega_1$.

Indeed, let $A \subset X$ and $\mathcal{U}$ be an open collection which covers $cl_\theta(A)$. There are two cases: I) $a \in cl_\theta(A)$; II) $a \notin cl_\theta(A)$.

I) If $a \in cl_\theta(A)$, there is $U \in \mathcal{U}$ so that $a \in U$, hence there is $\lambda < \omega_2$ and

$$U \supset \{a\} \cup (\omega_2 \setminus \lambda) \times (\{0,1\}^{\omega_2} \setminus \{(0,0,\ldots)\})$$

and

$$\overline{U} \supset \{a\} \cup (\omega_2 \setminus \lambda) \times \{0,1\}^{\omega_2}.$$

The set $cl_\theta(A) \cap (\lambda \times \{0,1\}^{\omega_2})$ is covered by closures of $\leq \omega_1$ members of $\mathcal{U}$. (Notice that if $\beta \in \lambda$, $t \in \{0,1\}^{\omega_2}$ and $\{\beta\} \times (V \setminus S)$ is such that V is a basic clopen neighborhood of t in the usual product topology and $t \notin S \subset \{0,1\}^{\omega_2}$, $|S| \leq \omega_2$, then $\overline{\{\beta\} \times (V \setminus S)} = \{\beta\} \times V$.)

II) If $a \notin cl_\theta(A)$, there is $\lambda < \omega_2$ and

$$(\{a\} \cup (\omega_2 \setminus \lambda) \times \{0,1\}^{\omega_2}) \cap A = \emptyset,$$

hence $A \subset \lambda \times \{0,1\}^{\omega_2}$. Since the last set is clopen and it is easy to prove that $cl_\theta(A)$ is equal to the closure of A in the usual product topology (λ with the discrete topology), there is $\mathcal{V} \subset \mathcal{U}$ with $|\mathcal{V}| \leq \omega_\infty$ and $A \subset \overline{\cup\mathcal{V}}$. ■

Theorem 1. *If X is a Urysohn space, then $|X| \leq 2^{sL_\theta(X)\kappa(X)}$.*

Proof. Applying the well-known method of Pol-Šapirovskii-Arhangel'skii, let $\kappa = sL_\theta(X)\kappa(X)$ and for each $x \in X$ let $\mathcal{B}_\S$ be a collection of closed neighborhoods of x such that $|\mathcal{B}_\S| \leq \kappa$ and every closed neighborhood of x contains a member of $\mathcal{B}_\S$. We shall define an increasing sequence $\{A_\alpha : \alpha \in \kappa^+\}$ of subsets of X and a sequence $\{\mathcal{U}_\alpha : \alpha \in \kappa^+\}$ of collections of open subsets of X such that:

(1) $|A_\alpha| \leq 2^\kappa, \forall \alpha < \kappa^+ \;\&\; A_\alpha \supset cl_\theta\left(\bigcup_{\beta<\alpha} A_\beta\right), \forall \alpha < \kappa^+$;

(2) $\mathcal{U}_\alpha = \left\{\rangle \backslash \sqcup(\mathcal{F}) : \mathcal{F} \in \cup\{\mathcal{B}_\S : \S \in \rfloor\updownarrow_\theta\left(\bigcup_{\beta<\alpha} A_\beta\right)\}\right\}, \quad \alpha < \kappa^+$;

(3) If $\mathcal{V} \in [\mathcal{U}_\alpha]^{\leq\kappa}$ and $\overline{\cup\mathcal{V}} \neq X$, then $A_{\alpha+1} \setminus \overline{\cup\mathcal{V}} \neq \emptyset, \quad \alpha < \kappa^+$.

Suppose that the sets A_β and $\mathcal{U}_\beta$, satisfying (1) - (3), have been defined for all $\beta < \alpha < \kappa^+$ and let us define A_α and $\mathcal{U}_\alpha$.

Since $cl_\theta\left(\bigcup_{\beta<\alpha} A_\beta\right)$ has cardinality $\leq 2^\kappa$, according to (2) $\mathcal{U}_\alpha$ has cardinality $\leq 2^\kappa$. For each $\mathcal{V} \in [\mathcal{U}_\alpha]^{\leq\kappa}$ such that $X \setminus \overline{\cup\mathcal{V}} \neq \emptyset$ fix a point $x_\mathcal{V} \in X \setminus \overline{\cup\mathcal{V}}$ and let A_α be the θ-closure of the union $\bigcup_{\beta<\alpha} A_\beta$ with all these $x_\mathcal{V}$.

Finally, put $A = \cup\{A_\alpha : \alpha < \kappa^+\}$. Then $cl_\theta(A) = A$. Indeed, let $y \in X$ so that the closure of each neighborhood of y intersects A; then for each $W \in \mathcal{B}_\dagger$ there is $\alpha_W < \kappa^+$ so that $W \cap A_{\alpha_W} \neq \emptyset$. Since $|\{\alpha_W : W \in \mathcal{B}_\dagger\}| \leq ||$, there is $\beta < \kappa^+$ so that $\beta > \alpha_W$ for every $W \in \mathcal{B}_\dagger$ and $y \in cl_\theta(A_\beta)$.

Now it is enough to show that $A = X$. On the contrary, there is $y \in X \setminus A$ then there is $W \in \mathcal{B}_\dagger$ so that $W \cap A = \emptyset$. For each $x \in A$ choose a $V_x \in \mathcal{B}_\S$ so that

$$V_x \subset X \setminus \text{int}(W).$$

$\{\text{int}(V_x) : x \in A\}$ is an open cover of $A = cl_\theta(A)$, so there is $B \subset A$ with $|B| \leq sL_\theta(X) \leq \kappa$ such that

$$A \subset \overline{\bigcup_{x\in B} \text{int}(V_x)}.$$

But since $|\mathcal{B}| \leq \kappa$, there is $\beta < \kappa^+$ so that $B \subset A_\beta$ and int $\mathcal{V} = \{\mathrm{int}(\mathcal{V}_\S) : \S \in \mathcal{B}\}$ is a convinient collection of open sets which appears at the step $\beta + 1$. Hence, $A_{\beta+1} \setminus \overline{\cup\mathcal{V}} \neq \emptyset$ and we have a contradiction. ■

Remark. In the precedent example we have:

$$2^{sL(X)\chi(X)} \geq 2^{\omega_3} \quad \text{and} \quad 2^{sL_\theta(X)\kappa(X)} = 2^{\omega_2}.$$

It is easy to prove the following theorem.

Theorem 2. *If X is a Hausdorff space, then $|X| \leq 2^{c(X)\kappa(X)}$. If X is a Urysohn space, then $|X| \leq 2^{c_\theta(X)\kappa(X)}$.*

Here $c_\theta(X)$ denotes the supremum of cardinalities of all collections of open subsets of X whose members have disjoint closures.

Definition 3. For a space X, the *θ-quasi-Menger number $qM_\theta(X)$* is the smallest cardinal number κ such that for every closed subset A of X and every collection $\{\mathcal{U}_\alpha : \alpha \leq \kappa\}$ of families of open subsets of X with $A \subset \bigcup_{\alpha<\kappa}(\cup\mathcal{U}_\alpha)$, there are finite subfamilies $\mathcal{V}_\alpha$ of $\mathcal{U}_\alpha$, $\alpha < \kappa$, such that $A \subset \bigcup_{\alpha<\kappa} cl_\theta(\cup\mathcal{V}_\alpha)$.

Theorem 3. *For every Urysohn space X, $|X| \leq 2^{qM_\theta(X)\kappa(X)}$.*

Proof. Let $qM_\theta(X)\kappa(X) = \kappa$ and let for each $x \in X$ $\mathcal{B}_\S$ be a collection of closed neighborhoods of x such that $|\mathcal{B}_\S| \leq \kappa$ and every closed neighborhood of x contains a member of $\mathcal{B}_\S$. We shall define an increasing sequence $\{F_\alpha : \alpha \in \kappa^+\}$ of subsets of X and a sequence $\{\mathcal{U}_\alpha : \alpha \in \kappa^+\}$ of collections of open subsets of X satisfying the following conditions:

(1) $|F_\alpha| \leq 2^\kappa$, for every $\alpha < \kappa^+$;

(2) $\mathcal{U}_\alpha = \left\{)\setminus\sqcup(\mathcal{B}) : \mathcal{B} \in \cup\{\mathcal{B}_\S : \S \in \rfloor\updownarrow_\theta \left(\bigcup_{\beta<\alpha}\mathcal{F}_\beta\right)\}\right\}$, for every $\alpha < \kappa^+$;

(3) If $\mathcal{V} \in [\mathcal{U}_\alpha]^{\leq\kappa}$ and $cl_\theta(\cup\mathcal{V}) \neq \mathcal{X}$, then $F_\alpha \setminus cl_\theta(\cup\mathcal{V}) \neq \emptyset$, $\alpha < \kappa^+$.

Suppose $\alpha < \kappa^+$ and the sets F_β and $\mathcal{U}_\beta$ satisfying (1) - (3) are already defined for all $\beta < \alpha$. We are going to define F_α and $\mathcal{U}_\alpha$.

Put $M_\alpha = cl_\theta\left(\bigcup_{\beta<\alpha} F_\beta\right)$. By the lemma, $|M_\alpha| \leq 2^\kappa$, hence $|\mathcal{U}_\alpha| \leq \in^\kappa$. For every $\mathcal{V} \in [\mathcal{U}_\alpha]^{\leq\kappa}$ such that $cl_\theta(\cup\mathcal{V}) \neq \mathcal{X}$ take a point $x_\mathcal{V} \in X \setminus cl_\theta(\cup\mathcal{V})$ and define F_α to be the θ-closure of the union of $\bigcup_{\beta<\alpha} F_\beta$ with the set of all these $x_\mathcal{V}$. Then $|F_\alpha| \leq 2^\kappa$.

Let $F = \cup\{F_\alpha : \alpha < \kappa^+\}$. Then $|F| \leq 2^\kappa$ and the proof will be finished if we prove that $cl_\theta(F) = X$ (taking into account the lemma once again). First, we show that $cl_\theta(F) = \bigcup_{\alpha<\kappa^+} cl_\theta(F_\alpha)$. Let $x \in cl_\theta(F)$. The closure of every neighborhood of x intersects F, so that for each $B \in \mathcal{B}_\S$ one can find some

$\alpha_B < \kappa^+$ for which $B \cap F_{\alpha_B} \neq \emptyset$. Since κ^+ is a regular cardinal and $|\{\alpha_B : B \in \mathcal{B}_\S\}| \leq \kappa$, there exists $\beta < \kappa^+$ such that $\beta > \alpha_B$ for every $B \in \mathcal{B}_\S$ and $x \in cl_\theta(F_\beta)$.

Suppose now $y \in X \setminus cl_\theta(F)$. Let $\mathcal{B}_\dagger = \{\mathcal{B}_\alpha : \alpha < \kappa\}$. For each $\alpha < \kappa$ let $\mathcal{W}_\alpha$ be the collection of all members $W \in \cup\{\mathcal{B}_\S : \S \in \rfloor\updownarrow_\theta(\mathcal{F})\}$ such that $B_\alpha \cap W = \emptyset$. Since X is a Urysohn space, $cl_\theta(F) \subset \bigcup_{\alpha<\kappa} \cup\{\mathrm{int}(W) : W \in \mathcal{W}_\alpha\}$. As $cl_\theta(F)$ is closed, one can choose $\mathcal{V}_\alpha \in [\mathcal{W}_\alpha]^{<\omega}$ for each $\alpha \in \kappa$ such that $cl_\theta(F) \subset \bigcup_{\alpha<\kappa} cl_\theta(\cup\{\mathrm{int}(V) : V \in \mathcal{V}_\alpha\})$. Clearly, for every $\alpha < \kappa$, $\cup\{\mathrm{int}(V) : V \in \mathcal{V}_\alpha\} \cap \mathcal{B}_\alpha = \emptyset$, hence $y \notin cl_\theta(\cup\{int(V) : V \in \mathcal{V}_\alpha\})$. This means $y \notin \bigcup_{\alpha<\kappa} cl_\theta(\cup\{\mathrm{int}(V) : V \in \mathcal{V}_\alpha\})$. There is a $\beta < \kappa^+$ such that all $\mathcal{V}_\alpha$, $\alpha < \kappa$, are contained in $\mathcal{U}_\beta$. Then by (3), $F_{\beta+1} \setminus \bigcup_{\alpha<\kappa} cl_\theta(\cup\{\mathrm{int}(V) : V \in \mathcal{V}_\alpha\}) \neq \emptyset$ which is a contradiction. ■

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Received: 27.07.1998

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