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Proportionality of k-th Derivative of Dirac Delta in the Hypercone

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We know that there is a relation between $\delta^{(n+2j-1)}(r)$ and $\triangle^{j}\{\delta(x)\}$ (Laplacian operator iterated j times) (cf. [7]), where $r = \sqrt[2]{x_1^2 + \therefore x_n^2}$.

In this paper we obtain relations between the distributions $\delta^{(k)}(P_+)$, $\delta^{(k)}(P_-)$, $\delta_1^{(k)}(P)$ and $\delta_2^{(k)}(P)$ in terms of the ultrahyperbolic operator iterated $(k+1-\frac{n}{2})$ times, where $P = P(x) = x_1^2 + \dots x_p^2 - x_{p+1}^2 - \dots x_{p+q}^2$, $p+q=n$ is the dimension of the space.

As an aplication of this formulae, we obtain a relation between the product $\delta^{(k)}(P_+)\cdot\delta^{(l)}(P_+)$ and $\delta^{(k+l+1)}(P_+)$ (see formula (60)) under conditions: p and q odd and $0 \le k+l-\frac{n}{2} < \frac{n}{2}$ for n even.

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1. Introduction

Let $m^2 + P$ be a quadratic form in n variables defined by

$$m^2 + P = m^2 + x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots x_{p+q}^2, \tag{1}$$

where $p+q=n$. The distributions $(m^2+P)_+^{\lambda}$ and $(m^2+P)_-^{\lambda}$ are defined by

$$(m^2+P)_+^{\lambda} = \int_{m^2+P>0} (m^2+P)^{\lambda}\,\varphi(x)dx \tag{2}$$

and

$$(m^2+P)_-^{\lambda} = \int_{m^2+P<0} (-(m^2+P))^{\lambda}\,\varphi(x)dx, \tag{3}$$

where λ is a complex number.

Bresters in [3] showed the following formulas:

$$\text{Res}_{\lambda=-k-1}(m^2+P)_+^\lambda = \frac{1}{k!}\delta^{(k)}(m^2+P) \tag{4}$$

and

$$\text{Res}_{\lambda=-k-1}(m^2+P)_+^\lambda = \frac{1}{k!}\delta^{(k)}(m^2+P). \tag{5}$$

On the other hand, let ϕ_s be a distribution of the variable s, and let $u(x) \in C^\infty(R^n)$ be such that the $(n-1)$ dimension manifold $u(x_1, x_2, ..., x_n) = 0$ has no critical points; $\phi_{u(x)}$ denotes the distribution defined on R^n by the formula (called the Leray formula, [2], p.102):

$$\int_{R^n} \phi_{u(x)} f(x)dx_1...dx_n = \int_{-\infty}^{+\infty} \phi_s ds \int_{u(x)=s} f(x)w_u(x, dx), \tag{6}$$

where w_u is an $(n-1)$ dimensional form on u defined as follows:

$$du \wedge dw = dx_1 \wedge dx_2 \wedge ...dx_n, \tag{7}$$

the manifold u(x)=s having the orientation such that $w(x,dx) > 0$.

Using (6) (the Leray formula) for $m^2 > 0$ we have,

$$\left\langle \delta^{(k)}(m^2+P), \varphi \right\rangle = \left\langle \delta^{(k)}(t), \gamma(t) \right\rangle = (-1)^k \psi^{(k)}(0), \quad ([8], \text{p. } 189), \tag{8}$$

where

$$\gamma(t) = \int_{m^2+P=t} \varphi w_{m^2+P}(x, dx) \quad ([8], \text{p.}189). \tag{9}$$

If

$$P = P(x) = x_1 + ... + x_p - x_{p+1} - ... - x_{p+q}, \tag{10}$$

we know that distribution

$$\delta^{(k)}(P) \text{ exist if } k < \frac{n}{2} - 1 \quad ([1], \text{p.}249). \tag{11}$$

If, on the other hand,

$$k \geq \frac{n}{2} - 1, \tag{12}$$

$\left\langle \delta_1^{(k)}(P), \varphi \right\rangle$ and $\left\langle \delta_2^{(k)}(P), \varphi \right\rangle$ defined by ([1], p.250):

$$\left\langle \delta_1^{(k)}(P), \varphi \right\rangle = \int_0^\infty \left[\left(\frac{\partial}{2s\partial s} \right)^k \left\{ s^{q-2} \cdot \frac{\psi(r,s)}{2} \right\} \right]_{s=r} r^{p-1} dr \tag{13}$$

and

$$(14)\quad \left\langle \delta_1^{(k)}(P),\varphi\right\rangle = (-1)^k \int_0^\infty \left[\left(\frac{\partial}{2r\partial r}\right)^k \left\{r^{p-2}\cdot\frac{\psi(r,s)}{2}\right\}\right]_{r=s} s^{q-1}ds,$$

are regularizations of $\langle\delta^{(k)}(\ P),\varphi\rangle$.

On the other hand, from ([1], p.278), if

$$(15)\qquad \delta^{(k)}(P_+) = (-)^k k!\,\mathrm{Res}_{\lambda=-1-k} P_+^\lambda$$

and

$$(16)\qquad \delta^{(k)}(P_-) = (-)^k k!\,\mathrm{Res}_{\lambda=-1-k} P_-^\lambda,$$

then

$$(17)\qquad \delta^{(k)}(P_+) = \delta_1^{(k)}(P)$$

and

$$(18)\qquad \delta^{(k)}(P_-) = \delta_1^{(k)}(-P),$$

if n is odd and if n is even and $k < \frac{n}{2} - 1$.

For the case $k \geq \frac{n}{2} - 1$, the relations (17) and (18) are not valid, for example, $\delta^{(k)}(P_+) - \delta_1^{(k)}(P)$ and $\delta^{(k)}(P_-) - \delta_1^{(k)}(-P)$ are generalized functions concentrated on the vertex of the P= 0 cone.

It is important to observe that for (4),(5),(8) and (9) to hold, m must be different from zero. Indeed, formulae (4),(5),(8) and (9) may not hold it we put in them $u(x) = P(x)$, where $P(x)$ is defined in (10). This is due to the fact that the cone $P(x) = 0$ has a critical point (namely, the origin).

For going from $\delta^{(k)}(m^2 + P)$ to $\delta^{(k)}(P)$, taking $\mathrm{m}^2 = 0$, we consider the conditions (11) and (12) and the formulae (13),(14),(15),(16),(17) and (18), therefore the following relations are valid:

$$(19)\qquad \delta^{(k)}(m^2 + P) = \delta_1^{(k)}(P) = \delta^{(k)}(P_+),$$

if $\mathrm{m}^2 = 0$ for n odd as well as for even n and $k < \frac{n}{2} - 1$. While taking into account (4),(5),(12),(13) and (14), the following relations are valid:

$$(20)\qquad \delta^{(k)}(m^2 + P) = \delta_1^{(k)}(P),$$

or

$$(21)\qquad (-1)^k\delta^{(k)}(-m^2 - P) = \delta_2^{(k)}(P),$$

if $\mathrm{m}^2 = 0$ for n even and $k \geq \frac{n}{2} - 1$, where $\delta_2^{(k)}(P) = (-1)^k\delta_1^{(k)}(-P)$ ([1], p.251).

2. Proportionality of k-th derivative of Dirac delta in the hypercone

In this section we obtain relations between the distributions $\delta^{(k)}(P_+)$, $\delta^{(k)}(P_-)$, $\delta_1^{(k)}(P)$ and $\delta_2^{(k)}(P)$, in terms of the ultrahyperbolic operator iterated $(k+1-\frac{n}{2})$ times under the conditions $k \geq \frac{n}{2}-1$, n even and k non-negative integer.

To obtain our result, we need the following formulae:

$$\delta^{(k-1)}(m^2+P) = \sum_{\nu=0}^{\frac{n}{2}-k-1} \frac{(m^2)^\upsilon}{\upsilon!}\delta^{(k+\nu-1)}(P_+)$$

$$(22)\ + \sum_{\upsilon \geq \frac{n}{2}-k} \frac{(m^2)^\upsilon}{\upsilon!}\Big\{\delta_1^{(k+\nu-1)}(P) + \frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}(-1)^{\nu+k-1}}{4^{\nu-\frac{n}{2}+k}(\upsilon-\frac{n}{2}+k)!}L^{\upsilon-\frac{n}{2}+k}\{\delta(x)\}\Big\},$$

$$\delta^{(k-1)}(-m^2-P) = (-1)^{k-1}\sum_{\nu=0}^{\frac{n}{2}-k-1} \frac{(m^2)^\upsilon}{\upsilon!}\delta^{(k+\nu-1)}(P_-)$$

$$+(-1)^{k-1}\sum_{\upsilon \geq \frac{n}{2}-k} \frac{(m^2)^\upsilon}{\upsilon!}\Big\{\delta_1^{(k+\nu-1)}(-P) + \frac{(-1)^{\frac{p}{2}}\pi^{\frac{n}{2}}(-1)^{\nu+k-1}}{4^{\nu-\frac{n}{2}+k}(\upsilon-\frac{n}{2}+k)!}(-L)^{\upsilon-\frac{n}{2}+k}\{\delta(x)\}\Big\},$$

(23)

if p and q are even (see [4]);

$$\delta^{(k-1)}(m^2+P) = \sum_{\nu=0}^{\frac{n}{2}-k-1} \frac{(m^2)^\upsilon}{\upsilon!}\delta^{(k+\nu-1)}(P_+)$$

$$(24)\quad + \sum_{\upsilon \geq \frac{n}{2}-k} \frac{(m^2)^\upsilon}{\upsilon!}\Big\{\delta_1^{(k+\nu-1)}(P) + \frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}(-1)^{\nu+k-1}}{4^{\nu-\frac{n}{2}+k}(\upsilon-\frac{n}{2}+k)!} \times [\psi(\frac{p}{2})-\psi(\frac{n}{2})](L)^{\upsilon-\frac{n}{2}+k}\{\delta(x)\}\Big\},$$

$$\delta^{(k-1)}(-m^2-P) = (-1)^{k-1}\sum_{\nu=0}^{\frac{n}{2}-k-1} \frac{(m^2)^\upsilon}{\upsilon!}\delta^{(k+\nu-1)}(P_-)$$

$$(25)\quad + \sum_{\upsilon \geq \frac{n}{2}-k} \frac{(m^2)^\upsilon}{\upsilon!}\Big\{\delta_1^{(k+\nu-1)}(-P) + \frac{(-1)^{\frac{p+1}{2}}\pi^{\frac{n}{2}-1}(-1)^{\nu+k-1}}{4^{\nu-\frac{n}{2}+k}(\upsilon-\frac{n}{2}+k)!}$$

$$\times [\psi(\frac{q}{2}) - \psi(\frac{n}{2})](-L)^{\upsilon-\frac{n}{2}+k}\{\delta(x)\}\Big\},$$

if p and q are odd (see [4]);

(26) $\delta^{(k)}(P_+) - \delta_1^{(k)}(P) = B_{k,p,q}L^{k-\frac{n}{2}+1}\{\delta(x)\}$, if $k \geq \frac{n}{2} - 1$ (see [6]),

(27) $\delta^{(k)}(P_-) - \delta_1^{(k)}(-P) = D_{k,p,q}L^{k-\frac{n}{2}+1}\{\delta(x)\}$, if $k \geq \frac{n}{2} - 1$ (see [6])

and

(28) $\delta_1^{(k)}(P) - \delta_2^{(k)}(P) = A_{k,p,q}L^{k-\frac{n}{2}+1}\{\delta(x)\}$, if $k \geq \frac{n}{2} - 1$ (see [6]).

Here:

(29) $$B_{k,p,q} = (-1)^k(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}\frac{1}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}$$

for p and q both even,

(30) $$B_{k,p,q} = (-1)^k(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}}\frac{1}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}[\psi(\frac{p}{2}) - \psi(\frac{n}{2})]$$

for p and q both odd,

(31) $$D_{k,p,q} = (-1)(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}\frac{1}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}$$

for p and q both even,

(32) $$D_{k,p,q} = (-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}}\frac{1}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}.[\psi(\frac{q}{2}) - \psi(\frac{n}{2})]$$

for p and q both odd,

(33) $$A_{k,p,q} = (-1)(-1)^k(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}\frac{1}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}$$

for p and q both even,

(34) $$A_{k,p,q} = (-1)(-!)^k(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}}\frac{1}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}[\psi(\frac{p}{2}) - \psi(\frac{q}{2})]$$

for p and q both odd;

(35) $$L^j = \left\{\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_{p+1}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2}\right\}^j,$$

also: $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ and for integral and half-integral values of the argument, $\psi(x)$ is given by:

$$\psi(s) = -C + 1 + \frac{1}{2} + ... + \frac{1}{s-1}, \quad s = 2, 3, ...,$$

$$\psi(s + \frac{1}{2}) = -C - 2\ln(2) + 2(1 + \frac{1}{3} + ... + \frac{1}{2s-1}), \quad s = 1, 2, ...,$$

where C is Euler's constant.

Remark 1. We observe that in the firs terms of the above formulas (22),(23),(24) and (25), $\frac{n}{2} - k - 1 \geq 0$, or equivalently, $k \leq \frac{n}{2} - 1$, while in the second ones, $k \geq \frac{n}{2}$.

Now putting $v = 0$ in the first terms of the above formulas, (22),(23),(24) and (25), and putting $l = v - \frac{n}{2} + k$ in the second ones, we have:

1. If p and q are even, then

$$(36) \qquad \delta^{(k-1)}(m^2 + P) = \delta^{(k-1)}(P_+) + \sum_{\nu=1}^{\frac{n}{2}-k-1} \frac{(m^2)^v}{v!} \delta^{(k+\nu-1)}(P_+)$$

$$+ \sum_{l \geq 0} \frac{(m^2)^{l+\frac{n}{2}-k}}{(l + \frac{n}{2} - k)!} \left\{ \delta_1^{(l+\frac{n}{2}-1)}(P) + \frac{(-1)^{\frac{q}{2}} \pi^{\frac{n}{2}} (-1)^{l+\frac{n}{2}-1}}{4^l l!} L^l \{\delta(x)\} \right\}$$

and

$$(37) \qquad \delta^{(k-1)}(-m^2 - P)$$

$$= (-1)^{k-1} \delta^{(k-1)}(P_-) + (-1)^{k-1} \sum_{\nu=1}^{\frac{n}{2}-k-1} \frac{(m^2)^v}{v!} \delta^{(k+\nu-1)}(P_-)$$

$$+ (-1)^{k-1} \sum_{l \geq 0} \frac{(m^2)^{l+\frac{n}{2}-k}}{(l + \frac{n}{2} - k)!} \left\{ \delta_1^{(l+\frac{n}{2}-1)}(-P) + \frac{(-1)^{\frac{p}{2}} \pi^{\frac{n}{2}} (-1)^{l+\frac{n}{2}-1}}{4^l l!} (-L)^l \{\delta(x)\} \right\}.$$

2. If p and q are odd, then

$$\delta^{(k-1)}(m^2 + P) = \delta^{(k-1)}(P_+) + \sum_{\nu=1}^{\frac{n}{2}-k-1} \frac{(m^2)^v}{v!} \delta^{(k+\nu-1)}(P_+)$$

$$(38) \qquad + \sum_{l\geq 0} \frac{(m^2)^{l+\frac{n}{2}-k}}{(l+\frac{n}{2}-k)!}\Big\{ \delta_1^{(l+\frac{n}{2}-1)}(P) + \frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}(-1)^{l+\frac{n}{2}-1}}{4^l l!} \times [\psi(\frac{p}{2}) - \psi(\frac{n}{2})](L)^l \{\delta(x)\} \Big\}$$

and

$$\delta^{(k-1)}(-m^2 - P) = (-1)^{k-1}\delta^{(k-1)}(P_-) + +(-1)^{k-1} \sum_{\nu=1}^{\frac{n}{2}-k-1} \frac{(m^2)^{\upsilon}}{\upsilon!}\delta^{(k+\nu-1)}(P_-)$$

$$(39) \quad +(-1)^{k-1} \sum_{l\geq 0} \frac{(m^2)^{l+\frac{n}{2}-k}}{(l+\frac{n}{2}-k)!}\Big\{ \delta_1^{(l+\frac{n}{2}-1)}(-P) + \frac{(-1)^{\frac{p+1}{2}}\pi^{\frac{n}{2}-1}(-1)^{l+\frac{n}{2}-1}}{4^l l!} \times [\psi(\frac{q}{2}) - \psi(\frac{n}{2})](-L)^l \{\delta(x)\} \Big\}.$$

Putting $m^2 = 0$ in (36), (37), (38), (39) and taking into account formulas (19), (20) and (21), we obtain the following formulas:

$$(40) \quad \delta_1^{(k-1)}(P) = \delta^{(k-1)}(P_+) + \delta_1^{(k-1)}(P) + \frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}(-1)^{k-1}}{4^{k-\frac{n}{2}}(k-\frac{n}{2})!} L^{k-\frac{n}{2}}\{\delta(x)\}$$

and

$$(41) \qquad \delta_2^{(k-1)}(P)$$

$$= \delta_1^{(k-1)}(P_-) + \delta_1^{(k-1)}(-P) + \frac{(-1)^{k-1}(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}(-1)^{k-1}}{4^{k-\frac{n}{2}}(k-\frac{n}{2})!} L^{k-\frac{n}{2}}\{\delta(x)\},$$

if p and q are even for $k \geq \frac{n}{2}$; and

$$(42) \qquad \delta_1^{(k-1)}(P) = \delta_1^{(k-1)}(P_+) + \delta_1^{(k-1)}(P) + \frac{(-1)^{k-1}(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}}(k-\frac{n}{2})!} \cdot [\psi(\frac{p}{2}) - \psi(\frac{n}{2})] L^{k-\frac{n}{2}}\{\delta(x)\}$$

and

$$(43) \qquad \delta_2^{(k-1)}(P) = \delta_1^{(k-1)}(P_-) + \delta_1^{(k-1)}(-P) + \frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}}(k-\frac{n}{2})!} \cdot [\psi(\frac{q}{2}) - \psi(\frac{n}{2})] L^{k-\frac{n}{2}}\{\delta(x)\},$$

if p and q are odd for $k \geq \frac{n}{2}$.

From (40) and considering the formula

(44) $$(-1)^k\delta^{(k)}(P_+) - \delta^{(k)}(P_-) = \frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}L^{k-\frac{n}{2}+1}\{\delta(x)\} \quad ([1], \text{p.279}),$$

we obtain the following formulas:

(45) $$\delta^{(k)}(P_+) = \frac{(-1)(-1)^k(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}L^{k-\frac{n}{2}+1}\{\delta(x)\}$$

and

(46) $$\delta^{(k)}(P_-) = \frac{(-2)(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}L^{k-\frac{n}{2}+1}\{\delta(x)\},$$

if p and q are both even and k$\geq \frac{n}{2} - 1$.

Similarly, from (42) and considering the formula

(47) $$\delta^{(k)}(P_-) = (-)^k\delta^{(k)}(P_+) \quad ([1], \text{p.279}),$$

we obtain the following formulas:

(48) $$\delta^{(k)}(P_+) = \frac{(-1)^{k-1}(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}\cdot[\psi(\frac{p}{2}) - \psi(\frac{n}{2})]L^{k-\frac{n}{2}+1}\{\delta(x)\}$$

and

$$\delta^{(k)}(P_-) = (-1)^k\delta^{(k)}(P_+) = \frac{-(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}\cdot[\psi(\frac{p}{2}) - \psi(\frac{n}{2})]L^{k-\frac{n}{2}+1}\{\delta(x)\},$$

(49)

if p and q are both odd and k$\geq \frac{n}{2}$.

On the other hand, from (43) and using (27) and (32), we have,

(50) $$\delta_2^{(k)}(P) = \delta_1^{(k)}(P_-) + (-1)^k\delta_2^{(k)}(P)$$

$$+ \frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}\cdot[\psi(\frac{q}{2}) - \psi(\frac{n}{2})]L^{k-\frac{n}{2}+1}\{\delta(x)\}$$

and

(51) $$\frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}.[\psi(\frac{q}{2}) - \psi(\frac{n}{2})]L^{k-\frac{n}{2}+1}\{\delta(x)\}$$

$$= \delta^{(k)}(P_-) - (-1)^k \delta_2^{(k)}(P).$$

Therefore, from (50) and (51) and using (49), we have

$$\delta_2^{(k)}(P) = 2\delta^{(k)}(P_-)$$

$$(52) \qquad = (-2)\frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}\cdot[\psi(\frac{p}{2}) - \psi(\frac{n}{2})]L^{k-\frac{n}{2}+1}\{\delta(x)\},$$

if p and q are both odd.

From (28) and using (34) and (52), we have

$$(53)\,\delta_1^{(k)}(P) = \delta_2^{(k)}(P) + \frac{(-1)^k(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}\cdot[\psi(\frac{q}{2}) - \psi(\frac{n}{2})]L^{k-\frac{n}{2}+1}\{\delta(x)\}$$

$$= (-2)\frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}.[\psi(\frac{p}{2}) - \psi(\frac{n}{2})]L^{k-\frac{n}{2}+1}\{\delta(x)\}$$
$$+ \frac{(-1)^k(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}\cdot[\psi(\frac{q}{2}) - \psi(\frac{n}{2})]L^{k-\frac{n}{2}+1}\{\delta(x)\},$$

if p and q both are odd.

On the other hand, from (26) and (45) and using (29), we have,

$$\delta_1^{(k)}(P) = \delta^{(k)}(P_+) - \frac{(-1)^k(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}(-1)^{k-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}L^{k-\frac{n}{2}+1}\{\delta(x)\}$$

$$(54) \qquad = \frac{(-2)(-1)^k(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}(-1)^{k-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}L^{k-\frac{n}{2}+1}\{\delta(x)\},$$

if p and q are both even.

Similarly, from (27) and (31), and using (46), we have,

$$\delta_2^{(k)}(P) = (-1)^k\delta^{(k)}(P_-) - (-1)^k\left[\frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}(-1)^{k-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}L^{k-\frac{n}{2}+1}\{\delta(x)\}\right]$$

$$= -\frac{2(-1)^k(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}(-1)^{k-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}L^{k-\frac{n}{2}+1}\{\delta(x)\} + \frac{(-1)^k(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}(-1)^{k-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}$$

$$\times L^{k-\frac{n}{2}+1}\left\{\delta(x)\right\} = -\frac{(-1)^k(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}(-1)^{k-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}L^{k-\frac{n}{2}+1}\left\{\delta(x)\right\}, \tag{55}$$

if p and q are both even.

Formulas (45), (46), (48), (49), (50), (52), (53), (54) and (55) represent the proportionality between $\delta^{(k)}(P_+)$, $\delta^{(k)}(P_-)$, $\delta_1^{(k)}(P)$ and $\delta_2^{(k)}(P)$, and the ultrahyperbolic operator iterated (k+1-$\frac{n}{2}$) times.

3. Application of basic formulas (48) and (49)

We know from ([5]) that the following formula is valid:

$$\delta^{(k-1)}(P_+)\cdot\delta^{(l-1)}(P_+) = C_{l,k,q,n}L^{k+l-\frac{n}{2}}\delta(x), \tag{56}$$

under the following conditions:

$$a)\ 0 \leq k+l-\frac{n}{2} < \frac{n}{2} \tag{57}$$

and

$$b)\ p \text{ and } q \text{ are both odd,} \tag{58}$$

where L^j is defined by (35) and

$$C_{l,k,q,n} = -\frac{1}{2}\frac{(-1)^{\frac{q-1}{2}}\pi^{\frac{n}{2}-1}(-1)^{k-1}}{4^{k+l-\frac{n}{2}}(k+l-\frac{n}{2})!}\cdot\frac{(k-1)!(l-1)!(-1)^{l-1}}{\Gamma(k+l)}. \tag{59}$$

From (56) and using (48), we obtain the following formula:

$$\delta^{(k)}(P_+).\delta^{(l)}(P_+) = C_{l+1,k+1,q,n}L^{k+l+1+1-\frac{n}{2}}\delta(x)$$

$$= -\frac{1}{2}\frac{(-1)^{\frac{q-1}{2}}\pi^{\frac{n}{2}-1}(-1)^k}{4^{k+l-\frac{n}{2}+2}(k+l-\frac{n}{2}+2)!}\cdot\frac{(k)!(l)!(-1)^l}{\Gamma(k+l+2)}\cdot\frac{4^{k+l-\frac{n}{2}+2}(k+l-\frac{n}{2}+2)!}{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}(-1)^{k+l+1}[\psi(\frac{p}{2})-\psi(\frac{n}{2})]}$$

$$\times\,\delta^{(k+l+1)}(P_+) = -\frac{1}{2}\frac{k!l!}{(k+l+1)!}\cdot\frac{1}{[\psi(\frac{p}{2})-\psi(\frac{n}{2})]}\delta^{(k+l+1)}(P_+), \tag{60}$$

under the conditions that p and q are both odd and $0 \leq k + l - \frac{n}{2} < \frac{n}{2}$.

Similarly, from (60) and using formula (47), we obtain

$$\delta^{(k)}(P_-).\delta^{(l)}(P_-) = (-1)^{k+l}\delta^{(k)}(P_+).\delta^{(l)}(P_+)$$

$$= (-1)^{k+l}\left[-\frac{1}{2}\frac{k!l!}{(k+l+1)!}.\frac{1}{[\psi(\frac{p}{2}) - \psi(\frac{n}{2})]}\right].(-1)^{k+l}\delta^{(k+l+1)}(P_+)$$

(61) $$= \frac{1}{2}\frac{k!l!}{(k+l+1)!}.\frac{1}{[\psi(\frac{p}{2}) - \psi(\frac{n}{2})]}\delta^{(k+l+1)}(P_-),$$

under the conditions that p and q are both odd and $0 \leq k + l + 2 - \frac{n}{2} < \frac{n}{2}$.

References

[1] I. M. Gelfand, G. E. Shilov. *Generalized Functions*, vol. **1**, Academic Press, N. York, 1964.

[2] J. Leray. *Hyperbolic Differential Equations*, Mimeographed Lecture Notes, Institute of Advanced Study, Princeton, 1952.

[3] D. W. Bresters. On distributions connected with quadratic forms. *SIAM J. Appl. Math.*, **16**,1968, 563-581.

[4] Manuel Aguirre T. The expansion of $\delta^{(k)}(m^2+P)$. *Integral Transforms and Special Functions* **8**, No 1-2, 1999.

[5] Manuel Aguirre T. The multiplicative product of Dirac delta in hypercone. In: *Memorias del V Encuentro Centroamericano de Matemática*, Universidad de Costa Rica, 1997; and: *Journal of Computational and Applied Mathematics*, To appear.

[6] Manuel Aguirre T. Relaciones de la derivada de orden k de la delta de Dirac en el hipercono con el operador ultra-hiperbólico. *Memorias del VII Encuentro Centroamericano de Matemática*, Universidad de Chiriquí, Rep. de Panamá, To appear.

[7] Manuel Aguirre T. Some multiplicative and convolution products between $\delta^{(n+2j-1)}(r)$ and $\Delta^j\delta(x)$. *Mathematica Balkanica (N.S.)*, To appear.

[8] D. A. González. On some heterodox distributional multiplicative products. *Revista de la Unión Matemática Argentina*, **29**, 1980.

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