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On Fuzzy Irresolute Functions and Fuzzy Weak Semicontinuity in Intuitionistic Fuzzy Topological Spaces

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The purpose of this paper is to introduce the concept of fuzzy irresolute and fuzzy weakly semicontinuity in intuitionistic fuzzy topological space defined by Çoker [4]. Several characterizations and certain properties of these functions are established.

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1. Introduction

In [1, 2, 3], Atanassov introduced the fundamental concept of intuitionistic fuzzy set which is a generalization of the concept of fuzzy set given by Zadeh [8]. Later, the concepts of intuitionistic fuzzy topological spaces and some other concepts were given by Çoker [4]. Gürçay, Eş and Çoker [5] introduced the notion of semicontinuity in intuitionistic fuzzy topological spaces. In this paper, we introduce and investigate fuzzy irresolute and fuzzy weakly semicontinuous functions which have been presented in [6] in intuitionistic fuzzy topological spaces.

First we present the fundamental definitions given by Atanassov:

Definition 1.1. ([3]) Let X be a nonempty fixed set. An intuitionistic fuzzy set (IFS, for short) A is an object having the form

$$A = \{< x, \mu_A(x), \gamma_A(x) >: x \in X\},$$

where the functions $\mu_A : X \to I$ and $\gamma_A \to I$ denote the degree of membership (namely, $\mu_A(x)$) and the degree of nonmembership (namely, $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Obviously, each fuzzy set A on a nonempty set X is an IFS having the form

$$A = \{< x, \mu_A(x), 1 - \mu_A(x) >: x \in X\}.$$

For the sake of simplicity, we use the symbol $A =< x, \mu_A, \gamma_A >$ for the IFS $A = \{< x, \mu_A(x), \gamma_A(x) >: x \in X\}$.

Definition 1.2. ([3]) Let X be a nonempty set, and the IFS's A and B be in the forms $A =< x, \mu_A, \gamma_A >, B =< x, \mu_B, \gamma_B >$, and let $\{A_i : i \in J\}$ be an arbitrary family of IFS's in X. Then we have:

(a) $A \subseteq B$ iff $x \in X$ $[\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)]$;
(b) $A = B$ iff $A \subseteq$ and $B \subseteq A$;
(c) $\overline{A} =< x, \gamma_A, \mu_A >$;
(d) $\cap A_i =< x, \wedge\mu_{A_i}, \vee\gamma_{A_i}$;
(e) $\cup A_i =< x, \vee\mu_{A_i}, \wedge\gamma_{A_i} >$;
(f) $0_\sim =< x, 0, 1 >$ and $1_\sim =< x, 1, 0 >$.

Proposition 1.3. ([4]) *Let A, B, C and $A_i (i \in J)$ be IFS's in X. Then the following are satisfied:*

(a) $A \subseteq B \Rightarrow A \cap B = A$ and $A \cup B = B$;
(b) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$;
(c) $B \cap (\cup A_i) = \cup(B \cap A_i)$;
(d) $B \cup (\cap A_i) = \cap(B \cup A_i)$;
(e) $A_i \subseteq B$ for all $i \in J \Rightarrow \cup A_i \subseteq B$;
(f) $B \subseteq A_i$ for alli $\in J \Rightarrow B \subseteq A_i$;
(g) $\overline{\cup A_i} = \cap \overline{A_i}$;
(h) $\overline{\cap A_i} = \cup \overline{A_i}$;
(i) $A \subseteq B \Leftrightarrow \overline{B} \subseteq \overline{A}$;
(j) $\overline{\overline{A}} = A$.

Now we define the image and preimage of IFS's. Let X, Y be two nonempty sets and $f : X \longrightarrow Y$ be a function.

Definition 1.4. ([4]) (a) If $B =< y, \mu_B, \gamma_B >$ is an IFS in Y, then the preimage of B under Y, denoted by $f^{-1}(B)$, is the IFS in X defined by

$$f^{-1}(B) =< x, f^{-1}(\mu_B), f^{-1}(\gamma_B) > .$$

(b) If $A =< x, \mu_A, \gamma_A >$ is an IFS in X, then the image of A under f, denoted by $f(A)$, is the IFS in Y defined by

$$f(A) =< y, f(\mu_A), f(\gamma_A) >,$$

where $f_{(}\gamma_A) = 1 - f(1 - \gamma_A)$.

Now we list the properties of images and preimages, some of which we shall frequently use in the following sections:

Corollary 1.5. ([4]) *Let A, A_i's $(i \in J)$ be IFS's in X, B, B_j's $(j \in K)$ be IFS's in Y and $f : X \longrightarrow Y$ a function. Then we have:*

(a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$;
(b) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$;
(c) $A \subseteq f^{-1}(f(A))$; *[If f is injective, then $A = f^{-1}(f(A))$];*
(d) $f^{-1}(f(B)) \subseteq B$; *[If f is surjective, then $f^{-1}(f(B)) = B$];*
(e) $f^{-1}(\cap B_j) = \cap f^{-1}(B_j)$;
(f) $f^{-1}(\cup B_j) = \cup f^{-1}(B_j)$;
(g) $f(\cup A_i) = \cup f(A_i)$;
(h) $f(\cap A_i) \subseteq \cap f(A_i)$; *[If f is injective, then $f(\cap A_i) = \cap f(A_i)$];*
(i) $f^{-1}(0_\sim) = 0_\sim$;
(j) $f^{-1}(1_\sim) = 1_\sim$;
(k) If f is surjective, then $f(1_\sim) = 1_\sim$;
(l) $f(0_\sim) = 0_\sim$;
(m) If f is surjective, then $\overline{f(A)} \subseteq f(\overline{A})$; *[If furthermore f is injective, then* $\overline{f(A)} = f(\overline{A})$*];*
(n) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

Definition 1.6. ([4]) An intuitionistic fuzzy topology (IFT, for short) on a nonempty set X is a family τ of IFS's in X containing $0_\sim, 1_\sim$ and closed under finite infima and arbitrary suprema. In this case the pair (X, τ) is called an intuitionistic fuzzy topological space (IFTS, for short) and any IFS in τ is known as an intuitionistic fuzzy open set (IFOS, for short) in X.

Definition 1.7. ([4]) The complement $\overline{A}$ of an IFTS A in an IFTS (X, τ) is called an intuitionistic fuzzy closed set (IFCS, for short) in X.

Definition 1.8. ([4]) Let (X, τ) be an IFTS and $A =< x, \mu_A, \gamma_A >$ be an IFS in X. Then the fuzzy interior and fuzzy closure of A are defined by

$$cl(A) = \cap\{K : K \text{ is an IFCS in } X \text{ and } A \subseteq K\}$$

and

$$int(A) = \cup\{G : G \text{ is an IFOS in } X \text{ and } G \subseteq A\}.$$

Proposition 1.9. ([4]) *For any IFS A in* (X, τ) *we have:*
(a) $cl(\overline{A}) = \overline{int(A)}$;
(b) $int(\overline{A}) = \overline{cl(A)}$.

Definition 1.10. ([5]) Let A be an IFS in an IFTS (X, τ). A is called:

(a) an intuitionistic fuzzy semi-open set (IFSOS, for short) of X if there exists $B \in \tau$ such that $B \subseteq A \subseteq cl(B)$;
(b) an intuitionistic fuzzy semi-closed set (IFSCS, for short) of X if there exists $\overline{B} \in \tau$ such that $int(B) \subseteq A \subseteq B$.

Theorem 1.11. ([5]) *The following are equivalent:*
(a) A is an intuitionistic fuzzy semi-open set;
(b) $\overline{B}$ *is an intuitionistic fuzzy semi-closed set;*
(c) $int(cl(\overline{A})) \subseteq \overline{A}$;
(d) $A \subseteq cl(int(A))$.

2. Intuitionistic fuzzy semi-interior and intuitionistic fuzzy semi-closure

Here we generalize the concepts of fuzzy semi-interior and fuzzy semi-closure of the fuzzy sets given by Yalvaç [7] to the intuitionistic case:

Definition 2.1. Let A be an IFS and define the following sets:

$$sclA = \cap\{B : A \subseteq B, B \text{ is an IFSCS in } X\},$$

$$sintA = \cup\{B : B \subseteq A, \text{ is an IFSOS in } X\}.$$

We call $sclA$ the intuitionistic fuzzy semi-closure of A and $sintA$ the intuitionistic fuzzy semi-interior of A.

It can be shown that $sintA$ is the greatest intuitionistic fuzzy semi-open set which is contained in A and $sclA$ is the lowest intuitionistic fuzzy semi-closed set which contains A, and we have:
(a) A is IFSOS $\Leftrightarrow A = sintA$;
(b) A is IFSCS $\Leftrightarrow A = sclA$.

Proposition 2.2. *For any IFS A in* (X, τ) *we have:*
(a) $A \subseteq sclA \subseteq clA$;
(b) $intA \subseteq sintA \subseteq A$.

Proof. (a) Let $A =< x, \mu_A, \gamma_A >$ and suppose that $\{< x, \mu_{F_i}, \gamma_{F_i} >: i \in I\}$ is the family of IFSCS's containing A. Furthermore, suppose that $\{< x, \mu_{G_k}, \gamma_{G_k} >: k \in K \subseteq I\}$ is the family of IFCS's containing A. By Definition 1.10,

$$\mu_{F_i} \leq \mu_{G_k} \text{ and } \gamma_{F_i} \geq \gamma_{G_k} \text{ for each } k \in K \subseteq I,$$

hence we have $\wedge\mu_{F_i} \leq \wedge\mu_{G_k}$ and $\vee\gamma_{F_i} \geq \vee\gamma_{G_k}$. Then we see that

$$sclA = \cap\{< x, \mu_{F_i}, \gamma_{F_i} >: i \in I\} \subseteq clA = \cap\{< x, \mu_{G_k}, \gamma_{G_k} >: k \in K\}.$$

The inclusion $A \subseteq sclA$ can be seen easily from Definition 2.1.

(b) Use a similar technique as above. ■

Proposition 2.3. *For any IFS A in (X, τ) we have:*
(a) $\overline{sintA} = scl\overline{A}$,
(b) $\overline{sclA} = sint\overline{A}$.

Proof. (a) Let $A =< x, \mu_A, \gamma_A >$ and suppose that the family of IFSOS's contained in A are indexed by the family $\{< x, \mu_{G_i}, \gamma_{G_i} >: i \in J\}$. Then we see that $sintA =< x, \vee\mu_{G_i}, \wedge\gamma_{G_i} >$ and hence $\overline{A} =< x, \wedge\gamma_{G_i}, \vee\mu_{G_i} >$. Since $\overline{A} =< x, \gamma_A, \mu_A >$ and $\mu_{G_i} \leq \mu_A$, $\gamma_{G_i} \geq \gamma_A$ for each $i \in J$, we obtain that $\{< x, \gamma_{G_i}, \mu_{G_i} >: i \in J\}$ is the family of IFSCS's containing $\overline{A}$, i.e. $scl\overline{A} =< x, \wedge\gamma_{G_i}, \vee\mu_{G_i} >$. Hence we have $scl\overline{A} = \overline{intA}$.

(b) This is analog (a). ■

Definition 2.4. ([4,5]) Let $f : (X, \tau) \longrightarrow (Y, \phi)$ be a function. Then,

(a) f is said to be fuzzy continuous, if $f^{-1}(B)$o is an IFOS in X, for each B IFOS in Y.

(b) f is called a fuzzy semicontinuous function, if $f^{-1}(B)$ is an intuitionistic fuzzy semiopen set in X, for each $B \in \phi$.

(c) f is called fuzzy almost continuous function, if $f^{-1}(B) \in \tau$ for each intuitionistic fuzzy regular open set B of Y.

Theorem 2.5. *The following are equivalent each to other:*
(a) $f : (X, \tau) \longrightarrow (Y, \phi)$ is fuzzy semicontinuous;
(b) $f^{-1}(B)$ is IFSCS, for each IFCS B in Y;
(c) $intclf^{-1}(A) \subseteq f^{-1}(clA)$, for each IFS B in Y;
(d) $f(intcl(B)) \subseteq clf(B)$, for each IFS B in Y.

Proof. Straightforward. ■

Theorem 2.6. *Let $f : X \longrightarrow Y$. f is IF semi-continuous, iff $f(scl(A)) \subseteq clf(A)$ for every IFS A in X.*

Proof. Let A be an intuitionistic fuzzy set of X. Since $clf(A)$ is an intuitionistic fuzzy closed set, $f^{-1}(clf(A))$ is an intuitionistic fuzzy semi-closed set in X. Clearly,

$$f^{-1}(clf(A)) = sclf^{-1}(clf(A)).$$

From Corollary 1.5, step by step we get

$$A \subseteq f^{-1}(f(A)) \Rightarrow sclA \subseteq sclf^{-1}(f(A)) \subseteq sclf^{-1}(clf(A)) = f^{-1}(clf(A))$$

$$\Rightarrow sclA \subseteq f^{-1}(clf(A)) \Rightarrow f(scl(A)) \subseteq clf(A).$$

Conversely, let A be an intuitionistic fuzzy closed set in Y. From the hypothesis, we have

$$f(sclf^{-1}(A)) \subseteq clA = A.$$

Then $sclf^{-1}(A) \subseteq f^{-1}(f(sclf^{-1}(A))) \subseteq f^{-1}(A)$. So $f^{-1}(A) = sclf^{-1}(A)$. Hence f is an intuitionistic fuzzy semi-continuous function. ■

Definition 2.7. A function $f : (X, \tau) \longrightarrow (Y, \phi)$ is called a fuzzy irresolute function, if $f^{-1}(A)$ is intuitionistic fuzzy semiopen in X for each intuitionistic fuzzy semiopen set A in Y.

It is evident that every intuitionistic fuzzy irresolute function is intuitionistic fuzzy semicontinuous. However we can see that the converse is not true by the following example.

Example 2.8. Let $X = \{a, b, c\}, Y = \{1, 2, 3\}$ and

$$G_1 = \langle x, (\frac{a}{.4}, \frac{b}{.4}, \frac{c}{.5}), (\frac{a}{.4}, \frac{b}{.4}, \frac{c}{.4})\rangle, \ G_2 = \langle x, (\frac{a}{.2}, \frac{b}{.3}, \frac{c}{.5}), (\frac{a}{.5}, \frac{b}{.5}, \frac{c}{.5})\rangle,$$

$$U_1 = \langle y, (\frac{1}{.5}, \frac{2}{.4}, \frac{3}{.5}), (\frac{1}{.4}, \frac{2}{.4}, \frac{3}{.3})\rangle, \ U_2 = \langle y, (\frac{1}{.4}, \frac{2}{.2}, \frac{3}{.4}), (\frac{1}{.5}, \frac{2}{.4}, \frac{3}{.5})\rangle$$

$$U_3 = \langle y, (\frac{1}{.4}, \frac{2}{.3}, \frac{3}{.5}), (\frac{1}{.5}, \frac{2}{.3}, \frac{3}{.5})\rangle.$$

Then the family $\tau = \{0_\sim, 1_\sim, G_1, G_2\}$ of IFS's in X is an IFT on X and the family $\phi = \{0_\sim, 1_\sim, U_1, U_2\}$ of IFS's in Y is an IFT on Y. If we define the function $f : X \longrightarrow Y$ as $f(a) = 2, f(b) = 3$ and $f(c) = 1$, then f is a fuzzy semi-continuous function, but is not a fuzzy irresolute function. Since U_3 is an IFSOS which is not an IFOS, then $f^{-1}(U_3) = \langle x, (\frac{a}{.3}, \frac{b}{.5}, \frac{c}{.4}), (\frac{a}{.3}, \frac{b}{.5}, \frac{c}{.5})\rangle$ is not an IFSOS in X.

Theorem 2.9. *Let $f : X \longrightarrow Y$. Then the following are equivalent:*
(a) f is fuzzy irresolute;
(b) $f^{-1}(B)$ is IFSCS in X, for each IFSCS B in Y;
(c) $f(scl(B)) \subseteq sclf(B)$, for each IFS B in Y;
(d) $sclf^{-1}(B) \subseteq f^{-1}(sclB)$, for each IFS B in Y.

Proof. For the implication $(a) \Leftrightarrow (c)$ one can make use of Theorem 1.11. The implications $(b) \Leftrightarrow (c)$, $(c) \Leftrightarrow (d)$ are similar to the fuzzy case. ■

Theorem 2.10. *$f : X \longrightarrow Y$ is an intuitionistic fuzzy irresolute function iff for each intuitionistic fuzzy set B in Y,*

$$f^{-1}(sintB) \subseteq sintf^{-1}(B).$$

Proof. The proof is obvious. ■

Theorem 2.11. *Let $f : X \longrightarrow Y$ be one-to-one and onto;f is a fuzzy irresolute function iff for every IFS A in X, $sintf(A) \subseteq f(sintA)$.*

Proof. It can be easily proved. ■

Definition 2.12. A function $f : (X, \tau) \longrightarrow (Y, \phi)$ is called a fuzzy weakly semicontinuous function if $f^{-1}(A) \subseteq sintf^{-1}(sclA)$ for each intuitionistic fuzzy open set A in Y.

Remark 2.13. Clearly, a fuzzy semicontinuous function is fuzzy weakly semicontinuous, but the converse need not be true as shown in the following example.

Example 2.14. Let $X = \{a, b\}, Y = \{1, 2\}$ and

$$G_1 = \langle x, (\frac{a}{.2}, \frac{b}{.2}), (\frac{a}{.4}, \frac{b}{.4})\rangle,\ G_2 = \langle x, (\frac{a}{.3}, \frac{b}{.3}), (\frac{a}{.3}, \frac{b}{.3})\rangle, U_1 = \langle y, (\frac{1}{.2}, \frac{2}{.2}), (\frac{1}{.5}, \frac{2}{.5})\rangle.$$

Consider the IFT's $\tau = \{0_\sim, 1_\sim, G_1, G_2\}$ and $\phi = \{0_\sim, 1_\sim, U_1\}$ on X and Y, respectively. We define the function $f : X \longrightarrow Y$ as $f(a) = 2$ and $f(b) = 1$. Then f is not fuzzy semicontinuous, since $f^{-1}(U_1) = \langle x, (\frac{a}{.2}, \frac{b}{.2}), (\frac{a}{.5}, \frac{b}{.5})\rangle$ is not an intuitionistic fuzzy semiopen set of X. It can be shown that f is weakly semicontinuous as follows:

$$IFSC(Y) = \{B : B = \langle y, (\frac{1}{\alpha_1}, \frac{2}{\alpha_2}), (\frac{1}{\beta_1}, \frac{2}{\beta_2})\rangle, \alpha_1, \alpha_2, \beta_1, \beta_2 \in [.2, .5]\}.$$

Hence,

$$sclU_1 = \langle y, (\frac{1}{.2}, \frac{2}{.2}), (\frac{1}{.5}, \frac{2}{.5}) \rangle = U_1, \ f^{-1}(sclU_1) = \langle x, (\frac{a}{.2}, \frac{b}{.2}), (\frac{a}{.5}, \frac{b}{.5}) \rangle.$$

Furthermore, since

$$IFSO(X) = \{A : A = \langle x, (\frac{a}{\alpha_1}, \frac{b}{\alpha_2}), (\frac{a}{\beta_1}, \frac{b}{\beta_2}) \rangle, \alpha_1, \alpha_2 \in [.2, .3] \quad \beta_2, \beta_2 \in [.3, .4]\}$$

and

$$sintf^{-1}(sclU_1) = \langle x, (\frac{a}{.2}, \frac{b}{.2}), (\frac{a}{.3}, \frac{b}{.3}) \rangle,$$

we have $f^{-1}(U_1) \subseteq sintf^{-1}(sclU_1)$. Thus f is fuzzy weakly semicontinuous.

Theorem 2.15. *If $f : (X, \tau) \longrightarrow (Y, \phi)$ is fuzzy almost continuous function, then f is fuzzy weakly semicontinuous function.*

Proof. Let B be an intuitionistic fuzzy open set in Y. Using Proposition 2.3 and Definition 2.12, we have

$$f^{-1}(B) \subseteq intf^{-1}(intclB) \subseteq sintf^{-1}(intclB) \subseteq sint(intclf^{-1}(sclB)).$$

Since $sclB$ is an intuitionistic fuzzy semiclosed set, by Theorem 1.11 we have

$$sint(intclf^{-1}(sclB)) \subseteq sint(f^{-1}(sclB)).$$

Hence we obtain $f^{-1}(B) \subseteq sintf^{-1}(sclB)$.

The converse of Proposition 2.15. may not be true.

Example 2.16. Refer to Example 2.14. Then f is fuzzy weakly semicontinuous, but not fuzzy almost continuous since $U_1 = intclU_1$ but $f^{-1}(U_1)$ is not intuitionistic fuzzy open in X.

Theorem 2.17. *The following are equivalent:*

(a) f is fuzzy weakly semicontinuous function,

(b) $scl(f^{-1}(sintB)) \subseteq f^{-1}(B)$ for each intuitionistic fuzzy closed B of Y,

(c) $f^{-1}(intB) \subseteq sint(f^{-1}(sclB))$ for each intuitionistic fuzzy set B of Y,

(d) $scl(f^{-1}(sintB)) \subseteq f^{-1}(clB)$ for each intuitionistic fuzzy set B of Y.

Proof. $(a) \Rightarrow (b)$. Let B be an intuitionistic fuzzy closed set of Y. Then $\overline{B} \in \phi$ from (a), we have $f^{-1}(\overline{B}) \subseteq sint(f^{-1}(scl\overline{B})$. By Proposition 2.3, we obtain

$$\overline{f^{-1}(B)} = f^{-1}(\overline{B}) \subseteq sint(\overline{f^{-1}(sintB)} = \overline{scl(f^{-1}(sintB))}.$$

Thus, $f^{-1}(B) \supseteq sclf^{-1}(sintB)$.

$(b) \Rightarrow (a)$ It is similiar to the previous proof. $(a) \Leftrightarrow (c), (b) \Leftrightarrow (d)$ is obvious.

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