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Regeneration in Clifford Analysis

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1. Introduction

In Complex Analysis of Several Variables, Matsugu [13] gave a necessary and sufficient condition for any pluriharmonic function g on a Riemann domain Ω over a Stein manifold to be the real part of a holomorphic function on Ω. The author [5] obtained similar results for a Riemann domain over a complex projective space, Fukushima-Watanabe [8] and Adachi-Fukushima-Watanabe [1] over a Grassmann manifold and the author [6]-[7] for domains over infinite dimensional spaces.

In Quaternionic Analysis, Nôno [15] gave a necessary and sufficient condition that any harmonic function f_1 on a domain Ω in $\mathbf{C}^2$ has a hyper-conjugate harmonic function f_2 so that the function $f_1 + f_2 j$ is hyperholomorphic on Ω. Marinov [12] developed systematically a theory of regenerations of regular functions. Li [11] added a regeneralization in Quaternionic Analysis, too. The main purpose of the present paper is to add a regeneration in the Clifford Analysis.

2. Regeneration

Let Ω be a complex manifold and f be a holomorphic function on Ω. Then its real part f_1 is a pluriharmonic function on Ω. Let (Ω, φ) be a Riemann domain over a Stein manifold S and $(\tilde{\Omega}, \tilde{\varphi})$ be its envelope of holomorphy over S. Then, Matsugu [13] proved that, for any pluriharmonic function f_1 on Ω, there exists a pluriharmonic function f_2 on Ω so that $f_1 + f_2 i$ is holomorphic on Ω if and only if there holds $\mathrm{H}^1(\tilde{\Omega}, \mathbf{Z}) = 0$, where $\mathbf{Z}$ is the ring of integers.

The field $\mathcal{H}$ of quaternions

$$z = x_1 + ix_2 + jx_3 + kx_4, \qquad x_1, x_2, x_3, x_4 \in \mathbf{R} \tag{1}$$

is a four dimensional non-commutative $\mathbf{R}$-field generated by four base elements 1, i, j and k with the following non commutative multiplication rule:

$$(2) \qquad i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

x_1, x_2, x_3 and x_4 are called, respectively, the real, i, j and k part of z. In the papers Nôno [14],[15], [16],[17] and Marinov [12] loco citato, two complex numbers

$$(3) \qquad z_1 := x_1 + ix_2, \quad z_2 := x_3 + ix_4 \in \mathbf{C}$$

are associated to (1), regarded as

$$(4) \qquad z = z_1 + z_2 j \in \mathcal{H}.$$

They identify $\mathcal{H}$ with $\mathbf{C}^2 \cong \mathbf{R}^4$, denotes a quaternion valued function f by $f = f_1 + f_2 j$ and use fully the theory of functions of several complex variables.

Using Laufer's results [10], Nôno [15] proved that the necessary and sufficient condition that, for any complex valued harmonic function f_1 on a domain Ω in $\mathbf{C}^2$, there exists a complex valued harmonic function f_2 on Ω so that $f_1 + f_2 j$ is hyperholomorphic on Ω is that Ω is a domain of holomorphy.

Marinov [12] named, those constructions of conjugate functions, regenerations and developed the theory of regenerations in Quaternionic Analysis, using $\bar{\partial}$-analysis of Hörmander [9]. The main purpose of the present paper is to add a regeneration in Clifford Analysis, using Dolbeault Isomorphism from resolution of sheaves.

3. Preliminaries

We use the definitions and notations in F. Brackx-W. Pincket [2]. Let $\mathcal{A}$ be the universal Clifford algebra constructed over a real n- dimensional quadratic vector space V with orthonormal basis $\{e_1, e_2, \cdots, e_n\}$. A basis for $\mathcal{A}$ is given by

$$(5) \ \{e_A : A = (h_1, h_2, \cdots, h_r) \in \mathcal{P}\{1, 2, \cdots, n\}; 1 \leq h_1 < h_2 < \cdots < h_r \leq n\},$$

where $e_\phi = e_0$ is the identity element 1.

Multiplication in $\mathcal{A}$ is defined by the following rule for the basis elements

$$(6) \qquad e_j e_i = -e_i e_j \quad (i \neq j), \qquad e_i e_i = -1.$$

An involution of $\mathcal{A}$ is given by

$$(7) \qquad \lambda = \sum_A \lambda_A e_A \mapsto \bar{\lambda} = \sum_A \lambda_A \bar{e}_A, \qquad \lambda_A \in \mathbf{R},$$

where $\bar{e}_A = (-1)^{n_A(n_A+1)/2} e_A, n_A$ is the cardinality of $\mathcal{A}$.

The inner product of two elements $\lambda = \sum_A \lambda_A e_A$ and $\mu = \sum_A \mu_A e_A$ of $\mathcal{A}$ is defined by

(8) $$< \lambda, \mu >= 2^n \sum_A \lambda_A \mu_A .$$

Then the norm of an element $\lambda = \sum_A \lambda_A e_A$ of $\mathcal{A}$ is defined by

(9) $$\|\lambda\| = 2^{n/2} \sqrt{\sum_A \lambda_A^2}$$

which turns $\mathcal{A}$ into a Banach algebra.

Let m and k be positive integers with $1 < m, k \leq n, m + k \leq n$. Let Ω be an open subset of $\mathbf{R}^m \times \mathbf{R}^k$ and

(10) $$f : \Omega \to \mathcal{A},$$

(11) $$\Omega \ni (x, y) = (x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_k) \rightsquigarrow f(x, y) = \sum_A f_A(x, y) e_A.$$

Now, we introduce the generalized Cauchy-Riemann operators

(12) $$\mathrm{D}_x = \sum_{i=1}^{m} e_i \partial_{x_i} \qquad \mathrm{D}_y = \sum_{j=1}^{k} e_{m+j} \partial_{y_j}$$

and their conjugated operators

(13) $$\bar{\mathrm{D}}_x := \sum_{i=1}^{m} \bar{e}_i \partial_{x_i} \qquad \bar{\mathrm{D}}_y := \sum_{j=1}^{k} \bar{e}_{m+j} \partial_{y_j},$$

which act, for a function $f : \Omega \to \mathcal{A}$ of class C^1, as

(14) $$\mathrm{D}_x f = \sum_{i=1}^{m} \sum_A e_i e_A \frac{\partial f_A}{\partial x_i}, \qquad f \mathrm{D}_y = \sum_{j=1}^{k} \sum_A e_A e_{m+j} \frac{\partial f_A}{\partial y_j} .$$

When $i \notin A$, let B tbe he permutation of the set $\{i\} \bigcup A$ in the order from small to large and ϵ_B^{iA} be the signature of the replacement $iA \hookrightarrow B$. When $i \in A$, let s be the positive integer, which denotes the number of the place of i in the permutation A with order from small to large, B the permutation of the set $A - \{i\}$ in the order from small to large, ϵ_B^{iA} be the number $(-1)^{s-r+1}$ and

$\{i\} \cup A = B$ stands for $A - \{i\} = B$. Then the above differential operators are represented more concretely,

$$(15) \quad \mathrm{D}_x f = \sum_B \Big(\sum_{\{i\} \cup A = B} \epsilon_B^{iA} \frac{\partial f_A}{\partial x_i} \Big) e_B, \qquad f\mathrm{D}_y = \sum_B \Big(\sum_{A \cup \{m+j\} = B} \epsilon_B^{Am+j} \frac{\partial f_A}{\partial y_j} \Big) e_B.$$

A function $f : \Omega \to \mathcal{A}$ of class C^1 is called *biregular* in Ω if:
(i) for each $y \in \mathbf{R}^k$ fixed, f is of C^1 in $x \in \Omega_y$ and satisfies $\mathrm{D}_x f = 0$,
(ii) for each $x \in \mathbf{R}^m$ fixed, f is of C^1 in $y \in \Omega_x$ and satisfies $f\mathrm{D}_y = 0$.

By Brackx-Pincket [2], a biregular function is real analytic.

4. Main theorems

Theorem 1. *Let m and k be positive integers with $1 < m, k \leq n$, $m + k \leq n$. Let $P = (h_1, h_2, \cdots, h_r)$ be a permutation with $1 \leq h_1 < h_2 < \cdots < h_r$. Let Ω be an open subset of $\mathbf{R}^m \times \mathbf{R}^k$ and*

$$(16) \qquad f : \Omega \to \mathcal{A},$$

$$(17) \quad \Omega \ni (x, y) = (x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_k) \rightsquigarrow f(x, y) = \sum_{A \neq P} f_A(x, y) e_A.$$

be a function of the class C^∞ on Ω. If there exists a function f_P of class C^∞ on Ω such that the $\mathcal{A}$ valued function $f = \sum_{A \neq P} f_A(x, y) e_A + f_P(x, y) e_P$ is a biregular function on Ω, the real valued functions $f_A (A \neq P)$ satisfy the integrability condition

$$(18) \qquad d\omega = 0$$

on Ω, where the differential form ω of degree 1 is given by

$$(19) \quad \omega = -\sum_{i=1}^{m} \Big(\sum_{\{i\}' \cup A = \{i\} \cup P, A \neq P} \epsilon_{iP}^{i'A} \frac{\partial f_A}{\partial x_{i'}} \Big) dx_i - \sum_{j=1}^{k} \Big(\sum_{A \cup \{m+j'\} = P \cup \{m+j\}, A \neq P} \epsilon_{Pm+j}^{Am+j'} \frac{\partial f_A}{\partial y_{j'}} \Big) dy_j.$$

Conversely, if $f_A (A \neq P)$ satisfies the integrability condition (18)-(19) on Ω and if the domain Ω satisfies $H^1(\Omega, \mathbf{Z}) = 0$ for the ring $\mathbf{Z}$ of integers, then there exits a function f_P of the class C^∞ on Ω such that the $\mathcal{A}$ valued function $f = \sum_{A \neq P} f_A(x, y) e_A + f_P(x, y) e_P$ is a biregular function on Ω.

Proof. If there exists a function f_P of the class C^∞ on Ω such that the $\mathcal{A}$ valued function $f = \sum_{A\neq P} f_A(x,y)e_A + f_P(x,y)e_P$ is a biregular function on Ω, by the definition, there hold

$$0 = D_x f = \sum_B \Big(\sum_{\{i\}\cup A=B} \epsilon_B^{iA} \frac{\partial f_A}{\partial x_i} \Big) e_B \tag{20}$$

and

$$0 = f\mathrm{D}_y = \sum_B \Big(\sum_{A\cup\{m+j\}=B} \epsilon_B^{Am+j} \frac{\partial f_A}{\partial y_j} \Big) e_B. \tag{21}$$

Hence, there exist the following relations between the derivatives of f_A's:

$$\sum_{\{i\}\cup A=B} \epsilon_B^{iA} \frac{\partial f_A}{\partial x_i} = 0 \tag{22}$$

and

$$\sum_{A\cup\{m+j\}=B} \epsilon_B^{Am+j} \frac{\partial f_A}{\partial y_j} = 0 \tag{23}$$

for any permutation B with order from small to large. Then the derivatives of the function f_P satisfy

$$\sum_{\{i'\}\cup A=B, A\neq P} \epsilon_B^{i'A} \frac{\partial f_A}{\partial x_{i'}} + \sum_{\{i\}\cup P=B} \epsilon_B^{iP} \frac{\partial f_P}{\partial x_i} = 0 \tag{24}$$

and

$$\sum_{A\cup\{m+j'\}=B, A\neq P} \epsilon_B^{Am+j'} \frac{\partial f_A}{\partial y_{j'}} + \sum_{P\cup\{m+j\}=B} \epsilon_B^{Pm+j} \frac{\partial f_P}{\partial y_j} = 0 \tag{25}$$

for any permutation B with order from small to large. The derivatives of f_P are given by

$$\frac{\partial f_P}{\partial x_i} = - \sum_{\{i\}'\cup A=\{i\}\cup P, A\neq P} \epsilon_{iP}^{i'A} \frac{\partial f_A}{\partial x_{i'}}. \tag{26}$$

Then, the differential ω of the function f_P

$$\omega = \sum_{i=1}^{m} \frac{\partial f_P}{\partial x_i} \mathrm{d}x_i + \sum_{j=1}^{k} \frac{\partial f_P}{\partial y_j} \mathrm{d}y_j \tag{27}$$

is represented by other $f_A(A \neq P)$ as follows:

$$\omega = -\sum_{i=1}^{m}(-\sum_{\{i\}'\cup A=\{i\}\cup P, A\neq P} \epsilon_{iP}^{i'A}\frac{\partial f_A}{\partial x_{i'}})dx_i - \sum_{j=1}^{k}(\sum_{A\cup\{m+j'\}=B, A\neq P} \epsilon_{Pm+j'}^{Am+j'}\frac{\partial f_A}{\partial y_{j'}})dy_j.$$

(28)

Of course, there holds the integrability condition $d\omega = 0$ since ω is the differential of the function f_P.

Let p be a non negative integer, $\mathbf{R}$ be the constant sheaf of real numbers over Ω, $\mathcal{E}^p$ be the sheaf of germs of differential forms of degree p with coefficients of class C^∞ over the domain $\Omega \subset \mathbf{R}^m \times \mathbf{R}^k$, d be the usual differential operator $d^p : \mathcal{E}^p \to \mathcal{E}^{p+1}$ and $\iota : \mathbf{R} \to \mathcal{E}^0$ be the canonical injection. Then, by the lemma of Poincaré, the above operators give a fine resolution

$$0 \to \mathbf{R} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \cdots \to \mathcal{E}^p \to \mathcal{E}^{p+1} \cdots \tag{29}$$

of the constant sheaf $\mathbf{R}$ over Ω. By the theorem of Dolbeault [3], we have the following Dolbeault's isomorphism

$$\mathrm{H}^p(\Omega, \mathbf{R}) \cong \mathrm{H}^p(\Omega, (d^p)^{-1}(0))/d^{p-1}(\mathrm{H}^0(\Omega, \mathcal{E}^{p-1})) \tag{30}$$

for any positive integer p. By the universal coefficient theorem [18], we have $\mathrm{H}^p(\Omega, \mathbf{R}) \cong \mathrm{H}^p(\Omega, \mathbf{Z}) \otimes \mathbf{R}$ and, hence, $\mathrm{H}^p(\Omega, \mathbf{R}) = 0$ if and only if $\mathrm{H}^p(\Omega, \mathbf{Z}) = 0$, for any positive integer p. Therefore, from the assumptions $\mathrm{H}^1(\Omega, \mathbf{Z}) = 0$ and (18)-(19), we have $\omega \in \mathrm{H}^0(\Omega, (d^1)^{-1}(0)) = d^0(\mathrm{H}^0(\Omega, \mathcal{E}^0))$ and there exists $f_P \in \mathrm{H}^0(\Omega, \mathcal{E}^0)$ such that $\omega = d^0 f_P$. The $\mathcal{A}$ valued function $f := f(x, y) = \sum_A f_A(x, y)e_A$ of class C^∞ on Ω satisfies $D_x f = 0$ by (18)-(19) and $\omega = d^0 f_P$, and $fD_y = 0$ by (18)-(19) and $\omega = d^0 f_4$. Hence the function f is the esired biregular function on Ω with f_P as e_P component for other components f_A given. ■

Corollary. *Let Ω be a domain in $\mathbf{R}^m \times \mathbf{R}^k$ with $H^1(\Omega, \mathbf{Z}) = 0$ for the ring $\mathbf{Z}$ of integers, $P = (h_1, h_2, \cdots, h_r)$ be a permutation with $1 \leq h_1 < h_2 < \cdots < h_r$, $f_A(A \neq P)$ be functions of class C^∞ on Ω satisfying the integrability condition (18)-(19). Then the f_A's are harmonic functions on Ω.*

Proof. By the theorem, there exists a real valued function f_P of class C^∞ on Ω such that the $\mathcal{A}$ valued function $f = \sum_A f_A(x, y)e_A$ is biregular on Ω. Since we have

$$\Delta_x f := \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} f = \overline{D_x} D_x f = 0, \tag{31}$$

$$f\Delta_y := f\sum_{j=1}^{k}\frac{\partial^2}{\partial y_k^2} = fD_y\overline{D_y} = 0,$$

f_A's are harmonic on Ω. ■

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