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Non-invariant Hypersurfaces of Kenmotsu Manifolds

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Non-invariant hypersurfaces of a Kenmotsu manifold are studied. It is proved that the fundamental 2-form of the induced (f, g, u, v, λ)-structure on the non-invariant hypersurface of a Kenmotsu manifold is closed. A sufficient condition for certain vector field on the hypersurface to be harmonic is given. A necessary and sufficient condition for a totally umbilical non-invariant hypersurface of a Kenmotsu manifold to be totally geodesic is proved. Finally, a sufficient condition for the induced (f, g, u, v, λ)-structure to be normal and quasi-normal is found.

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1. Introduction

There are two well known classes of almost contact manifolds, viz. Sasakian manifolds [1] and Kenmotsu manifolds [2]. On the other hand, in [3] it is introduced the notion of (f, g, u, v, λ)-structure on a manifold. It is known that on a non-invariant hypersurface of an almost contact metric manifold there always exists a (f, g, u, v, λ)-structure. Motivated by this fact, in this paper non-invariant hypersurfaces of a Kenmotsu manifold are studied. The paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3, some properties of non-invariant hypersurfaces of a Kenmotsu manifold are given. It is proved that the fundamental 2-form of the induced (f, g, u, v, λ)-structure on the non-invariant hypersurface of a Kenmotsu manifold is closed. A sufficient condition for certain vector field on the hypersurface to be harmonic is given. In Section 4 we find a necessary and sufficient condition for a totally umbilical non-invariant hypersurface of a Kenmotsu manifold to be totally geodesic. For a non-invariant hypersurface with the induced (f, g, u, v, λ)-structure of a Kenmotsu manifold the second fundamental form is also calculated provided f is parallel. In the last section a sufficient condition for the induced (f, g, u, v, λ)-structure to be normal and quasi-normal is found.

2. Preliminaries

Let $\overline{M}$ be an almost contact metric manifold [1] with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on $\overline{M}$ such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X) \tag{3}$$

for all $X, Y \in T\overline{M}$.

An almost contact metric manifold is known to be a *Kenmotsu manifold* [2], if

$$(\overline{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{4}$$

where $\overline{\nabla}$ is the operator of covariant differentiation with respect to g. From (4) it follows that

$$\overline{\nabla}_X \xi = -\phi^2 X = X - \eta(X)\xi, \qquad X \in T\overline{M}. \tag{5}$$

Let M be a hypersurface of a Riemannian manifold $\overline{M}$ with a Riemannian metric g. Then Gauss and Wiengarten formulae are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y)N, \quad (X, Y \in TM), \tag{6}$$

$$\overline{\nabla}_X N = -HX + \omega(X)N, \tag{7}$$

where $\overline{\nabla}$, ∇ are respectively the Riemannian and induced Riemannian connections in $\overline{M}$ and M; N is the unit normal vector in the normal bundle $T^{\perp}M$; ω is a 1-form on M and h is the second fundamental form related to H by

$$h(X, Y) = g(HX, Y). \tag{8}$$

The hypersurface M is known to be *totally geodesic* in $\overline{M}$ if $h = 0$, and *totally umbilical* in $\overline{M}$ if $H = \alpha I$.

If $\overline{M}$ is an almost contact metric manifold and M its hypersurface, then defining

$$\phi X = fX + u(X)N, \tag{9}$$

$$\phi N = -U, \tag{10}$$

$$\xi = V + \lambda N, \tag{11}$$

$$\eta(X) = v(X) \tag{12}$$

for $X \in TM$, we get an induced (f, g, u, v, λ)-structure [3] on the non-invariant hypersurface such that

$$f^2 = -I + u \otimes U + v \otimes V, \tag{13}$$

$$fU = -\lambda V, \quad fV = \lambda U, \tag{14}$$

$$u \circ f = \lambda v, \quad v \circ f = -\lambda u, \tag{15}$$

$$u(U) = 1 - \lambda^2, \quad u(V) = v(U) = 0, \quad v(V) = 1 - \lambda^2, \tag{16}$$

$$g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y), \tag{17}$$

$$g(X, fY) = -g(fX, Y), \quad g(X, U) = u(X), \quad g(X, V) = v(X) \tag{18}$$

for all $X, Y \in TM$, where $\lambda = \eta(N)$.

3. Some properties of non-invariant hypersurfaces

First, we state the following lemma whose proof is straight forward and hence is omitted.

Lemma 3.1. *Let M be a non-invariant hypersurface with (f, g, u, v, λ)-structure of an almost contact metric manifold $\overline{M}$. Then*

$$\begin{aligned}(\overline{\nabla}_X \phi)Y &= ((\nabla_X f)Y - u(Y)HX + h(X, Y)U) \\ &\quad + ((\nabla_X u)Y + h(X, fY) - u(Y)\omega(X))N,\end{aligned} \tag{19}$$

$$\overline{\nabla}_X \xi = (\nabla_X V - \lambda HX) + (h(X, V) - X\lambda + \lambda\omega(X))N, \tag{20}$$

$$(\overline{\nabla}_X \phi)N = (-\nabla_X U + fHX + \omega(X)U) + (-h(X, U) + u(HX))N, \tag{21}$$

$$(\overline{\nabla}_X \eta)Y = (\nabla_X v)Y - \lambda h(X, Y). \tag{22}$$

Proposition 3.2. *For a non-invariant hypersurface M with (f, g, u, v, λ)-structure of a Kenmotsu manifold $\overline{M}$, we have*

$$(\nabla_X f)Y = u(Y)HX - h(X, Y)U - v(Y)fX - g(X, fY)V, \tag{23}$$

$$(\nabla_X u)Y = -\lambda g(X, fY) - h(X, fY) - \omega(X)u(Y) - u(X)v(Y), \tag{24}$$

$$\nabla_X V = \lambda HX + X - v(X)V, \tag{25}$$

$$h(X, V) = -\lambda v(X) - X\lambda - \lambda\omega(X), \tag{26}$$

(27) $$\nabla_X U = \lambda f X + f H X + \omega(X) U - u(X) V,$$

(28) $$(\nabla_X v) Y = \lambda h(X,Y) + g(X,Y) - v(X) v(Y).$$

Proof. Using (4), (9) and (11) in (19) and equating tangential and normal parts we get (23) and (24) respectively. Using (5) and (11) in (20) and equating tangential and normal parts we get (25) and (26) respectively. Using (4), (10) and (11) in (21) and equating tangential parts we get (27). Lastly, (28) follows from (22). ■

Theorem 3.3. *If M is a non-invariant hypersurface M with (f,g,u,v,λ)-structure of a Kenmotsu manifold, then the 2-form F on M given by*

$$F(X,Y) \equiv g(X, fY)$$

is closed.

Proof. From (23), we get

$$(\nabla_X F)(Y,Z) = v(Y) g(fX,Z) - v(Z) g(fX,Y) - u(Y) h(X,Z) + u(Z) h(X,Y),$$

which gives

$$(\nabla_X F)(Y,Z) + (\nabla_Y F)(Z,X) + (\nabla_Z F)(X,Y) = 0,$$

that is, $dF = 0$. ■

Theorem 3.4. *Let M be a non-invariant hypersurface M with (f,g,u,v,λ)-structure of a Kenmotsu manifold. If $H = -\lambda I$, then V is harmonic.*

Proof. From (28), we get

$$(\nabla_X v) Y - (\nabla_Y v) X = 0.$$

Moreover, if $H = -\lambda I$ then from (25) we get $\nabla_V V = 0$. Thus, V is harmonic. ■

4. Totally geodesic non-invariant hypersurfaces

Theorem 4.1. *Let M be a totally umbilical non-invariant hypersurface with (f,g,u,v,λ)-structure of a Kenmotsu manifold. Then it is totally geodesic if and only if*

(29) $$\omega = -v - d(\log \lambda).$$

P r o o f. If M is totally umbilical, putting $H = \alpha I$, the equation (26) gives

$$-\lambda v(X) - X\lambda - \lambda\omega(X) = \alpha v(X). \tag{30}$$

Then M is totally geodesic, that is $\alpha = 0$, if and only if (29) is correct. ∎

Theorem 4.2. *Let M be a non-invariant hypersurface with (f, g, u, v, λ)-structure of a Kenmotsu manifold. If f is parallel, then we get*

$$h(X,Y) = \frac{\mu u(X)u(Y) - \lambda(1-\lambda^2)v(X)v(Y)}{(1-\lambda^2)^2}, \tag{31}$$

$$\omega = -d(\log\lambda), \tag{32}$$

where $\mu = h(U,U)$. Consequently, M is totally geodesic if and only if

$$\mu u^2 = \lambda(1-\lambda^2)v^2. \tag{33}$$

P r o o f. If f is parallel, then (23) gives

$$(1-\lambda^2)h(X,Y) = u(Y)u(HX) - \lambda v(X)v(Y). \tag{34}$$

From here we get

$$(1-\lambda^2)h(X,U) = (1-\lambda^2)u(HX) = \mu u(X).$$

that is,

$$\mu u(X) = (1-\lambda^2)u(HX). \tag{35}$$

From (34) and (35), eliminating $u(HX)$, we get (31). The equation (32) follows from (31) and (26). The last part is obvious. ∎

5. Normal (f, g, u, v, λ)–structure

In this section we find a sufficient condition for the induced (f, g, u, v, λ)-structure on the non-invariant hypersurface of a Kenmotsu manifold to be normal and quasi-normal.

Theorem 5.1. *Let M be a non-invariant hypersurface with (f, g, u, v, λ)-structure of a Kenmotsu manifold. If f commutes with H and $v = \omega$, then the induced (f, g, u, v, λ)-structure on M is normal.*

Proof. In view of (23), (24) and (28), we get

$$\begin{aligned} S(X,Y) &\equiv [f,f](X,Y) + du(X,Y)U + dv(X,Y)V \\ &= u(X)(fHY - HfY) - u(Y)(fHX - HfX) \\ &\quad + u(X)(\omega(Y) - v(Y))U - u(Y)(\omega(X) - v(X))U. \end{aligned}$$

Now, if $fH = Hf$ and $v = \omega$ then S vanishes, that is, the induced (f,g,u,v,λ)-structure becomes normal. ∎

Now we recall the definition of a quasi-normal (f,g,u,v,λ)-structure. The (f,g,u,v,λ)-structure is called *quasi-normal* [4], if

$$g(S(X,Y),Z) - ((dF)(fX,Y,Z) - (dF)(fY,X,Z)) = 0.$$

Then in view of Theorem 3.3 we get the following theorem.

Theorem 5.2. *Let M be a non-invariant hypersurface with (f,g,u,v,λ)-structure of a Kenmotsu manifold. If (f,g,u,v,λ)-structure on M is normal then it is also quasi-normal.*

In view of Theorems 5.1 and 5.2 we get the following corollary.

Corollary 5.3. *Let M be a non-invariant hypersurface with (f,g,u,v,λ)-structure of a Kenmotsu manifold. If f commutes with H and $v = \omega$, then the induced (f,g,u,v,λ)-structure on M is quasi-normal.*

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