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A Representation of a Nonnegative Algebraic Polynomial of Degree Four as a Sum of Two Squares of Real Polynomials

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In this paper, for every nonnegative algebraic polynomial of degree four, we find an explicit representation as a sum of two squares (Theorem 1). We discover necessary and sufficient conditions for nonnegativity of an algebraic polynomial of degree four (Theorem 2). Examples illustrating the application of Theorem 1 are given. Applications of Theorem 2 to the sixth and seventh coefficients of the univalent functions of the class S are given as well. Finally, we derive another method for solution of the problem.

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Key Words: nonnegative algebraic polynomial of degree four, sum of two squares, necessary and sufficient conditions for nonnegativity, class S of univalent functions, the sixth and seventh coefficients

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Let a_0, a_1, a_2, a_3, a_4 ($a_0 > 0$, $a_4 \geq 0$) be arbitrary real numbers such that the polynomial

$$y = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \tag{1}$$

is nonnegative for every real x, i.e. $0 \leq y \leq +\infty$ if $-\infty < x < +\infty$. If such a polynomial would have real zeros of order 1 or 3, then it would change its sign which is impossible. Hence this polynomial can only have either one double real zero and two complex conjugate zeros or one fourfold real zero or two double complex conjugate zeros or two pairs complex conjugate zeros. It is well known [1] that the polynomial (1) can be represented as a sum of the squares of two other real polynomials of degree two at the most and such a representation is not unique. In this paper we find two explicit representations of the polynomial (1) as a sum of the square of real polynomials of degree one at most.

Theorem 1. *If the real numbers a_0, a_1, a_2, a_3, a_4 $(a_0 > 0,\ a_4 \geq 0)$ are such that the polynomial (1) is nonnegative for every real x, then it has the following two representations*

$$y = (x^2\sqrt{a_0} + x\frac{a_1}{2\sqrt{a_0}} + t_{1,2}\sqrt{a_0})^2 + \left(x\sqrt{a_2 - \frac{a_1^2}{4a_0} - 2a_0 t_{1,2}} \pm \sqrt{a_4 - a_0 t_{1,2}^2}\right)^2, \tag{2}$$

where the radicals are arithmitic and t_1 and t_2 $(t_1 < t_2)$ are the two lesser real roots of the Ferrary resolving cubic equation

$$8a_0^3t^3 - 4a_0^2a_2t^2 + 2(a_0a_1a_3 - 4a_0^2a_4)t + 4a_0a_2a_4 - a_0a_3^2 - a_1^2a_4 = 0 \tag{3}$$

of (1) which in this case always has three real roots t_1, t_2 and t_3 $(t_1 \leq t_2 \leq t_3)$, where the sign $+$ is taken for such t_1 or t_2 for which

$$a_3 - a_1t_{1,2} > 0, \tag{4}$$

the sign $-$ is taken for such t_1 or t_2 for which

$$a_3 - a_1t_{1,2} < 0, \tag{5}$$

and the two signs $\pm$ are taken for such t_1 or t_2 for which

$$a_3 - a_1t_{1,2} = 0, \tag{6}$$

in which case the two signs $\pm$ are reduced to the sign $+$ because at least one of the radicals in (2) vanishes.

If $t_1 = t_2 \leq t_3$, the two representations (2) are reduced to one representation and the polynomial (1) has either one double real zero and two complex conjugate zeros (if $t_1 = t_2 < t_3$) or two double real zeros (if $t_1 = t_2 < t_3$ and the two radicands in (2) vanish; in this case the representation (2) is a perfect square) or one fourfold real zero (if $t_1 = t_2 = t_3$). If $t_1 < t_2 \leq t_3$, the two representations (2) are different and the polynomial (1) has either two double complex conjugate zeros (if $t_1 < t_2 = t_3$; in this case the two radicands in (2) for t_2 vanish and (2) is a perfect square for t_2) or two pairs complex conjugate zeros (if $t_1 < t_2 < t_3$). These five cases are unique.

Proof. (i) Let $x_{1,2} = \alpha_1 \pm i\beta_1$, $x_{3,4} = \alpha_2 \pm i\beta_2$ be zeros of the polynomial (1) where $\alpha_{1,2}$ and $\beta_{1,2}$ are real numbers. In particular, β_1 and β_2 can vanish;

this corresponds to double real zeros of (1). Then the polynomial (1) can be written in the form

$$(7) \qquad \begin{aligned} y &= a_0(x-x_1)(x-x_2)(x-x_3)(x-x_4) \\ &= a_0[x^2-(\alpha_1+\alpha_2)x+\alpha_1\alpha_2-\beta_1\beta_2]^2 \\ &\quad +a_0[(\beta_1+\beta_2)x-\alpha_1\beta_2-\alpha_2\beta_1]^2. \end{aligned}$$

The identification of (1) and (7) yields the system of equations

$$(8) \qquad \begin{aligned} \alpha_1+\alpha_2 &= -\frac{a_1}{2a_0}, \\ \alpha_1^4+4\alpha_1\alpha_2+\alpha_2^2+\beta_1^2+\beta_2^2 &= \frac{a_2}{a_0}, \\ \alpha_1(\alpha_2^2+\beta_2^2)+\alpha_2(\alpha_1^2+\beta_1^2) &= -\frac{a_3}{2a_0}, \\ (\alpha_1^2+\beta_1^2)(\alpha_2^2+\beta_2^2) &= \frac{a_4}{a_0}. \end{aligned}$$

If we set

$$(9) \qquad t_1 = \alpha_1\alpha_2-\beta_1\beta_2 = \frac{x_1x_3+x_2x_4}{2},$$

then from the fourth equation in (8) and (9) we obtain

$$(10) \qquad \alpha_1\beta_2+\alpha_2\beta_1 = \pm\sqrt{\frac{a_4}{a_0}-t_1^2},$$

where the radical is arithmetic. From the first and the second equations in (8) we obtain

$$(11) \qquad \beta_1^2+\beta_2^2 = \frac{a_2}{a_0}-\frac{a_1^2}{4a_0^2}-2\alpha_1\alpha_2.$$

With the help of (11) and (9) it follows that

$$(12) \qquad \beta_1+\beta_2 = \pm\sqrt{\frac{a_2}{a_0}-\frac{a_1^2}{4a_0^2}-2t_1},$$

where the radical is arithmetic. Now (12), (10), (9) and the first equation in (8) transform (7) into (2) for t_1. The identification of (1) and (2) in this case yields the two irrational equations

$$(13) \qquad a_3-a_1t_1 = \pm2\sqrt{a_2-\frac{a_1^2}{4a_0}-2a_0t_1}\cdot\sqrt{a_4-a_0t_1^2},$$

where the radicals are arithmetic. From (13) we obtain that t_1 satisfies the equation (3). The assertions in (4), (5) and (6) for t_1 follows from (13) as well.

(ii) Now we can write (7) in the form

$$y = a_0(x - x_1)(x - x_2)(x - x_3)(x - x_4) \tag{14}$$

$$= a_0[x^2 - (\alpha_1 + \alpha_2)x + \alpha_1\alpha_2 + \beta_1\beta_2]^2 + a_0[(\beta_2 - \beta_1)x - (\alpha_1\beta_2 - \alpha_2\beta_1)]^2.$$

The same procedure leads us to (8) and to the values

$$t_2 = \alpha_1\alpha_2 + \beta_1\beta_2 = \frac{x_1x_4 + x_2x_3}{2}, \tag{15}$$

$$\alpha_1\beta_2 - \alpha_2\beta_1 = \pm\sqrt{\frac{a_4}{a_0} - t_2^2}, \tag{16}$$

$$\beta_2 - \beta_1 = \pm\sqrt{\frac{a_2}{a_0} - \frac{a_1^2}{4a_0} - 2t_2}. \tag{17}$$

The relations (14)–(17) yield the representation (2) in this case. The identification of (1) and (2) for t_2 again leads us to the two irrational equations in (13) but for t_2, i.e. t_2 satisfies the equation (3). The assertions in (4), (5) and (6) for t_2 follow from (13) by the change of t_1 with t_2.

(iii) The third root of (3) is determined by the equation $t_3 = (a_2/2a_0) - t_1 - t_2$ with the help of the second equation (8), (9) and (15), i.e.

$$t_3 = \frac{1}{2}(\alpha_1^2 + \beta_1^2) + \frac{1}{2}(\alpha_2^2 + \beta_2^2) = \frac{x_1x_2 + x_3x_4}{2}. \tag{18}$$

It follows from (9), (15) and (18) that the three roots $t_{1,2,3}$ of the equation (3) are real and that $t_1 \le t_3$ and $t_2 \le t_3$ since

$$t_3 - t_1 = \frac{(\alpha_1 - \alpha_2)^2 + (\beta_1 + \beta_2)^2}{2}, \quad t_3 - t_2 = \frac{(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2}{2}. \tag{19}$$

If we assume that $\beta_{1,2} \ge 0$, then from (9), (15) and (19) we obtain $t_1 \le t_2 \le t_3$ and the last part of the assertions of Theorem 1.

(iv) According to the Ferrari classic method, any polynomial (1) can be represented in the identical form

$$y = (x^2\sqrt{a_0} + x\frac{a_1}{2\sqrt{a_0}} + t\sqrt{a_0})^2$$

$$-[(2a_0t - a_2 + \frac{a_1^2}{4a_0})x^2 + (a_1t - a_3)x + a_0t^2 - a_4] \tag{20}$$

for any t. If some t satisfies the condition

$$(a_1t - a_3)^2 - 4(2a_0t - a_2 + \frac{a_1^2}{4a_0})(a_0t^2 - a_4) = 0, \tag{21}$$

then (21) leads us to the equation (3) which has three real roots in the examined case, i.e. such t is equal to any root of the equation (3). Then (20) takes the form

$$y = (x^2\sqrt{a_0} + x\frac{a_1}{2\sqrt{a_0}} + t\sqrt{a_0})^2 - \left(x\sqrt{2a_0t - a_2 + \frac{a_1^2}{4a_0}} + \sqrt{a_0t^2 - a_4}\right)^2 \tag{22}$$

where the product of the radicals in the second term has the sign of $a_1t - a_3$. From the comparison of (2) with (22) and (13) with (21) it is clear that the third root t_3 of (3) can also be used for the representation (2) but then the polynomial in the second term of (2) will have pure imaginary coefficients if $t_1 \leq t_2 < t_3$. In this case the product of the two radicals in the second term of (2) has the sign of $a_3 - a_1t_3$ as well. Really, it follows from (8) and (18) that

$$a_2 - \frac{a_1^2}{4a_0} - 2a_0t_3 = -a_0(\alpha_1 - \alpha_2)^2 \leq 0 \tag{23}$$

and

$$a_4 - a_0t_3^2 = -\frac{a_0}{4}(\alpha_1^2 + \beta_1^2 - \alpha_2^2 - \beta_2^2)^2 \leq 0. \tag{24}$$

The equality signs in (23)–(24) simultaneously hold only if $t_3 = t_2$ according to (19). In addition, if $t_3 = t_2$ from (19), (23) and (24) we again obtain that (2) as well as (22) are reduced to a perfect square.

This completes the proof of Theorem 1. ∎

Remark 1. The inverse assertion that if the three roots of (3) are real, then (1) is nonnegative, is not always true. Really, from (21) we conclude that the two radicands in the second term in (22) have the same signs if they are nonvanishing for a real root t of (3). In general, let t', t'', t''' be the roots of (3). Then it follows from (22) that

$$x'x'' = t' + \sqrt{t'^2 - \frac{a_4}{a_0}}, \quad x'''x^{iv} = t' - \sqrt{t'^2 - \frac{a_4}{a_0}}, \tag{25}$$

(26) $$x'x''' = t'' + \sqrt{t''^2 - \frac{a_4}{a_0}}, \quad x''x^{iv} = t'' - \sqrt{t''^2 - \frac{a_4}{a_0}},$$

(27) $$x'x^{iv} = t''' + \sqrt{t'''^2 - \frac{a_4}{a_0}}, \quad x''x''' = t''' - \sqrt{t'''^2 - \frac{a_4}{a_0}}$$

for a suitable choice among the zeros x', x'', x''', x^{iv} of (1). Thus from (25)–(27) we obtain the roots

(28) $$t' = \frac{x'x'' + x'''x^{iv}}{2}, \quad t'' = \frac{x'x''' + x''x^{iv}}{2}, \quad t''' = \frac{x'x^{iv} + x''x'''}{2}.$$

With the help of (28) and (1) we get

(29) $$2a_0t' - a_2 + \frac{a_1^2}{4a_0} = \frac{a_0}{4}(x' + x'' - x''' - x^{iv})^2$$

and

(30) $$a_0t'^2 - a_4 = \frac{a_0}{4}(x'x'' - x'''x^{iv})^2,$$

(31) $$2a_0t'' - a_2 + \frac{a_1^2}{4a_0} = \frac{a_0}{4}(x' + x'' - x''' - x^{iv})^2$$

and

(32) $$a_0t''^2 - a_4 = \frac{a_0}{4}(x'x''' - x''x^{iv})^2,$$

(33) $$2a_0t''' - a_2 + \frac{a_1^2}{4a_0} = \frac{a_0}{4}(x' + x^{iv} - x'' - x''')^2$$

and

(34) $$a_0t'''^2 - a_4 = \frac{a_0}{4}(x'x^{iv} - x''x''')^2.$$

Now it follows from (29) and (30) that, for a real root t', the two corresponding radicals in (22) are simultaneously nonnegative or negative if $x' + x'' - x''' - x^{iv}$ and $x'x'' - x'''x^{iv}$ are simultaneously real or pure imaginary numbers, respectively. It is clear for the first case that among x', x'', x''', x^{iv} there can be real roots of odd multiplicities and hence the polynomial (1) cannot be negative. For the second case the representation (22) takes the form (2) and hence the polynomial (1) is nonnegative. Analolous conclusions follow from (31)–(32) and (33)–(34), respectively. For example, the polynomial

$$y = a_0x^4 + a_1x^3 + a_2x^2, \quad a_{0,1,2} > 0 \quad (a_3 = a_4 = 0), \quad a_1^2 - 4a_0a_2 > 0$$

has the roots

$$x' = 0, \quad x'' = \frac{-a_1 + \sqrt{a_1^2 - 4a_0a_2}}{2a_0}, \quad x''' = 0, \quad x^{iv} = \frac{-a_1 - \sqrt{a_1^2 - 4a_0a_2}}{2a_0}$$

and hence it is nonnegative while the corresponding equation (3) is

$$2a_0t^3 - a_2t^2 = 0$$

with the three real roots

$$t' = 0, \quad t'' = \frac{a_2}{2a_0}, \quad t''' = 0.$$

From Theorem 1 and Remark 1 we obtain the following theorem.

Theorem 2. *The polynomial (1) is nonnegative if and only if equation (3) has three real roots* $t_1 \le t_2 \le t_3$*, and the following four conditions are simultaneously fulfilled:*

$$a_2 - \frac{a_1^2}{4a_0} - 2a_0t_{1,2} \ge 0 \tag{35}$$

and

$$a_4 - a_0t_{1,2}^2 \ge 0. \tag{36}$$

Corollary. *Under the hypothesis of Theorem 2, it is necessary that*

$$a_2 - \frac{a_1^2}{4a_0} - 2a_0t_3 \le 0 \tag{37}$$

and

$$a_4 - a_0t_3^2 \le 0. \tag{38}$$

The following examples illustrate the applications of Theorems 1 and 2.

Example 1. For the polynomial

$$y = x^4 - 6x^3 + 14x^2 - 14x + 5 \quad (t_1 = t_2 = 2, \quad t_3 = 3),$$

we have

$$y = (x^2 - 3x + 2)^2 + (x - 1)^2$$

and

$$y = (x^2 - 3x + 3)^2 - (x - 2)^2 = (x - 1)^2(x^2 - 4x + 5).$$

Example 2. For the polynomial

$$y = x^4 + 2x^3 - 3x^2 - 4x + 4 \quad (t_1 = t_2 = -2, \quad t_3 = \frac{5}{2}),$$

we have

$$y = (x^2 + x - 2)^2$$

and

$$y = (x^2 + x + \frac{5}{2})^2 - (3x + \frac{3}{2})^2 = (x - 1)^2(x + 2)^2.$$

Example 3. For the polynomial

$$y = x^4 - 8x^3 + 24x^2 - 32x + 16 \quad (t_1 = t_2 = t_4 = 4),$$

we have

$$y = (x^2 - 4x + 4)^2 = (x - 2)^4.$$

Example 4. For the polynomial

$$y = x^4 + 2x^3 + 5x^2 + 4x + 4 \quad (t_1 = -\frac{3}{2}, \quad t_2 = t_3 = 2),$$

we have

$$y = (x^2 + x - \frac{3}{2})^2 + (x\sqrt{7} + \frac{\sqrt{7}}{2})^2$$

and

$$y = (x^2 + x + 2)^2.$$

Example 5. For the polynomial

$$y = x^4 - 4x^3 + 11x^2 - 14x + 10 \quad (t_1 = -1, \quad t_2 = 3, \quad t_3 = \frac{7}{2}),$$

we have

$$y = (x^2 - 2x - 1)^2 + (3x - 3)^2,$$

$$y = (x^2 - 2x + 3)^2 + (x - 1)^2,$$

and

$$y = (x^2 - 2x + \frac{7}{2})^2 - (\frac{3}{2})^2 = (x^2 - 2x + 5)(x^2 - 2x + 2).$$

Remark 2. With the help of the identity

$$p^2 + q^2 = \frac{1}{2}(p + q)^2 + \frac{1}{2}(p - q)^2$$

some of the representations (2) can be expressed as a sum of two squares of quadreatic trinomials. For example, from the two representations of y in Example 5 we can obtain the corresponding new representations

$$y = \frac{1}{2}(x^2 + x - 4)^2 + \frac{1}{2}(x^2 - 5x + 2)^2,$$

and

$$y = \frac{1}{2}(x^2 - x + 2)^2 + \frac{1}{2}(x^2 - 3x + 4)^2.$$

Applications to the univalent functions of the class S

In the problems about the sixth and seventh coefficients of the univalent functions of the famous class S it has to establish the positiveness of the polynomials [2]

$$y = 6x^4 - 16x^3 + \frac{108}{7}x^2 - \frac{32}{5}x + 1, \tag{39}$$

$$y = \frac{99}{7}x^4 - \frac{216}{7}x^3 + \frac{169}{7}x^2 - 8x + 1 \tag{40}$$

and

$$y = \frac{55}{14}x^4 - \frac{80}{7}x^3 + \frac{110}{9}x^2 - \frac{40}{7}x + 1. \tag{41}$$

Now we use our criterion, expressed by Theorem 2. For the polynomial (39), the eqation (3) is

$$t^3 - \frac{9}{7}t^2 + \frac{49}{90}t - \frac{719}{9450} = 0$$

with roots $t_1 < t_2 < t_3$, lying in the open intervals

$$(0.3, 0.39), \quad (0.39, \frac{25}{63} = 0.3968253\ldots), \quad (\frac{1}{\sqrt{6}} = 0.4082482\ldots, 0.6),$$

respectively, for which the corresponding conditions (35) and (36) are fulfilled, i.e.

$$\frac{25}{63} - t_{1,2} > 0. \quad \frac{1}{6} - t_{1,2}^2 > 0,$$

as well as the corresponding conditions (37) and (38) are fulfilled, i.e.

$$\frac{25}{63} - t_3 < 0, \quad \frac{1}{6} - t_3^2 < 0.$$

Hence, the polynomial (39) has two pairs complex conjugate zeros and it is positive for every real x. Analogously, for the polynomial (40), the equation (3) is

$$t^3 - \frac{169}{198}t^2 + \frac{259}{1089}t - \frac{1561}{71874} = 0,$$

with roots $t_1 < t_2 < t_3$, separated in the open intervals

$$(0.2, 0.25), \quad (0.25, \frac{563}{2178} = 0.258494\ldots), \quad (\sqrt{\frac{7}{99}} = 0.2659078\ldots, 0.4),$$

respectively, for which the conditions (35) and (36) are satisfied, i.e.

$$\frac{563}{2178} - t_{1,2} > 0, \quad \frac{7}{99} - t_{1,2}^2 > 0,$$

as well as the conditions (37) and (38) are satisfied, i.e.

$$\frac{563}{2178} - t_3 < 0, \quad \frac{7}{99} - t_3^2 < 0.$$

Therefore, the polynomial (40) has two pairs complex conjugate zeros and it is positive for every real x. Finally, for the polynomial (41), the equation (3) is

$$t^3 - \frac{14}{9}t^2 + \frac{486}{605}t - \frac{8252}{59895} = 0$$

with roots $t_1 < t_2 < t_3$ in the open intervals

$$(0.4, 0.49), \quad (0.49, \frac{542}{1089} = 0.4977043\ldots), \quad (\sqrt{\frac{14}{55}} = 0.5045249\ldots, 0.6),$$

respectively, for which the conditions (35) and (36) hold, i.e.

$$\frac{542}{1089} - t_{1,2} > 0, \quad \frac{14}{55} - t_{1,2}^2 > 0,$$

as well as the conditions (37) and (38) hold, i.e.

$$\frac{542}{1089} - t_3 < 0, \quad \frac{14}{55} - t_3^2 < 0.$$

Consequently, the polynomial (41) has two pairs complex conjugate zeros and it is positive for every real x.

Another method for solution of the problem

The Horner transformation

$$y = \xi - \frac{a_1}{4a_0} \tag{42}$$

reduces the polynomial (1) to the form

$$y = a_0\xi^4 + p\xi^2 + q\xi + r, \tag{43}$$

where

$$p = -\frac{3a_1^2}{8a_0} + a_2, \tag{44}$$

$$q = \frac{a_1^2}{8a_0^2} - \frac{a_1a_2^2}{2a_0} + a_3,$$

$$r = a_0(\frac{a_1}{4a_0})^4 - a_1(\frac{a_1}{4a_0})^3 + a_2(\frac{a_1}{4a_0})^2 - a_3\frac{a_1}{4a_0} + a_4.$$

According to Descartes' solution of the quartic equation (see, for examples, Dickson [3, pp.42-43] or [4, pp. 52-53]) we set

$$y = (\sqrt{a_0}\xi^2 + k\xi + l)(\sqrt{a_0}\xi^2 - k\xi + m), \tag{45}$$

where k, l, m are new unknown coefficients. The comparision of (43) and (45) yields

$$l\sqrt{a_0} + m\sqrt{a_0} - k^2 = p, \quad k(m-l) = q, \quad lm = r. \tag{46}$$

It follows from the first two equations of (46) that

$$l = \frac{1}{2}(\frac{p+k^2}{\sqrt{a_0}} - \frac{q}{k}), \quad m = \frac{1}{2}(\frac{p+k^2}{\sqrt{a_0}} + \frac{q}{k}). \tag{47}$$

From the third equation in (46) and (47) we obtain the Descartes resolving equation

$$\tau^3 + 2p\tau^2 + (p^2 - 2a_0 r)\tau - a_0 q^2 = 0, \quad k^2 = \tau. \tag{48}$$

With the help of (48) we can verify that (43) can be written in the form

$$y = (\sqrt{a_0}\xi^2 + \frac{p+\tau}{2\sqrt{a_0}})^2 + (\sqrt{-\tau}\xi + \frac{q}{2\sqrt{-\tau}})^2. \tag{49}$$

It follows from (49) that *the polynomial y is nonnegative if and only if the root τ of the equation (48) is negative.* Inserting (42) and (44) in (49), we obtain the corresponding representation of the polynomial (1) as a sum of two squares of polynomials of degree two at the most. See in Dickson [3, p.45, the Theorem] other criteria for the nonnegativity of the polynomial (43).

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