Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 14, 2000, Fasc. 3-4

On a Nonlocal Boundary Value Problem for a Quasilinear Equation of Mixed Type

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Using the Faedo - Galerkin method we prove the existence of a generalized solution of a nonlocal boundary-value problem for one second order quasilinear equation of mixed type in a bounded multidimensional cylindrical domain.

AMS Subj. Classification: Primary 35M10, Secondary 35D05

Key Words: quasilinear equation of mixed type, nonlocal boundary-value problem, existence of a generalized solution

1. Introduction

Let D be a bounded domain in the space $\mathbf{R}^{m-1}$ of points $x' = (x_1, \ldots, x_{m-1})$, where $m \geq 2$ is an integer. Let $G = \{x = (x', x_m) \in \mathbf{R}^m : x' \in D, 0 < x_m < h\}$, $S = \{x \in \mathbf{R}^m : x' \in \partial D, 0 < x_m < h\}, h = const$, $S \in C^1$.

We consider the operator

$$\mathcal{L}u = \sum_{i,j=1}^{m-1} a_{ij}(x)u_{x_ix_j} + k(x)u_{x_mx_m} + \sum_{i=1}^{m} b_i(x)u_{x_i} + c(x)u - u|u|^{\rho} - f(x,u) ,$$

where $a_{ij} \in C^2(\overline{G})$, $a_{ij} = a_{ji}$ for $i,j = 1, \ldots, m-1$; $\sum_{i,j=1}^{m-1} a_{ij}(x)\xi_i\xi_j \geq a_0 \sum_{i=1}^{m-1} \xi_i^2$, $\forall x \in \overline{G}$ and $\forall \xi' \in \mathbf{R}^{m-1}$, $a_0 = const > 0$; $k \in C^2(\overline{G})$, $k(x',0) = k(x',h) = 0$ $\forall x' \in \overline{D}$; $c, b_i \in C^1(\overline{G})$ for $i = 1, \ldots, m$; $\rho = const > 0$. The function $f(x,t)$

* This research is partially supported by the Ministry of Education and Sciences of Bulgaria under Contract MM 904/99

is defined in $G \times \mathbf{R}$ and $f \in \mathbf{CAR}$, i.e. $f(x,t)$ is continuous with respect to t for almost every $x \in G$ and it is measurable with respect to x in G for every $t \in \mathbf{R}$ (see [5], 12.2). All the functions in this paper are real-valued.

The operator $\mathcal{L}$ is elliptic, hyperbolic, parabolic at a point $x \in G$, if $k(x) > 0$, $k(x) < 0$, $k(x) = 0$ respectively. In our case $\mathcal{L}$ is an operator of mixed type in G, because there are no restrictions on the sign of $k(x)$ for $x \in G$.

We consider the following boundary value problem:

Find a function $u(x)$ in $\overline{G}$ such that

$$\mathcal{L}u = 0 \text{ in } G, \tag{1}$$

$$u = 0 \text{ on } S,\ u(x',h) = \lambda u(x',0) \text{ in } D, \tag{2}$$

where $\lambda \neq 0$ is a given real constant.

A nonlocal problem for the linear equation

$$\mathcal{L}u = \sum_{i,j=1}^{m-1} a_{ij}(x)u_{x_i x_j} + k(x)u_{x_m x_m} + \sum_{i=1}^{m} b_i(x)u_{x_i} + c(x)u = f(x),$$

where $k(x',0) = k(x',h) = 0\ \ \forall x' \in \overline{D}$, with the boundary conditions (2) is investigated in [4] for $0 < \lambda \leq 1$, in [7, 8] for $0 < |\lambda| < 1$, in [9] for $\lambda \neq 0$. This problem is investigated also in [17] in the case where $-1 < \lambda \leq 1$, $a_{ij}(x) = \delta_i^j$, δ_i^j is the Kronecker's symbol, $b_i = 0$, $i,j = 1,\ldots,m-1$, $k = k(x_m)$ and $k(h) \geq 0 \geq k(0)$.

Nonlocal boundary value problems for different nonlinear equations of second order of mixed type are considered in [4], [6].

Let $Z = W_2^1(G) \cap L_{\rho+2}(G)$ be the linear normed space with a norm

$$\|u\| = (\|u\|^2_{W_2^1(G)} + \|u\|^2_{L_{\rho+2}(G)})^{1/2}, \tag{3}$$

where $\|v\|_{L_r(G)} = (\int_G |v|^r\,dx)^{1/r}$, $1 \leq r < \infty$, and $\|u\|_{W_2^1(G)} = [\int_G (u^2 + \sum_{i=1}^{m} u^2_{x_i})\,dx]^{1/2}$ are the norms in $L_r(G)$ and in the Sobolev space $W_2^1(G)$, respectively. Let X be the closure in the norm (3) of the set $\tilde{C}^2 = \{u \in C^2(\bar{G}) :\ u \text{ satisfies } (2)\}$ and let Y be the closure in the norm (3) of the set of all functions belonging to $C^2(\bar{G})$ and vanishing on S. In the sequel we suppose that $f(x,u(x)) \in L_2(G)\ \ \forall u \in L_{\rho+2}(G)$ and that for some constants $F_1 > 0$, $F_2 > 0$, $\sigma > 1$ the inequality

$$\|f(x,u)\|^2_{L_2(G)} \le F_1 + F_2\|u\|^{(\rho+2)/\sigma}_{L_{\rho+2}(G)} \tag{4}$$

holds for every $u \in L_{\rho+2}(G)$. It is not difficult to establish (4), using 12.4 and 12.11 from [5] and the imbedding $L_{2\sigma}(G) \hookrightarrow L_2(G)$ (see [1]), if for example $\kappa \in L_{2\sigma}(G)$ and $F_0 = const \ge 0$ exist such that $|f(x,t)| \le \kappa(x) + F_0|t|^{(\rho+2)/2\sigma}$ for almost every $x \in G$ and for every $t \in \mathbf{R}$. Since $u|u|^\rho \in L_\eta(G)$ for $u \in L_{\rho+2}(G)$ with $\eta = (\rho+2)/(\rho+1)$ where $\rho > 0$, then for u, $v \in Z$ we denote

$$B[u,v] \equiv -\int_G [(kv)_{x_m}u_{x_m} + \sum_{i,j=1}^{m-1}(a_{ij}v)_{x_j}u_{x_i}]\,dx$$

$$+\int_G [\sum_{i=1}^{m} b_i u_{x_i} + cu - u|u|^\rho - f(x,u)]v\,dx. \tag{5}$$

Definition. A function $u(x)$ is called a generalized solution of problem (1), (2), if $u \in X$ and

$$B[u,v] = 0 \quad \forall v \in Y. \tag{6}$$

The main result in this paper is the following theorem.

Theorem 1. *Let $k(x', x_m) > 0$ for $x' \in \bar{D}$ and $h_+ \le x_m < h$ where $h_+ = const$, $0 < h_+ < h$; $a_{ij}(x', h) = a_{ij}(x', 0)$ in $\bar{D}$, $i,j = 1,\dots,m-1$; $c(x) = -M + g(x)$, where $M = const > 0$ and $g(x)$ does not depend on M; $2b_m - k_{x_m} > 0$ in $G_0 = \{x \in \bar{G} : \ k(x) = 0\}$; and (4) be satisfied. Then a positive constant $\bar{M}$ exists such that problem (1), (2) has a generalized solution for every $M \ge \bar{M}$.*

Sections 2 and 3 of this paper contain some preliminary results. The proof of Theorem 1 is given in Section 4. It consists of three steps: 1) using the Faedo - Galerkin method we construct a sequence belonging to the Banach space X, which sequence is bounded in the norm (3); 2) that implies the existence of a subsequence weakly convergent in X to a function $u \in X$; 3) taking a limit in some integral equalities for this subsequence, we obtain (6).

Similar schemes of proofs are used for local boundary-value problems for quasilinear equations of mixed type in [2, 3, 15], for a nonlinear degenerating hyperbolic equation in [16], for quasilinear hyperbolic - parabolic equations in [12, 14] and others.

Some of the results of the present paper are published without proofs in [10].

2. Preliminary results

Lemma 1. *i) Z is a Banach, separable and reflexive space.*

ii) The spaces X and Y with the norm (3) are Banach, separable and reflexive.

Proof. i) The spaces $W_2^1(G)$ and $L_{\rho+2}(G)$ are Banach ones and $L_{\rho+2}(G) \hookrightarrow L_2(G)$ for $\rho > 0$ (see [1]). Then Z also is a Banach space. Let $V = \{w = (w_0, w_1, \ldots, w_m, w_{m+1}) : \ w_j \in L_2(G),\ j = 0, 1, \ldots, m,\ w_{m+1} \in L_{\rho+2}(G)\}$. Then V is a separable and reflexive Banach space with a norm $\|w\|_V = (\sum_{j=0}^{m} \|w_j\|^2_{L_2(G)} + \|w_{m+1}\|^2_{L_{\rho+2}(G)})^{1/2}$ ([1], 1.22). From the imbedding $L_{\rho+2}(G) \hookrightarrow L_2(G)$ for $\rho > 0$ and the definition of a generalized derivative (see [1]) it follows that the space $\tilde{V} = \{(v, v_{x_1}, \ldots, v_{x_m}, v) : \ v \in Z\}$ is a closed subspace of V. Therefore $\tilde{V}$ is a separable and reflexive Banach space under the norm $\|.\|_V$ ([1], 1.21). Let consider the linear one-to-one mapping $\mathcal{M} : \ Z \to \tilde{V}$ such that $\mathcal{M}v = (v, v_{x_1}, \ldots, v_{x_m}, v)\ \forall v \in Z$. From (3) it follows $\|\mathcal{M}v\|_V = \|v\|\ \forall v \in Z$. Hence Z is a separable and reflexive space.

ii) The proof is a consequence of i) and the fact that X and Y are closed subspaces of Z. ■

Lemma 2. *A countable linearly independent set $\mathcal{R}$, consisting of functions from $C^2(\bar{G})$ vanishing on S, exists such that its linear span is dense in Y.*

Proof. Let the set $\{v_j\}_{j=1}^{\infty} \subset Y$ be dense in Y. Let $\varepsilon_n > 0\ \forall n \in \mathbf{N}$ and $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$. For $j, n \in \mathbf{N}$ a function $\varphi_{jn} \in C^2(\bar{G})$, $\varphi_{jn}|_S = 0$, exists such that $\|v_j - \varphi_{jn}\| < \varepsilon_n$. Let $v \in Y$ and $\delta > 0$. We take $\varepsilon_{n_\delta} < \frac{\delta}{2}$ and v_{j_δ} with the properties $j_\delta, n_\delta \in \mathbf{N}$, $\|v - v_{j_\delta}\| < \frac{\delta}{2}$. Then $\|v - \varphi_{j_\delta n_\delta}\| \le \|v - v_{j_\delta}\| + \|v_{j_\delta} - \varphi_{j_\delta n_\delta}\| < \frac{\delta}{2} + \varepsilon_{n_\delta} < \delta$. Arranging the countable set $\{\varphi_{jn}\}_{j,n=1}^{\infty}$ in a sequence $\psi_1, \psi_2, \ldots$ we see that it is dense in Y and $\psi_j \in C^2(\bar{G})$, $\psi_j|_S = 0\ \forall j \in \mathbf{N}$.

Let now Φ_1 be the first element of the sequence $\{\psi_j\}_{j=1}^{\infty}$ which is not equal to zero in $\bar{G}$. Let Φ_2 be the next element of $\{\psi_j\}_{j=1}^{\infty}$ such that Φ_1 and Φ_2 are linearly independent in $\bar{G}$. Let Φ_3 be the next element of $\{\psi_j\}_{j=1}^{\infty}$ such that Φ_1, Φ_2 and Φ_3 are linearly independent in $\bar{G}$ and so on. The set $\mathcal{R} = \{\Phi_1, \Phi_2, \Phi_3, \ldots\}$ is the needed because: 1) $\Phi_j \in C^2(\bar{G})$, $\Phi_j|_S = 0\ \forall j \in \mathbf{N}$, 2) $\mathcal{R}$ is an infinite, countable, linearly independant set, 3) the linear span of $\mathcal{R}$ is dense in Y. ■

Lemma 3. ([13], Ch. 1, Lemma 1.3) *If $\{w_r\}_{r=1}^{\infty}$ is a bounded sequence in $L_\eta(G)$, $1 < \eta < +\infty$, $w \in L_\eta(G)$ and $w_r \underset{r\to\infty}{\longrightarrow} w$ almost everywhere in G, then $w_r \underset{r\to\infty}{\longrightarrow} w$ weakly in $L_\eta(G)$.*

Lemma 4. *Let* $u_1, u_2, \ldots, u_n \in \tilde{C}^2$, $\varphi \in C^2(\bar{G})$, $\varphi|_S = 0$. *Let* $\{\vec{\gamma}^r\}_{r=1}^{\infty} \in \mathbf{R}^n$ *be convergent to* $\vec{\gamma}^0 \in \mathbf{R}^n$ *in the norm* $|\vec{\gamma}| = [\sum_{s=1}^{n}(\gamma_s)^2]^{1/2}$ *where* $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$. *Then,*

$$B[\sum_{s=1}^{n}\gamma_s^r u_s,\ \varphi] \underset{r\to\infty}{\longrightarrow} B[\sum_{s=1}^{n}\gamma_s^0 u_s,\ \varphi]. \tag{7}$$

Proof. We set $w_r(x) = \sum_{s=1}^{n}\gamma_s^r u_s(x)$ for $r \in \mathbf{N}\bigcup\{0\}$ and $\eta = \frac{\rho+2}{\rho+1}$, where $\rho > 0$. Using the Minkowski and Hölder inequalities we obtain

$$\begin{aligned}\|w_r|w_r|^\rho\|_{L_\eta(G)} = \|w_r\|_{L_{\rho+2}(G)}^{\rho+1} &\le (\sum_{s=1}^{n}|\gamma_s^r|\|u_s\|_{L_{\rho+2}(G)})^{\rho+1}\\ &\le c_1|\vec{\gamma}^r|^{\rho+1} \le c_2, \quad \forall r \in \mathbf{N},\end{aligned}$$

because the convergent sequence $\{\vec{\gamma}^r\}_{r=1}^{\infty}$ is bounded. Here c_1 and c_2 are positive constants nondepending on $r \in \mathbf{N}$. Clearly $w_r(x) \underset{r\to\infty}{\longrightarrow} w_0(x)$ $\forall x \in \bar{G}$. Then Lemma 3 implies

$$w_r|w_r|^\rho \underset{r\to\infty}{\longrightarrow} w_0|w_0|^\rho \text{ weakly in } L_\eta(G). \tag{8}$$

The Minkowski inequality and the boundness of the sequence $\{\vec{\gamma}^r\}_{r=1}^{\infty}$, from (4), are used to get $\|f(x,w_r)\|_{L_2(G)}^2 \le c_3$ $\forall r \in \mathbf{N}$. Since $1 < \eta < 2$, it follows that $\|f(x,w_r)\|_{L_\eta(G)} \le c_4$ $\forall r \in \mathbf{N}$, where $c_4 = const > 0$ does not depend on $r \in \mathbf{N}$. The property $f \in \mathbf{CAR}$ implies $f(x,w_r(x)) \underset{r\to\infty}{\longrightarrow} f(x,w_0(x))$ for almost every $x \in G$. Then, inview of Lemma 3,

$$f(x,w_r(x)) \underset{r\to\infty}{\longrightarrow} f(x,w_0(x)) \text{ weakly in } L_\eta(G). \tag{9}$$

The convergences (8) and (9) and the fact that $w_r \underset{r\to\infty}{\longrightarrow} w_0$, $\frac{\partial w_r}{\partial x_i} \underset{r\to\infty}{\longrightarrow} \frac{\partial w_0}{\partial x_i}$ in $L_2(G)$, $i = 1, \ldots, m$, imply (7). ∎

We denote $\mathcal{P}(x_m) = (x_m^2 - 2hx_m + \delta)\exp(-\nu h)$, where $\nu = const > 0$ is such that

$$\min(\lambda^2, |\lambda|^{\rho+2}) > \exp(-\nu h) \tag{10}$$

and the constant δ satisfies the inequalities

$$\min[\alpha(|\lambda|^{\rho+2}), \alpha(\lambda^2)] > \delta > h^2 \tag{11}$$

with $\alpha(y) = h^2 y(y - \exp(-\nu h))^{-1}, y \neq \exp(-\nu h)$. Let $\mu = const > 0$, $\psi \in C^2(\bar{G})$, $\psi(x) = 0$ for $x \in G_0 \cup G_-$ and $\psi(x) > 0$ for $x \in G_+$, where $G_+ = \{x \in \overline{G} : k(x) > 0\}$, $G_- = \{x \in \overline{G} : k(x) < 0\}$. We denote $p(x_m) = \mathcal{P}(x_m)\exp(\nu x_m)$, $q(x) = -\mu\psi(x)\exp(\nu x_m)$ and $l(u) \equiv p(x_m)\dfrac{\partial u}{\partial x_m} + q(x)u$.

Lemma 5. *Let $\nu > 0$ satisfying (10) be fixed and $k(x', x_m) > 0$ for $x' \in \bar{D}$ and $h_+ \leq x_m < h$, where $0 < h_+ < h$. Let $\mu > \frac{\lambda}{H_0}$, if $\lambda > 1$, where $H_0 = \min\limits_{x' \in \bar{D}} \displaystyle\int_{h_+}^{h} \frac{\psi(x', t)}{\mathcal{P}(t)}\, dt$. Then for every $\varphi \in C^2(\bar{G})$, which vanishes on S, there exists a unique function $u_\varphi \in \tilde{C}^2$ such that*

$$l(u_\varphi) \equiv p\frac{\partial u_\varphi}{\partial x_m} + qu_\varphi = \varphi \quad \text{in } G.$$

Remark. Obviously $\mathcal{P}(x_m) \geq (\delta - h^2)\exp(-\nu h) > 0 \ \forall x_m$. Hence $H_0 > 0$.

Proof. Under our assumptions for μ we get for every $x' \in \bar{D}$

$$-\lambda + \exp\Big(\int_0^h \frac{\mu\psi(x', t)}{\mathcal{P}(t)}\, dt\Big) \geq -\lambda + \exp(\mu H_0) \geq -\lambda + 1 + \mu H_0 > 0.$$

Then one can verify that the function

$$u_\varphi(x', x_m) = \Big\{\int_0^{x_m} \frac{\varphi(x', \theta)}{\mathcal{P}(\theta)\exp(\nu\theta)} \exp\Big(\int_\theta^0 \frac{\mu\psi(x', t)}{\mathcal{P}(t)}\, dt\Big)\, d\theta$$
$$-\Big[\int_0^h \frac{\varphi(x', \theta)}{\mathcal{P}(\theta)\exp(\nu\theta)} \exp\Big(\int_\theta^h \frac{\mu\psi(x', t)}{\mathcal{P}(t)}\, dt\Big)\, d\theta\Big]\Big[-\lambda$$
$$+\exp\Big(\int_0^h \frac{\mu\psi(x', t)}{\mathcal{P}(t)}\, dt\Big)\Big]^{-1}\Big\} \exp\Big(\int_0^{x_m} \frac{\mu\psi(x', t)}{\mathcal{P}(t)}\, dt\Big)$$

possesses the needed properties. Let u_φ, $\tilde{u}_\varphi$ be two functions with these properties and $U = u_\varphi - \tilde{u}_\varphi$. Then $U \in \tilde{C}^2$ and $l(U) = 0$ in G, i.e. for fixed $x' \in D$

$$\frac{\partial U}{\partial x_m} - \frac{\mu\psi(x', x_m)}{\mathcal{P}(x_m)} U = 0, \ 0 < x_m < h; \ U(x', h) = \lambda U(x', 0).$$

Solving this nonlocal problem we find

$$U(x', x_m) = C(x')\exp\Big(\int_0^{x_m} \frac{\mu\psi(x', t)}{\mathcal{P}(t)}\, dt\Big), \ 0 \leq x_m \leq h,$$

where the constant $C(x')$ satisfies

$$C(x')[-\lambda + \exp(\int_0^h \frac{\mu\psi(x',t)}{\mathcal{P}(t)}\,dt)] = 0.$$

Hence $C(x') = 0$ and $u_\varphi = \tilde{u}_\varphi$ in $\bar{G}$. ■

3. An a-priori estimate

Lemma 6. *Let the assumptions of Theorem 1 hold. Then, with the notations introduced before in Lemma 5, constants $\nu > 0$ and $\mu > 0$ satisfying the conditions of Lemma 5 exist such that the a-priory estimate*

$$B[u, l(u)] \geq \beta_1 ||u||^2_{W_2^1(G)} + \beta_2 ||u||^{\rho+2}_{L_{\rho+2}(G)} - \beta_3 \tag{12}$$

takes place $\forall u \in \tilde{C}^2$ and $\forall M \geq \bar{M}(\nu, \mu)$, where β_1, β_2, β_3 are positive constants nondepending on u.

Proof. Taking $v = l(u)$ in (5) for $u \in \tilde{C}^2$ and integrating by parts in $B[u, l(u)]$, we get

$$\begin{aligned} B[u, l(u)] &= \int_G [-(kp)_{x_m} - kq + b_m p] u^2_{x_m}\,dx \\ &- \int_G kp u_{x_m x_m} u_{x_m}\,dx + \int_G p u_{x_m} \sum_{j=1}^{m-1} (-\sum_{i=1}^{m-1} a_{ijx_i} \\ &+ b_j) u_{x_j}\,dx - \int_G p \sum_{i,j=1}^{m-1} a_{ij} u_{x_i x_m} u_{x_j}\,dx \\ &- \int_G q \sum_{i,j=1}^{m-1} a_{ij} u_{x_i} u_{x_j}\,dx - \int_G (M-g) q u^2\,dx \\ &+ \int_G [b_m q - (kq)_{x_m} - (M-g)p] u_{x_m} u\,dx \\ &+ \int_G q u \sum_{j=1}^{m-1} (b_j - \sum_{i=1}^{m-1} a_{ijx_i}) u_{x_j}\,dx \\ &- \int_G u \sum_{i,j=1}^{m-1} a_{ij} q_{x_i} u_{x_j}\,dx - \int_G q|u|^{\rho+2}\,dx \end{aligned} \tag{13}$$

$$-\int_G p|u|^\rho u u_{x_m}\,dx - \int_G f(x,u)(pu_{x_m}+qu)\,dx = \sum_{s=1}^{12} I_s.$$

Integrating by parts and using (2), (10) and (11) we have

$$I_4 = \frac{1}{2}\int_D \sum_{i,j=1}^{m-1} a_{ij}(x',0)u_{x_i}(x',0)u_{x_j}(x',0)[p(0)-\lambda^2 p(h)]\,dx'$$

$$+\frac{1}{2}\int_G \sum_{i,j=1}^{m-1}(a_{ij}p)_{x_m}u_{x_i}u_{x_j}\,dx \geq \frac{1}{2}\int_G \sum_{i,j=1}^{m-1}(a_{ij}p)_{x_m}u_{x_i}u_{x_j}\,dx,$$

$$I_2 = \frac{1}{2}\int_G (kp)_{x_m}u_{x_m}^2\,dx.$$

The zeroes of $Q(t) = t^2+2(\frac{1}{\nu}-h)t+\delta-\frac{2h}{\nu}$ are $t_1 = h-\frac{1}{\nu}-\sqrt{\mathcal{D}}$ and $t_2 = h-\frac{1}{\nu}+\sqrt{\mathcal{D}}$, where $\mathcal{D} = \frac{1}{\nu^2}+h^2-\delta$. Let $\bar{\nu} = \max[\frac{2}{(h-h_+)}, \frac{6}{h\lambda^2}, -\frac{1}{h}\ln(\min(\lambda^2, |\lambda|^{\rho+2}))]$ and $\nu > \bar{\nu}$. Then (10) holds and from (11) it follows

$$h^2 < \delta < h^2\lambda^2(\lambda^2-\exp(-\nu h))^{-1}. \tag{14}$$

Since $\exp(\nu h) > 1+\nu h+2^{-1}\nu^2h^2+6^{-1}\nu^3h^3$, then $\lambda^2\exp(\nu h)-1 > \nu^2h^2$. Hence $h^2\lambda^2(\lambda^2-\exp(-\nu h))^{-1} < h^2+\nu^{-2}$ and from (14) follows that $0 < \mathcal{D} < \nu^{-2}$. Thus $h_+ < h-\frac{2}{\nu} < t_1 < t_2 < h \ \ \forall\nu > \bar{\nu}$. Since $Q(h) = \delta - h^2 > 0$, then

$$\nu\mathcal{P}(t)+\mathcal{P}'(t) = \nu\exp(-\nu h)Q(t) < 0 \text{ for } t_1 < t < t_2 \text{ and } \nu > \bar{\nu}.$$

Therefore for every $\nu > \bar{\nu}$ a constant $\bar{\mu}(\nu) > 0$ exists with the property $p'(x_m) - 2q(x) > 0$ in $\bar{G}\ \forall\mu > \bar{\mu}(\nu)$.

In the same way as in the proof of Theorem 1 in [9], we show that for some $\bar{\bar{\nu}} = const > 0$ and for arbitrary fixed $\nu > \bar{\bar{\nu}}$ a constant $\bar{\bar{\mu}}(\nu) > 0$ can be found such that for every $\mu > \bar{\bar{\mu}}(\nu)$

$$\sum_{s=1}^{5} I_s \geq \int_G p(\tilde{c}_1 u_{x_m}^2 + \tilde{c}_2\sum_{i=1}^{m-1} u_{x_i}^2)\,dx\,, \tag{15}$$

where $\tilde{c}_1 = \tilde{c}_1(\mu,\nu) > 0,\ \tilde{c}_2 = \tilde{c}_2(\nu) > 0.$

Since $k(x',0) = k(x',h) = 0$ in $\bar{D}$, then $q(x',0) = q(x',h) = 0$ in $\bar{D}$. Integrating by parts and using (2), we obtain

$$I_7 = \frac{1}{2}\int_G [(M-g)p - b_m q + (kq)_{x_m}]_{x_m} u^2\,dx + \frac{1}{2}\int_D \{[p(0)$$

$$-\lambda^2 p(h)]M + [\lambda^2 p(h)g(x',h) - p(0)g(x',0)]\}u^2(x',0)\,dx',$$

$$I_8 = -\frac{1}{2}\int_G u^2 \sum_{j=1}^{m-1}(qb_j - \sum_{i=1}^{m-1} a_{ijx_i}q)_{x_j}\,dx,$$

$$I_9 = \frac{1}{2}\int_G u^2 \sum_{i,j=1}^{m-1}(a_{ij}q_{x_i})_{x_j}\,dx.$$

We fix $\nu > \max(\bar{\nu},\ \bar{\bar{\nu}})$ and $\mu > \max(\bar{\mu}(\nu),\ \bar{\bar{\mu}}(\nu))$. Then a constant $\bar{M}(\nu,\mu) > 0$ exists such that

$$\sum_{s=6}^{9} I_s = \frac{1}{2}M\{\int_G (p' - 2q)u^2\,dx + \int_D [p(0)$$

$$-\lambda^2 p(h)]u^2(x',0)\,dx'\} + \tilde{I} \geq \tilde{c}_3 \int_G pu^2\,dx \tag{16}$$

$\forall M > \bar{M}(\nu,\mu)$, where $\tilde{c}_3 = \tilde{c}_3(\mu,\nu,M) > 0$. Let us note that in (16) the term $\tilde{I}$ does not depend on M.

Further, using (2), (10) and (11) we find

$$I_{11} = -\frac{1}{\rho+2}\int_G p\frac{\partial}{\partial x_m}(|u|^{\rho+2})\,dx = \frac{1}{\rho+2}\{\int_G p'|u|^{\rho+2}\,dx +$$

$$+\int_D |u(x',0)|^{\rho+2}[p(0) - |\lambda|^{\rho+2}p(h)]\,dx'\} \geq \frac{1}{\rho+2}\int_G p'|u|^{\rho+2}\,dx.$$

Obviously $p'(x_m) - (\rho+2)q(x) \geq p'(x_m) - 2q(x)$ in $\bar{G}$. Then for $\nu > \max(\bar{\nu},\ \bar{\bar{\nu}})$ and $\mu > \max(\bar{\mu}(\nu),\ \bar{\bar{\mu}}(\nu))$ the estimate

$$I_{10} + I_{11} \geq \tilde{c}_4 \int_G |u|^{\rho+2}\,dx \tag{17}$$

holds with $\tilde{c}_4 = \tilde{c}_4(\mu,\nu) > 0$.

Using the Hölder inequality, the inequality $|ab| \leq \frac{\varepsilon}{r}|a|^r + \frac{1}{r'}\varepsilon^{\frac{-r'}{r}}|b|^{r'}$ with $\varepsilon > 0$, $a,b \in \mathbf{R}$, $r > 1$, $\frac{1}{r} + \frac{1}{r'} = 1$, and the inequality (4), we obtain

$$\int_G |f(x,u)|(|pu_{x_m}| + |qu|)\,dx \leq (\frac{\tilde{c}_5 F_2}{\varepsilon_1})^{\sigma'}\frac{1}{\sigma'}\varepsilon_2^{\frac{-\sigma'}{\sigma}}$$

$$+\frac{\tilde{c}_5 F_1}{\varepsilon_1} + \frac{\varepsilon_2}{\sigma}\int_G |u|^{\rho+2}\,dx + \frac{\varepsilon_1}{2}\int_G (u_{x_m}^2 + u^2)\,dx\ , \tag{18}$$

where $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\frac{1}{\sigma'} + \frac{1}{\sigma} = 1$, $\tilde{c}_5 > 0$ depends only on p and q.

Thus for $\nu > \max(\bar{\nu}, \bar{\bar{\nu}})$, $\mu > \max(\bar{\mu}(\nu), \bar{\bar{\mu}}(\nu))$ and $M \geq \bar{M}(\nu, \mu)$ from (13) and (15) - (18), with ε_1, ε_2 sufficiently small, it follows the estimate (12). ■

4. Proof of Theorem 1

Lemma 5 implies the existence of a function $u_j \in \tilde{C}^2$ such that

$$l(u_j) \equiv p\frac{\partial u_j}{\partial x_m} + qu_j = \Phi_j \text{ in } G \tag{19}$$

for $j = 1, 2, \ldots$, where the set $\{\Phi_j\}_{j=1}^{\infty} = \mathcal{R}$ is given in Lemma 2. We consider the nonlinear algebraic system

$$B[\sum_{r=1}^{n} \gamma_r u_r,\ \Phi_i] = 0,\ i = 1, 2, \ldots, n. \tag{20}$$

where the real constants $\gamma_1, \ldots, \gamma_n$ are unknown. Let

$$|||\vec{\gamma}||| = \|\sum_{r=1}^{n} \gamma_r u_r\|_{L_{\rho+2}(G)}, \tag{21}$$

where $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$. We have $l(\sum_{r=1}^{n} \gamma_r u_r) = \sum_{r=1}^{n} \gamma_r \Phi_r$ in view of (19). Since $\mathcal{R}$ is a linearly independent set, then $\gamma_1 u_1(x) + \ldots + \gamma_n u_n(x) = 0$ in $\bar{G}$ if and only if $\gamma_1 = \gamma_2 = \ldots = \gamma_n = 0$. One can verify that (21) is a norm in $\mathbf{R}^n$. Then for some positive constants N_1 and N_2 the inequalities $N_1|\vec{\gamma}| \leq |||\vec{\gamma}||| \leq N_2|\vec{\gamma}|$ hold $\forall \vec{\gamma} \in \mathbf{R}^n$, where the notation $|.|$ is given in Lemma 4. These inequalities, (12) and (21) imply

$$B[\sum_{r=1}^{n} \gamma_r u_r, l(\sum_{r=1}^{n} \gamma_r u_r)] \geq 0\ \forall \vec{\gamma} \in \mathbf{R}^n :\ |\vec{\gamma}| \geq K_n \tag{22}$$

with $K_n = N_1^{-1}(\beta_3 \beta_2^{-1})^{1/(\rho+2)}$.

Let consider the operator $A : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$A(\vec{\gamma}) = (B[\sum_{r=1}^{n} \gamma_r u_r,\ \Phi_1], \ldots,\ B[\sum_{r=1}^{n} \gamma_r u_r,\ \Phi_n]).$$

Using the linearity of $B[u,v]$ with respect to v and (22), we get the estimate

$$\langle A(\vec{\gamma}), \vec{\gamma}\rangle \geq 0 \; \forall \vec{\gamma} \in \mathbf{R}^n : |\vec{\gamma}| \geq K_n,$$

where $\langle .,.\rangle$ is the scalar product in $\mathbf{R}^n$ generating the norm $|.|$. Lemma 4 shows that the operator A is continuous. Then Lemma 4.3, Ch. 1, [13] implies the existence of a vector $\vec{\gamma}^n \in \mathbf{R}^n$ such that $|\vec{\gamma}^n| \leq K_n$ and $A(\vec{\gamma}^n) = \vec{0}$. Hence $\vec{\gamma}^n$ is a solution of the system (20).

Setting $U_n = \sum_{r=1}^{n} \gamma_r^n u_r$ we have $U_n \in \tilde{C}^2$ and

$$0 = \sum_{i=1}^{n} B[U_n, \Phi_i]\gamma_i^n = B[U_n, l(U_n)] \; \forall n \in \mathbf{N}.$$

Then (12) gives $\beta_1 \|U_n\|^2_{W_2^1(G)} + \beta_2 \|U_n\|^{\rho+2}_{L_{\rho+2}(G)} \leq \beta_3 \; \forall n \in \mathbf{N}$. Therefore the following estimates

$$\|U_n\|_{W_2^1(G)} \leq \left(\frac{\beta_3}{\beta_1}\right)^{1/2}, \; \|U_n\|_{L_{\rho+2}(G)} \leq \left(\frac{\beta_3}{\beta_2}\right)^{1/(\rho+2)}, \tag{23}$$

and

$$\|U_n\| \leq \left[\frac{\beta_3}{\beta_1} + \left(\frac{\beta_3}{\beta_2}\right)^{2/(\rho+2)}\right]^{1/2} \equiv \beta_4 \tag{24}$$

hold $\forall n \in \mathbf{N}$. The set $\{v \in X : \|v\| \leq \beta_4\}$ is a closed, bounded and convex subset of the reflexive Banach space X (see Lemma 1, ii)). Then it is weakly closed ([5], 25.2) and hence, it is weakly compact in X ([5], 24.8 and 25.6). Therefore the inequality (24) implies the existence of a subsequence $\{U_{n_j}\}_{j=1}^{\infty}$ weakly convergent in X to $u \in X$. Our aim is to show that u is a generalized solution of problem (1), (2).

Let consider the linear continuous functional

$$\mathcal{F}_\varphi(v) = (v, \varphi)_1 \; \forall v \in W_2^1(G),$$

where $(.,.)_1$ is the scalar product in $W_2^1(G)$, generating the norm $\|.\|_{W_2^1(G)}$, and $\varphi \in W_2^1(G)$ is arbitrary and fixed. If $v \in X$, then $v \in W_2^1(G)$ and the inequalities

$$|\mathcal{F}_\varphi(v)| \leq \|v\|_{W_2^1(G)} \|\varphi\|_{W_2^1(G)} \leq \|v\| \|\varphi\|_{W_2^1(G)} \; \forall v \in X$$

imply that $\mathcal{F}_\varphi \in X^*$. Hence $\mathcal{F}_\varphi(U_{n_j}) \underset{j\to\infty}{\longrightarrow} \mathcal{F}_\varphi(u) \; \forall \varphi \in W_2^1(G)$, i.e. $U_{n_j} \underset{j\to\infty}{\longrightarrow} u$ weakly in $W_2^1(G)$ ([1], Riesz representation theorem 1.11). That is equivalent to $U_{n_j} \underset{j\to\infty}{\rightharpoonup} u$ and $\dfrac{\partial U_{n_j}}{\partial x_i} \underset{j\to\infty}{\rightharpoonup} \dfrac{\partial u}{\partial x_i}$ weakly in $L_2(G)$, $i = 1, \ldots, m$ (see [11],Ch. 1, §5).

The first inequality of (23) and Rellich - Kondrashov imbedding theorem (see [1]) show that the subsequence $\{U_{n_j}\}_{j=1}^{\infty}$ can be choosen also convergent to u strongly in $L_2(G)$. Then a subsequence $\{U_{n_{j_r}}\}_{r=1}^{\infty}$ exists, which is convergent to u almost everywhere in G and which is bounded in $L_{\rho+2}(G)$ due to (23). Since for $\eta = (\rho+2)/(\rho+1)$

$$\|v|v|^{\rho}\|_{L_\eta(G)} = \|v\|_{L_{\rho+2}(G)}^{\rho+1} \; \forall v \in L_{\rho+2}(G),$$

then Lemma 3 implies that $U_{n_{j_r}}|U_{n_{j_r}}|^{\rho} \underset{r\to\infty}{\rightharpoonup} u|u|^{\rho}$ weakly in $L_\eta(G)$. Using the inequalities $1 < \eta < 2$, (4) and the second one of (23), we obtain the estimate

$$\|f(x, U_{n_{j_r}}\|_{L_\eta(G)} \le c_1[F_1 + F_2\|U_{n_{j_r}}\|_{L_{\rho+2}(G)}^{(\rho+2)/\sigma}]^{1/2} \le c_2$$

$\forall r \in \mathbf{N}$, where c_1, c_2 are positive constants, nondepending on $\{U_{n_{j_r}}\}_{r=1}^{\infty}$. We have $f(x, U_{n_{j_r}}(x)) \underset{r\to\infty}{\rightharpoonup} f(x, u(x))$ almost everywhere in G, because $f \in$ **CAR**. Due to Lemma 3 this convergence is also weak in $L_\eta(G)$.

Therefore $B[U_{n_{j_r}}, \Phi_i] \underset{r\to\infty}{\rightharpoonup} B[u, \Phi_i]$ $\forall i \in \mathbf{N}$ and then $B[u, \Phi_i] = 0$ $\forall i \in \mathbf{N}$. It is not difficult to obtain the estimate

$$B[w, v] \le (c_3\|w\| + \|f(x, w)\|_{L_2(G)})\|v\| \quad \forall w, v \in Z,$$

where $c_3 = const > 0$ is nondepending on w and v. Since $B[w, v]$ is a linear form with respect to v and the linear span of $\{\Phi_j\}_{j=1}^{\infty}$ is dense in Y, according to Lemma 2, then (6) holds. Theorem 1 is proved. ■

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