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Estimates for Nonlinear Parabolic Equations

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We study smooth non-negative solutions of the equation $u_t = \partial_j a_j(u, \nabla u) + a_0(u, \nabla u)$ in the strip $S_T = R^d \times (0, T)$, $d \geq 1$. A regularizing effect, pointwise estimates and gradient estimates are obtained. Applications to the regularized degenerated nonlinear parabolic equations as well as Barenblatt-type solutions are shown.

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1. Introduction

The equation we will base on in this paper and which we will refer to below as "double powered", is

$$w_t = \partial_j \left(\left| |w|^\lambda \frac{\nabla w}{w} \right|^{p-2} |w|^\lambda \partial_j w \right) + \left(\frac{r}{p} - 1 \right) \frac{w}{|w|^\lambda} \left| |w|^\lambda \frac{\nabla w}{w} \right|^p \tag{1.1}$$

for $\lambda, r \in R$, $p > 1$.

This nonlinear degenerate or singular parabolic equation appears in different physical settings where the solution of the corresponding problem is naturally nonnegative. Its properties, appriory estimates, quantitative effects are matter of study in many works, see the survey of Kalashnikov [12].

Replacing $\frac{1}{p}|q|^p$ by $\bar{a}(q)$ - nonnegative convex function in R^d, and $|s|^\lambda$ by $\varphi(s)$ nonnegative function, we transform (1.1) into two parameters' equation (1.2)

$$w_t = w \partial_j \bar{a}^j \left(\varphi(w) \frac{\nabla w}{w} \right) + k \frac{w}{\varphi(w)} \bar{a} \left(\varphi(w) \frac{\nabla w}{w} \right) - \lambda d \frac{w}{\varphi(w)} \bar{a}_1 \left(\varphi(w) \frac{\nabla w}{w} \right),$$

where $\bar{a}^j(q) = \dfrac{\partial}{\partial q_j}\bar{a}(q)$, $\bar{a}_1(q) = q_j\bar{a}^j(q) - \bar{a}(q)$ and $k, \lambda \in R$. In the p-homogeneous case of (1.1) since $\bar{a}_1 = (p-1)\bar{a}$, the parameters k, λ are connected by the relation $k = r + \lambda(p-1)d$.

To a more general form equation

$$u_t = \partial_j a_j(u, \nabla u) + a_0(u, \nabla u), \tag{1.3}$$

that we will deal with in a part of this paper, we can come in different ways. For instance, if w from (1.2) is a positive sufficiently smooth function then the change $w = e^u$ leads to (1.3) with $a_j(u,q) = \bar{a}^j(\eta(u)q)$, $\eta(u) = \varphi(e^u)$ and corresponding a_0. Note also that since in (1.3) with $a_j(u,q)$ need not to be a gradient of a convex function $a(u,q)$, the equation (1.3) includes the case of "polytropical filtration", see [12].

Let us look at the equation (1.2) with $\varphi(w) = |w|^\lambda$. Denote by $\bar{\theta}(s)$ the Young conjugate of $\bar{a}(q)$ and by $[s]_+ = \max(s, 0)$. Then the nonnegative continuous functions

$$B(\lambda, k) = \left[C^\lambda \left(1 + k\frac{t}{\tau}\right)^{-\frac{\lambda d}{k}} - \lambda(kt+\tau)\bar{\theta}\left(\frac{x - x_0}{kt+\tau}\right)\right]_+^{\frac{1}{\lambda}} \tag{1.4}$$

with the standard changes for $k = 0$ or (and) $\lambda = 0$, are Barenblatt-type solutions of (1.2). The B's with "+" are such in a weak (integral) sence. In (1.4) $C > 0$, τ are constants ($\tau > 0$ for $\lambda < 0$) and since $\left(1 + k\dfrac{t}{\tau}\right)$ must be positive some of the solutions are defined only for $t \in [0, T)$, for some $T = T(\tau, k)$. These solutions illustrate typical for nonlinear parabolic equations effects like finite speed of propagation ($\lambda > 0$), finite blow-up time ($\lambda \geq 0$, $k > 0$), finite time of extinction ($k < 0$).

However in this study we will point out some other, obviously nonisolated from these above-mentioned properties of the solutions, which we will discuss as follows:

(i) By the "regularizing effect" for a solution u of the equation (1.3) we mean an inequality of the form

$$u_t - b(u, \nabla u) = \partial_j a_j(u, \nabla u) + c(u, \nabla u) \geq (\leq) f(t),$$

where c is a part of a_0, i.e. $b + c = a_0$. It includes of course an estimate of u_t (when $c = a_0$) also known as "regularizing effect". Such "definition" may be generalized admitting u and ∇u as arguments of the right hand side function f.

An example of such an inequality gives the smooth Barenblatt-type solutions of (1.2) with powered φ, satisfying

$$W(t,x) \equiv \partial_j \bar{a}_j \left(\frac{\nabla B^\lambda}{\lambda}\right) = -\frac{d}{kt+r} = \frac{W(0,x)}{1-\frac{k}{d}tW(0,x)}.$$

So, for instance, if $W(0,x) \geq -\omega_0 (\leq \omega_0)$, $\omega_0 > 0$ and $k > 0 (< 0)$, then

$$\text{(1.5)} \quad \begin{gathered} W(t,x) \geq -\frac{\omega_0}{1+\frac{k}{d}t\omega_0} > -\frac{d}{kt}, \quad t > 0, \\ \left(W(t,x) \leq \frac{\omega_0}{1-\frac{k}{d}t\omega_0} < \frac{d}{|k|t}, \quad t > 0\right), \end{gathered}$$

and it remains true for the weak B – solutions, but in sence of D'. Is such an estimate valuable for a class of solutions of the Cauchy problem for the equation (1.2), is a question that we will try particularly to answer.

There were steps in this path as in the porous medium case and its generalizations, see for example Aronson, Benilan [1], Crandall, Pierre [3], Dahlberg, Kenig [4], Di Benedetto [5], and Fabricant, Marinov, Rangelov [9], so in the one dimensional double powered equation, see Esteban, Vazquez [6].

The regularizing effect for w_t is known for roughly speaking abstract Cauchy problems $w_t = Aw$ with homogeneous operator A, see Benilan, Crandall [2], Evans [8] and Kalashnikov [12].

(ii) A non-popular but essential in our look consequence of the regularizing effect is the pointwise estimate. For $\nu = \frac{1}{\lambda}B^\lambda$ in the double powered case and $k > 0$, $r > 0$ it becomes

$$\text{(1.6)} \quad s^\alpha \nu(s,x) \leq t^\alpha \nu(t,y) + k\left(t^\delta - s^\delta\right)\bar{\theta}\left(\frac{y-x}{k\left(t^\delta - s^\delta\right)}\right), \quad 0 < s < t < T,$$

where $\alpha = \frac{\lambda d}{k}$, $\delta = \frac{r}{k}$, $\bar{\theta}(\sigma) = \frac{1}{p'}|\sigma|^{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$.

The analogue of this estimate for $\lambda = 0$ corresponds to the classical result of Mozer for the linear case, see Mozer [14], and may be seen as a more detailed Harnak type inequality. The capacities of (1.6) are shown in Fabricant, Marinov, Rangelov [10], where it leads to $L^\infty - L^1$, the interface and the initial trace estimates.

In the one-dimensional double powered Cauchy problem the boundness of the derivative with respect to x for bounded u was proved in Esteban, Vazquez [5] by Berstein method. We will show that this is an immediate product of the regularizing effect.

(iii) Let w is a smooth positive solution of (1.2) in S_T and $z(t,x) = \dfrac{1}{\varphi(w)}\bar{a}_1\left(\varphi(w)\dfrac{\nabla w}{w}\right)$, $z(t) = \sup_{x\in R^d} z(t,x)$. If w is Barenblatt-type solution, then

$$z(t,x) \le \frac{z(0)}{1-\lambda k t z(0)} \tag{1.7}$$

for $t < \min(T, T^*)$, $T^* = \dfrac{1}{\lambda k z(0)}$ if $\lambda k > 0$, and $T^* = \infty$ if $\lambda k \le 0$. The example shows that T^* is the exact blow-up moment.

Since our aim is to demonstrate the properties cited above for more general equations and to discuss the methods and the variants, we do not trace the limiting process for the original equation (1.1). However we will say a few words about the possible appproximation. Some authors use $\left(|q|^2+\varepsilon\right)^{\frac{p}{2}}$ as a regularization of $|q|^p$ and base the proof on Theorem 4.1, Ch. 6 in Ladyzenskaja, Solonnikov, Ural'tzeva [13]. But in the case when the coefficients depend on w these theorems suggest hard structural conditions, non-satisfied by the equation even after a change of the variables. So we tie ourselves down to Theorem 8.1, Ch. 5 of [13] and withdraw as from the degeneration, so from the singularity. The approximation proposed in Section 6 for the equation (1.1) reserves the needed estimates.

The plan of the exposition is: In Section 2 we derive the equations satisfied by quantities like W and z and through algebraic inequalities come to differential inequalities for them. In Section 3 we cite theorems from [13] which give existence and sufficient smoothness of the solutions and prove some comparision results for differential inequalities.

In Sections 4 and 5 we apply all this to the more special case of equation (1.2) and get the basic theorems and corollaries.

2. Preliminaries, differential inequalities

The purpose of this section is to study some a priori differential inequalities for appropriate expressions of the first and second derivatives with respect to x and first derivative with respect to t for smooth solutions of the equation

$$u_t = \partial_j a_j(u,\nabla u) + a_0(u,\nabla u) \tag{2.1}$$

in the strip $S_T = R^d \times [0,T)$, $d \ge 1$. Here we shall use the notations: $\partial_j = \dfrac{\partial}{\partial x_j}$, $u_j = \partial_j u$ and $f^j(u,q) = \dfrac{\partial}{\partial q_j} f(u,q)$, $j = 1,\ldots,d$, $f^0(u,q) = \dfrac{\partial}{\partial u} f(u,q)$ as for functions or vector functions f so for matrices f. Suppose that:

Functions $a_j(u,q), a_0(u,q)$ have continuous derivatives up to the second order with respect to u, q.

(2.2) *Function $u(t,x)$ is a solution of the equation (2.1) such that: $u(t,x)$ has 4 continuous derivatives with respect to x and $u_t(t,x)$ has 2 continuous derivatives with respect to x.*

Denote by $Lf \equiv f_t - \partial_j a_j^k \partial_k f - (a_j^0 + a_0^j)\partial_j f$ the parabolic differential operator and $e_j = a_j^l u_l b^0 - a_j^0(u_l b^l - b)$, $e_0 = (a_0^l u_l - a_0)b^0 - a_0^0(u_l b^l - b)$, where b is an arbitrary smooth function of u and ∇u.

The next lemma holds:

Lemma 2.1. *Let the coefficients a_j, a_0 and the solution $u(t,x)$ of the equation (2.1) are as in (2.2). Then the next equalities take places:*

(i) Let $W = \partial_j a_j + c$, $c = a_0 - b$, then

$$LW = (\partial_k a_j)(\partial_j b^k) + W\partial_j a_j^0 + \partial_j e_j + e_0 + c^0 W + c^0 b. \tag{2.3}$$

(ii) Function u_t satisfies

$$L(u_t) = u_t(\partial_j a_j^0 + a^0). \tag{2.4}$$

(iii) For arbitrary smooth function $b = b(u, \nabla u)$, we have:

$$Lb = -(\partial_k a_j)(\partial_j b^k) + b\partial_j a_j^0 + b^0 \partial_j a_j - \partial_j e_j - e_0 + a_0^0 b. \tag{2.5}$$

Proof. (i) We have $u_t = W + b$ and the differentiation with respect to t gives

$$W_t = \partial_j \left[a_j^k \partial_k (W+b) + a_j^0 (W+b)\right] + c^j \partial_j (W+b) + c^0 (W+b)$$

$$= \partial_j a_j^k \partial_k W + \partial_j(a_j^k \partial_k b + a_j^0 b) + (a_j^0 + a_0^j)\partial_j W + W\partial_j a_j^0 - b^j \partial_j W + c^j \partial_j b + c^0 W + cb^0.$$

The first and third terms in the line above are as in the expression of LW, and the forth and seventh are from the right hand side of (2.3). The sum of the second and the fifth terms can be written as

$$\partial_j(a_j^k b^l u_{lk} + a_j^k b^0 u_k + a_j^0 b) - b^j \partial_j W$$

$$= \partial_j \left[b^l \partial_l a_j + a_j^k u_k b^0 - a_j^0(u_k b^k - b)\right] - b^l \partial_l(\partial_j a_j + c)$$

$$= (\partial_l a_j)(\partial_j b^l) - b^l \partial_l c + \partial_j e_j.$$

It is clear that the rest two terms, the sixth and the eight, give

$$c^j\partial_j b + c^0 b = b^j\partial_j c + c^j u_j b^0 - c^0(u_j b^j - b) = b^j\partial_j c + e_0 + cb^0.$$

Summarizing these expresions, we get (2.3).

(ii) We choose $b = 0$ in (i). Then $c = a_0$, $e_j = e_0 = 0$, $W = u_t$ and the right hand side of (2.3) take the form $u_t\partial_j a_j^0 + a_0^0 u_t$, so we get (2.4).

(iii) Since $W = u_t - b$ and the operator L is linear, from (i) with $c = a_0 - b$ and from (ii) we get

$$Lb = Lu_t - LW$$

$$= (W + b)(\partial_j a_j^0 + a_0^0) - (\partial_k a_j)(\partial_j b^k) - W\partial_j a_j^0 - \partial_j e_j - e_0 - (a_0^0 - b^0)W - cb^0$$

$$= b\partial_j a_j^0 + a_0^0 - (\partial_k a_j)(\partial_j b^k) - \partial_j e_j - e_0 + b^0(W - c),$$

but $b^0(W - c)$ is equal to $b^0\partial_j a_j$ so (2.5) is true. ■

We are going to use a comparison principle for solutions of differential inequalities. In order to be in a position to do this we need a suitable form with respect to the functions in question in the right hand sides of the equalities (2.3) – (2.5). The next lemma allows doing this.

Lemma 2.2. *Let for a_j and b from Lemma 2.1 : $\tilde{A} = \frac{1}{2}\left\{a_j^k + a_k^j\right\}$ is a strictly positive matrix, $B = \{b^{jk}\}$ is a strictly positive (negative) matrix, $\tilde{A}^{-1}$, B^{-1} are theirs inverse matrices, $D = \{d_{jk}\}$ is an arbitrary matrix, d_{0j}, $j = 1, \ldots, d$ are vectors, and their arguments are u and ∇u. Then the next inequality holds:*

$$(2.6)\qquad (\partial_k a_j)(\partial_j b^k) + d_{jk}u_{kj} + d_{0j}u_j \geq (\leq) \frac{\left[\partial_j a_j + \frac{1}{2}TrB^{-1}(S' + D)\right]^2}{TrB^{-1}\tilde{A}}$$

$$-\frac{1}{4}Tr\tilde{A}^{-1}(D - S)B^{-1}(D - S)^* - TrB^{-1}Q,$$

where S, S', Q are matrices with elements respectively

$$s_{jk} = a_j^l u_l b^{0k} - a_j^0 u_l b^{lk}, \quad s'_{jk} = a_l^j u_l b^{0k} - a_j^0 u_l b^{lk}, \quad q_{jk} = d_{lj}u_l b^{0k} - d_{j0}u_l b^{lk}.$$

The sign ($\leq$) corresponds to the case of strictly negative B.

P r o o f. We will prove the inequality (2.6) for B strictly positive, if B is strictly negative then the proof is the same.

Denote

$$\alpha_j = a_j^0 u_k, \quad \beta_{jk} = b^{0j}u_k, \quad \delta_{jk} = d_{j0}u_k, \quad A = \left\{a_j^k\right\},$$

then

$$S = A\beta^* - \alpha B, \quad S' = A^*\beta^* - \alpha B, \quad Q = D^*\beta^* - \delta B.$$

With a suitable change of the indices we have:

$$(\partial_l a_j)(\partial_j b^l) + d_{jl}u_{lj} + d_{j0}u_j$$

$$= a_j^k u_{kl} b^{lm} u_{mj} + (a_j^k u_j b^{0l} + a_k^0 u_j b^{jl} + d_{lk})u_{kl} + (a_j^0 u_j b^{l0} u_l) + d_{j0}u_j$$

$$= M_1 + M_2 + M_3 + M_4,$$

where

$$M_1 = Tr\, A\partial^2 u B \partial^2 u = Tr A^* \partial^2 u B \partial^2 u = Tr\tilde{A}\partial^2 u B\partial^2 u$$

$$M_2 = Tr(A^*\beta^* + \alpha B + D)\partial^2 u$$

$$= Tr(2\tilde{A}\beta^* + D - S)\partial^2 u = Tr 2\tilde{A} G B \partial^2 u$$

with

$$G = \frac{1}{2}\tilde{A}^{-1}(2\tilde{A}\beta^* + D - S)B^{-1} = \beta^* B^{-1} + \frac{1}{2}\tilde{A}^{-1}(D - S)B^{-1}.$$

So,

(2.7) $$M_1 + M_2 = Tr\tilde{A}(\partial^2 u + G)B(\partial^2 u + G)^* - Tr\tilde{A}GBG^*.$$

Let us compute $Tr\tilde{A}GBG^*$:

$$Tr\tilde{A}GBG^* = \frac{1}{2}Tr(2\tilde{A}\beta^* + D - S)B^{-1}\left[\beta + \frac{1}{2}(D - S)^*\tilde{A}^{-1}\right]$$

$$= \frac{1}{4}Tr\tilde{A}^{-1}(D - S)B^{-1}(D - S)^* + Tr\beta^* B^{-1} D^* + Tr\beta^* B^{-1}(\beta\tilde{A} - S^*)$$

$$= M_{21} + M_{22} + M_{23}.$$

Since

$$S^* = \beta A^* - B\alpha^*, \quad \beta\tilde{A} - S^* = \frac{1}{2}\beta(A - A^*) + B\alpha^*$$

and $A - A^*$ is an antisymetric matrix, then

$$M_{23} = \frac{1}{2}Tr\beta^* B^{-1}\beta(A - A^*) + Tr\beta^*\alpha^* = (a_j^0 u_j)(b^{0l}u_l) = M_3.$$

Since $D^*\beta^* = Q + \delta B$, then

$$M_{22} = Tr\, B^{-1}Q + Tr B^{-1}\delta B$$

$$= Tr B^{-1} Q + M_4.$$

So we obtain

$$M_3 + M_4 = M_{23} + M_{22} - Tr\ B^{-1}Q$$

$$= Tr\tilde{A}GBG^* - \frac{1}{4}Tr\tilde{A}^{-1}(D-S)B^{-1}(D-S)^* - TrB^{-1}Q. \tag{2.8}$$

From (2.7) and (2.8) we get the expression of the left hand side of (2.6) in the form

$$Tr\tilde{A}(\partial^2 u + G)B(\partial^2 u + G)^* - \frac{1}{4}Tr\tilde{A}^{-1}(D-S)B^{-1}(D-S)^* - Tr\ B^{-1}Q.$$

To this expression we apply the algebraic inequality

$$Tr(\tilde{A}QBQ^*)Tr(\tilde{A}^{-1}CB^{-1}C^*) \geq (TrCQ^*)^2 \tag{2.9}$$

for strictly positive matrices $\tilde{A}$, B and arbitrary C and Q. Choosing C as $\tilde{A}$ and Q as $\partial^2 u + G$ we get

$$(\partial_k a_j)(\partial_j b^k) + d_{jk}u_{kj} + d_{0j}u_j$$

$$\geq \frac{\left[Tr\tilde{A}(\partial^2 u + G)\right]^2}{TrB^{-1}\tilde{A}} - \frac{1}{4}Tr\tilde{A}^{-1}(D-S)B^{-1}(D-S)^* - TrB^{-1}Q.$$

But since

$$Tr\tilde{A}(\partial^2 u + G) = TrA\partial^2 u + TrB^{-1}\left[\tilde{A}\beta^* + \frac{1}{2}(D-S)\right]$$

$$= \partial_j a_j - Tr\alpha + TrB^{-1}\frac{1}{2}(S' + D + 2\alpha B)$$

$$= \partial_j a_j + \frac{1}{2}TrB^{-1}(S' + D)$$

we obtain the inequality (2.6). ■

Remark 2.1. Note that the right hand side of the inequality (2.6) is the maximum in σ for strictly positive B (and minimum for strictly negative B) of the quadratic form

$$K(\sigma) = \sigma\partial_j a_j + \frac{\sigma}{2}Tr\ B^{-1}(S' + S)$$

$$-\frac{1}{4}Tr\tilde{A}^{-1}(D - S - \sigma\tilde{A})B^{-1}(D - S - \sigma\tilde{A})^* - TrB^{-1}Q. \tag{2.10}$$

We will use the above form $K(\sigma)$ further.

The application of Lemma 2.2 to Lemma 2.1 gives some useful inequalities, which are summarized in the next lemma.

Lemma 2.3. *If the function $b(u, \nabla u)$ from Lemma 2.1 is strictly convex (concave) with respect to ∇u, then:*

(i) There exist functions f, g, h of u and ∇u such that

$$LW \geq (\leq) fW^2 + gW + h \tag{2.11}$$

(the sign $(\leq)$ is for a concave function $b(u, \nabla u)$).

(ii) There exists a function $l(u, \nabla u)$ such that

$$Lb \leq (\geq) l(u, \nabla u) \tag{2.12}$$

(the sign $(\geq)$ is for a concave function $b(u, \nabla u)$).

Proof. (i) We compute the coefficients at u_{jk} in (2.3) and then choose them as d_{jk}. So, $D = WA^0 + E$, where $E = \{e_j^k\}$. Let $b_1 = u_l b^l - b$ and $A_1 = u_l A^l + A$. Since

$$e_j^k = \left[a_j^l u_l b^0 - a_j^0(u_l b^l - b)\right]^k = (a_j^{lk} u_l + a_j^k) b^0 - a_j^{0k}(u_l b^l - b) + a_j^l u_l b^{0k} - a_j^0 u_l b^{lk}$$

$$= b^0 a_{1j}^k - b_1 a_j^{0k} + s_{jk},$$

then $E = H + S$, with $H = b^0 A_1 - b_1 A^0$, $D - S = WA^0 + H$. Let $d_{j0} = Wa_j^0 + e_j^0$, $\varepsilon_{jk} = e_j^0 u_k$, then $Q = W(A^0\beta^* - \alpha^0\beta) + (E^*\beta^* - \varepsilon B)$. Since $\partial_j a_j = W - c$, from (2.3) and (2.6) we obtain the inequality

$$LW \geq \frac{\left[W\left(1 + \frac{1}{2}Tr\ B^{-1}A^0\right) + \frac{1}{2}Tr\ B^{-1}(H + S + S')\right]^2}{Tr\ B^{-1}\tilde{A}}$$

$$-\frac{1}{4}Tr\tilde{A}^{-1}(WA^0 + H)B^{-1}(WA^0 + H)^* - WTr\ B^{-1}(A^0\beta^* - \alpha^0 B)$$

$$-TrB^{-1}(E^*\beta^* - \varepsilon B) + c^0 W + cb^0 + e_0$$

$$= fW^2 + gW + h.$$

For example, the principal coefficient f has the form

$$f = \frac{\left(1 + \frac{1}{2}Tr\ B^{-1}A^0\right)^2}{Tr\ B^{-1}\tilde{A}} - \frac{1}{4}Tr\ \tilde{A}^{-1}A^0 B^{-1}A^{0*}.$$

(ii) Our aim now is to get rid of the second derivatives of u in the right hand side of (2.5). Again we have to choose as d_{jk} the coefficients of u_{jk} in (2.5) and apply Lemma 2.2 in a view of Remark 2.1, i.e. to apply the inequality

$$(\partial_j a_k)(\partial_j b_k) + d_{jk}u_{kj} + d_{0j}u_j \geq K(0) = -\frac{1}{4}Tr\tilde{A}^{-1}(D - S)B^{-1}(D - S)^* - B^{-1}Q^*.$$

It is clear from (2.5) that $d_{jk} = -ba_j^{k0} - b^0 a_j^k + e_j^k$. So $D = -bA^0 - b^0\tilde{A} + E$ and $E = H + S$. Choosing $d_{j0} = -ba_j^{00} - b^0 a_j^0 + e_j^0$, we get

$$(2.13)\quad Lb \le a_0^0 b - e_0 + \frac{1}{4} Tr \tilde{A}^{-1}(D-S)B^{-1}(D-S)^* + Tr\ B^{-1}Q = l(u, \nabla u)$$

with the corresponding for this case Q.

In the case of concave function $b(u, \nabla u)$ the proof of (2.11) with a sign ($\le$) and (2.12) with a sign ($\ge$) is similar. ■

R e m a r k 2.2. A sufficient condition for the existence of a constant f_0 such that $f \ge f_0$ in (i) is:

$$(2.14)\qquad \frac{1}{d}B \ge f_0\tilde{A} - \tilde{A}^0 + \frac{1}{4(f_0-\mu)}(A^{0*} - \mu\tilde{A})\tilde{A}^{-1}(A^0 - \mu\tilde{A})$$

for some $\mu = \mu(u,q) < f_0$. Indeed, for the function f we have

$$\begin{aligned} f &= \frac{(1 + \frac{1}{2}Tr\ B^{-1}A^0)^2}{Tr\ B^{-1}\tilde{A}} - \frac{1}{4}Tr\tilde{A}^{-1}A^0B^{-1}A^{0*} \\ &= \sup_\sigma \left[\sigma - \frac{1}{4}Tr\ B^{-1}(A^{0*} - \sigma\tilde{A})\tilde{A}^{-1}(A^0 - \sigma\tilde{A})\right] \\ &= \sup_\sigma Tr\ B^{-1}\left[\frac{\sigma - f_0}{d}B - \frac{1}{4}(A^{0*} - \sigma\tilde{A})\tilde{A}^{-1}(A^0 - \sigma\tilde{A})\right] + f_0, \end{aligned}$$

so if there exists $\sigma > f_0$ such that

$$(2.15)\qquad \frac{\sigma - f_0}{d}B > \frac{1}{4}(A^{0*} - \sigma\tilde{A})\tilde{A}^{-1}(A^0 - \sigma\tilde{A}),$$

then $f \ge f_0$. Let $\sigma = 2f_0 - \mu$, then

$$\frac{1}{d}B > \frac{1}{4(f_0-\mu)}\left[A^{0*} - (2f_0-\mu)\tilde{A}\right]\tilde{A}^{-1}\left[A^0 - (2f_0-\mu)\tilde{A}\right]$$

$$= (f_0-\mu)\tilde{A} - \frac{1}{2}(A^{0*} - \mu\tilde{A}) - \frac{1}{2}(A^0 - \mu\tilde{A}) + \frac{1}{4(f_0-\mu)}(A^{0*} - \mu\tilde{A})\tilde{A}^{-1}(A^0 - \mu\tilde{A})$$

and we get the sufficient condition (2.14).

It is clear from the proof that f increases to ∞ as a function of positive matrix B. The corresponding result for negative matrix B is obvious.

In Section 4 we shall use Remark 2.2 in studying regularizing effect estimates.

3. Cauchy problem, comparison results

Of course the differential inequalities in Lemma 2.3 are only the first step toward the comparison theorem. We need much more information about the coefficients of the operator L (some of them depend on the second derivatives of u), about the bounds of the functions in the right hand side of the inequalities for the subsolutions W or supersolution b.

Consider the Cauchy problem for equation (2.1)

$$\left| \begin{array}{l} u_t = \partial_j a_j(u, \nabla u) + a_0(u, \nabla u), \quad (t,x) \in S_T, \\ u(0,x) = u_0(x), \quad x \in R^d \end{array} \right. . \tag{3.1}$$

The existence of a sufficiently smooth solution $u(t,x)$ of (3.1) follows from the result of [13] under some structural restrictions on a_j, a_0 and some smoothness' conditions on $u_0(x)$.

Let a_j, a_0 satisfy:

(i) There exists $\Lambda > 1$ *such that* $\Lambda^{-1}|\xi|^2 \le a_j^k \xi_j \xi_k \le \Lambda |\xi|^2$.

(ii) There exist positive constants b_1, b_2 *such that* $u a_0(u,0) \le b_1 u^2 + b_2$.

(3.2) *(iii) For* $|u| \le M$ *and arbitrary* q

$$\sum_1^d \left(|a_j| + |a_j^0|\right)(1 + |q|) + |a_0| \le m(1 + |q|)^2 \quad \textit{for some positive } m.$$

Let $u_0(x)$ *satisfy:*

(i) $u_0(x) \in H^{2+\beta}(R^d)$,

(3.3) *Holder continuous second derivatives with power* β.

(ii) $\max_{R^d} |u_0(x)| < \infty, \max_{R^d} |\partial_j u_0(x)| < \infty$ *and* $\max_{R^d} |\partial_{ij} u_0(x)| < \infty$.

Lemma 3.1. *Under the conditions (3.2), (3.3) the Cauchy problem (3.1) has unique solution* $u(t,x) \in H^{2+\beta}$ *with bounded derivatives up to the second order.*

This lemma follows directly from [13], Theorem 8.1, Ch. 5. Moreover, due to Theorem 12.1, Ch. 4 from [13], if $u_0(x) \in H^{n+\beta}(R^d)$, then $u(t,x) \in H^{n+\beta}(R^d)$ for $n \ge 2$.

Denote by

$$LW = W - c_{jk}(t,x)W_{jk} - c_j(t,x)W_j$$

a differential operator in the strip $S_T = (0,T) \times R^d$ with coefficients c_{jk}, c_j continuous, bounded functions in S_T, $c_{jk}\xi_j\xi_k \geq 0$, $\xi \in R^d$.

Lemma 3.2. *Let $W(t,x)$ is a smooth bounded solution of the differential inequality*

$$LW \geq fW^2 + gW + h \tag{3.4}$$

in the strip S_T, where $\lim_{t\to 0} W(t,x) = W_0(x) \geq -\omega_0$ in R^d, $\omega_0 = const \geq 0$, $f \geq f_0 > 0$ and there is a constant μ such that $g + \sqrt{(g^2 - 4fh)_+} \leq 2f\mu$, then

$$W \geq -\mu - \frac{(\omega_0 - \mu)_+}{1 + f_0 t(\omega_0 - \mu)_+} \quad in \quad S_T. \tag{3.5}$$

Proof. Since

$$L(W+\mu) = LW \geq f(W+\mu)^2 + (g - 2f\mu)(W+\mu) + f\left(\mu^2 - \frac{g}{f}\mu + \frac{h}{f}\right)$$

$$\geq f_0(W+\mu)^2 + g_1(W+\mu) + h_1$$

with $g_1 \leq 0$, $h_1 \geq 0$ and

$$\lim_{t\to+0}[W(t,x) + \mu] \geq -\omega_0 + \mu \geq -(\omega_0 - \mu)_+,$$

it is sufficient to assume $f = f_0$, $g \leq 0$, $h \geq 0$ with ω_0 replaced by $(\omega_0 - \mu)_+$ in (3.4). We shall prove the inequality (3.5) in every cylinder $S_{T,R} = (0,T)\times\{|x| \leq R\}$ and then under limiting process we will obtain the corresponding inequality for S_T.

For fixed R we define in $S_{T,R}$ the function

$$p_R(t,x) = -\frac{\omega_0}{1 + f_0 t\omega_0} - \frac{C_0}{R^2}\left(|x|^2 + k\right)e^t,$$

where $k > \max_{S_T} \sum_j \left(2c_{jj} + c_j^2\right)$ is a constant and $C_0 \geq 0$, $-C_0 \leq W$ in S_T. Then applying operator L to p_R we get

$$Lp_R - f_0 p_R^2 - g p_R - h$$

$$= \frac{f_0\omega_0^2}{(1 + f_0 t\omega_0)^2} - \frac{C_0}{R^2}\left(|x|^2 + k\right)e^t + \frac{2C_0}{R^2}c_{jj}e^t + 2c_j x_j \frac{C_0}{R^2}e^t$$

$$-\frac{f_0\omega_0^2}{(1+f_0t\omega_0)^2} - \frac{2f_0\omega_0}{1+f_0t\omega_0}\frac{C_0}{R^2}\left(|x|^2+k\right)e^t - \frac{f_0C_0^2}{R^4}\left(|x|^2+k\right)^2 e^{2t}$$
$$+g\left(\frac{\omega_0}{1+f_0t\omega_0} + \frac{C_0}{R^2}\left(|x|^2+k\right)e^t\right) - h$$
$$\le -\frac{C_0}{R^2}e^t\left(|x|^2+k-2c_{jj}-2c_jx_j\right) \le 0.$$

Since

$$Lp_R \le f_0p_R^2 + gp_R + h, \quad LW \ge f_0W^2 + gW + h$$

and

$$p_R|_{t=0} \le -\omega_0 \le W|_{t=0}, \quad p_R|_{|x|=R} \le -C_0 \le W|_{|x|=R},$$

we conclude under the comparison principle for boundary-value problems (see Friedman [11]), that $W \ge p_R$ in $S_{T,R}$. Passing $R \to \infty$ the inequality (3.5) is established. ∎

Lemma 3.3. *Let $b(t,x)$ is a smooth, bounded solution of differential inequality*

$$Lb \le l(b) \quad \textit{in} \quad S_T \tag{3.6}$$

and $\lim_{t\to 0} b(t,x) \le b_0 = const.$ *Let*

$$\begin{aligned} &l(b) \text{ is a smooth function defined on finite interval } I = (\alpha,\beta],\\ &b,\ b_0 \in I,\ l(s) > 0 \quad \text{on } I. \end{aligned} \tag{3.7}$$

Then,

$$t > \int_{b_0}^{b(t,x)} \frac{ds}{l(s)} \quad \textit{in} \quad S_T. \tag{3.8}$$

Proof. Under the condition (3.7) there exist a constant $M = M_I > 0$ such that

$$\sup_I l'(s) \le M \text{ and } \int_\sigma^\beta \frac{ds}{l(s)} < \infty \text{ for } \sigma \in I.$$

Let $\varphi(\sigma) = \exp\left(-M\int_\sigma^\beta \frac{ds}{l(s)}\right)$ for $\sigma \in I$. Then $\varphi' = \frac{M}{l}\varphi > 0$ and $\varphi'' = \frac{M(M-l')}{l^2}\varphi \ge 0$, so φ is a convex function. If b is the solution in the question and

$$F(t,x) = \varphi(b(t,x)), \; LF = \varphi'(b)Lb - \varphi''(b)c_{jk}b_jb_k \le \varphi'(b)l(b) = MF.$$

Also $F|_{t=0} = \varphi(b)|_{t=0} \leq \varphi(b_0)$ since φ is an increasing function and $F \leq 1$ everywhere since $b \leq \beta$ and $l(b) > 0$. As in the previous Lemma 3.2 we define in $S_{T,R}$ the function

$$q_R(t,x) = \varphi(b_0)e^{Mt} + \frac{|x|^2+k}{R^2}e^{(M+1)t}, \quad \text{where } k > \sum_1^d (2c_{jj} + c_j^2).$$

Then

$$Lq_R = Mq_R + \frac{e^{(M+1)t}}{R^2}\left(|x|^2 + k - 2c_{jj} - 2c_jx_j\right) > Mq_R.$$

Since $q_R|_{t=0} = \varphi(b_0) + \frac{|x|^2+k}{R^2} > \varphi(b_0) \geq F|_{t=0}$ and

$$q_R|_{|x|=R} \geq \varphi(b_0)e^{Mt} + \left(1 + \frac{k}{R^2}\right)e^{(M+1)t} > 1 > F,$$

we conclude as above (see [11]) that

$$F \leq q_R \text{ in } S_{T,R}\,,$$

i.e.

$$\exp\left(-M\int_{b(t,x)}^{\beta}\frac{ds}{l(s)}\right) \leq \exp M\left(t - \int_{b_0}^{\beta}\frac{ds}{l(s)}\right) + \frac{|x|^2+k}{R^2}e^{(M+1)t}.$$

Passing $R \to \infty$ we obtain (3.8) ■

Lemma 3.4. *Let $b(t,x)$ be a smooth, bounded solution of the differential inequality*

(3.9) $$Lb \leq l(b) < 0 \quad in \quad S_T$$

and $\lim_{t\to 0} b(t,x) \leq b_0 = const.$ *Let*

$l(b)$ is a smooth function defined on finite interval $I = (\alpha, b_0]$,

(3.10) $$l(s) < 0 \quad on \quad I.$$

Then,

(3.11) $$t + \int_b^{b_0}\frac{ds}{l(s)} \leq 0 \quad in \quad S_T.$$

Proof. From Lemma 3.3, since $Lb < \varepsilon$, $\varepsilon > 0$, then $b(t,x) - b_0 < \varepsilon t$ in S_T. Let $\varphi(\sigma) = \exp\left(M\int_\sigma^{b_0}\frac{ds}{l(s)}\right)$, for $\sigma \in I$ and $M > 0$, $\inf_I l'(s) \geq -M$. Then $\varphi' = -\frac{M}{l}\varphi > 0$ and $\varphi'' = \frac{M(M+l')}{l^2}\varphi \geq 0$. Denote by $F(t,x) = \varphi(b(t,x))$. Then $F \leq 1$ everywhere and

$$LF = \varphi'(b)Lb - \varphi''(b)c_{jk}b_jb_k \leq \varphi'(b)l(b) = -MF.$$

Define in $S_{T,R}$ the function $q_R(t,x) = \varphi(b_0)e^{Mt} + \frac{|x|^2+k}{R^2}e^{(M+1)t}$, where $kM > \sum_1^d\left(2c_{jj} + \frac{1}{M}c_j^2\right)$. We get

$$Lq_R = -Mq_R + \frac{e^{(M+1)t}}{R^2}\left(M|x|^2 + kM - 2c_{jj} - 2c_jx_j\right) > -Mq_R.$$

Since $q_R|_{t=0} = 1 + \frac{|x|^2+k}{R^2} > 1 \geq F|_{t=0}$ and $q_R|_{|x|=R} \geq e^{-Mt} + 1 + \frac{k}{R^2} > 1 \geq F|_{|x|=R}$ we conclude (see [11]) that $F \leq q_R$ in $S_{T,R}$. Passing $R \to \infty$ we obtain (3.11). ∎

4. Regularizing effect and pointwise estimate

Let us consider a Cauchy problem for a particular case of (2.1):

$$(4.1)\qquad \left|\begin{array}{l} u_t = \partial_j\bar{a}_j(\eta(u)\nabla u) + \frac{1}{\eta(u)}\bar{b}(\eta(u)\nabla u), \\ u(0,x) = u_0, \end{array}\right.$$

where $\eta(u) > 0$ and $\bar{b}(q)$ is a smooth convex function. Suppose that the conditions (3.2), (3.3) hold, so under Lemma 3.1 the solution u have bounded second order derivatives. Denote $\frac{\eta'}{\eta} = \lambda(u)$.

We follow the notations of Section 2 for the equation (2.1). Here $a_j(u,q) = \bar{a}_j(\eta(u)q)$ and $a_0(u,q) = b(u,q) = \frac{1}{\eta(u)}\bar{b}(\eta(u)q)$, i.e. $c = 0$, and $A^0 = \lambda A_1$ $H = b^0A_1 - b_1A^0 = 0$, $S = 0$, $e_j = 0$, $Tr\ B^{-1}S' = 0$, $Tr\ B^{-1}(A^0\beta^* - \alpha^0B) = -\lambda'\eta(u)\bar{a}_j^lu_ju_l$. The inequality (2.11) then has the form

$$(4.2)\qquad \begin{aligned} LW \geq & \left[\frac{\left(1+\frac{\lambda}{2}Tr\ B^{-1}A_1\right)^2}{Tr\ B^{-1}\tilde{A}} - \frac{\lambda^2}{4}Tr\ B^{-1}A_1\tilde{A}^{-1}A_1^*\right]W^2 \\ & + \lambda'\eta(u)\bar{a}_i^lu_iu_lW = fW^2 + gW. \end{aligned}$$

The next regularizing effect for the solutions of the Cauchy problem (4.1) takes place.

Theorem 4.1. *Let $\lambda' \leq 0$ and let there exist a constant $f_0 > 0$ and a function μ such that $\mu\lambda < f_0$ and*

$$\frac{1}{d}B \geq f_0\tilde{A} - \lambda\tilde{A}_1 + \frac{\lambda^2}{4(f_0 - \lambda\mu)}(\mu\tilde{A} - A_1^*)\tilde{A}^{-1}(\mu\tilde{A} - A_1). \tag{4.3}$$

Let $\partial_j\bar{a}_j(\eta(u_0)\nabla u_0) \geq -\omega_0$ for $\omega_0 = const \geq 0$. Then

$$\partial_j\bar{a}_j(\eta(u)\nabla u) \geq -\frac{\omega_0}{1 + f_0\omega_0 t}. \tag{4.4}$$

Proof. Under the condition $\lambda' \leq 0$ it follows that $g = \lambda'\eta(u)\bar{a}_j^l u_j u_l \leq 0$. The condition (4.3) and Remark 2.2 give $f \geq f_0 > 0$. So we apply Lemma 3.2 with $g_0 = h_0 = 0$ and obtain the regularizing effect (4.4). ∎

Remark 4.1. Note that formally if $\bar{a}_j = \dfrac{\partial}{\partial q_j}\dfrac{1}{p}|q|^p$ and $\eta(u) = e^{\lambda u}$, $\lambda = const$, then $A_1 = (p-1)A$ and for $\mu = p - 1$ the condition (4.3) becomes

$$\frac{1}{d}B \geq [f_0 - \lambda(p-1)]A. \tag{4.5}$$

In (1.1) $B = rA$ so $f_0 = \dfrac{k}{d}$ and regularizing effect estimate (1.5) holds if $r > 0$, $k > 0$.

Remark 4.2. The condition on λ, $\lambda' \leq 0$ is not optimal. It is verified if $\eta_\delta(u) = \left(\sum c_k e^{-\lambda_k \delta u}\right)^{-\frac{1}{\delta}}$, $c_k > 0$, $\lambda_k \in R^1$ with $\delta > 0$ but not with $\delta < 0$. This condion can be weakened, as in Fabricant, Marinov, Rangelov [10], where the differential inequality for $W_1 = \dfrac{1}{H(u)}W$ with an appropriate $H(u) > 0$ gives more chances for the η functions.

Using Theorem 4.1 we shall prove the next pointwise estimate for the solutions of (4.1).

Theorem 4.2. *Let $u(t,x)$ be a solution of (4.1), where $\bar{b}(q)$ is a smooth convex function, $\eta(u) > 0$, $\eta'\eta = \lambda(u)$ and $\lambda'(u) \leq 0$. Suppose that the* *regularizing effect for (4.2) takes place, so there exists a function* $f(t)$, $f(t) \geq 0$ *such that*

$$\partial_j \bar{a}_j(\eta(u)\nabla u) \geq -f(t). \tag{4.6}$$

Then for all $0 < t_0 < t < T$, $x_0, x \in R^d$ *and* $\sigma \in (\min u, \max u)$,

$$\int_\sigma^{u(t_0,x_0)} \eta(s)ds \leq e^{\lambda(\sigma)g(t)} \int_\sigma^{u(t,x)} \eta(s)ds$$

$$+ \frac{e^{\lambda(\sigma)g(t)} - 1}{\lambda(\sigma)}\eta(\sigma) + H_{\lambda(\sigma)}(t_0, t, x_0, x) \tag{4.7}$$

with $g(t) = \int_{t_0}^t f(s)ds$ *and*

$$H_{\lambda(\sigma)}(t_0, t, x_0, x) = \inf_{z(s)} \left\{ \int_{t_0}^t e^{\lambda(\sigma)g(s)} \bar{\beta}(\dot{z}(s))ds : z(t_0) = x_0, z(t) = x \right\},$$

with $\bar{\beta}$ *the Young conjugate to* $\bar{b}$.

If $\lambda(m) = \lambda(\min u) < \infty$, *the estimate (4.7) may be replaced by*

$$\int_{u(t_0,x_0)}^{u(t,x)+g(t)} \eta(s)ds + H_{\lambda(m)}(t_0, t, x_0, x) \geq 0. \tag{4.8}$$

Proof. If we apply the estimate (4.6) to the equation (4.1) we obtain

$$u_t \geq -f(t) + \frac{\bar{b}(\eta(u)\nabla u)}{\eta(u)}. \tag{4.9}$$

Let us define the function

$$E(\tau) = e^{\lambda(\sigma)g(\tau)} \int_\sigma^{u(\tau,z(\tau))} \eta(s)ds + \frac{e^{\lambda(\sigma)g(\tau)}}{\lambda(\sigma)}\eta(\sigma) + \int_{t_0}^\tau e^{\lambda(\sigma)g(s)}\bar{\beta}(\dot{z}(s))ds,$$

where $z(s)$ - vector function and $z(t_0) = x_0$, $z(t) = x$. Then since $\dot{g}(\tau) = f(\tau)$, we obtain

$$\dot{E}(\tau) = e^{\lambda(\sigma)g(\tau)}[f(\tau)\lambda(\sigma) \int_\sigma^u \eta(s)ds$$

$$+\eta(u)u_\tau + \eta(u)u_j\dot{z}_j + \eta(\sigma)f(\tau) + \bar{\beta}(\dot{z}(\tau))]$$

$$\geq e^{\lambda(\sigma)g(\tau)} f(\tau)[\eta(\sigma) - \eta(u) + \lambda(\sigma) \int_\sigma^u \eta(s)ds]$$

$$+e^{\lambda(\sigma)g(\tau)}[\eta(u)u_j\dot{z}_j + \bar{b}(\eta(u)\nabla u) + \bar{\beta}(\dot{z}(\tau))].$$

The expression in the first brackets is nonnegative, i.e. $\int_\sigma^u \eta(s)[\lambda(\sigma) - \lambda(s)]ds \geq 0$, because $\lambda(s)$ is a decreasing function. The second expression is nonnegative due to the Young inequality. So $E(t) \geq E(t_0) = \int_\sigma^{u(t_0,z(t_0))} \eta(s)ds$, for $t > t_0$. Now we go to infimum over all smooth curves $\{z(s)\}$ such that $z(t_0) = x_0$, $z(t) = x$ and obtain (4.7). To prove (4.8) note that the conditions $\lambda'(u) \leq 0$ and $\lambda(m) < \infty$ give $\eta(u+g) \leq \eta(u)e^{\lambda(u)g} \leq \eta(u)e^{\lambda(m)g}$ since $g \geq 0$ and $\ln(\eta(u))$ is concave. Also, the function $\bar{b}(rq)r$ is non-decreasing in r since the function $\bar{b}_1(s) = s_j\bar{b}^j(s) - \bar{b}(s)$ is nonnegative. So

$$\frac{\bar{b}(\eta(u)\nabla u)}{\eta(u)} \geq \frac{\bar{b}(\eta(u+g)e^{-\lambda g}\nabla u)}{\eta(u+g)e^{-\lambda g}},$$

where $\lambda = \lambda(m)$ and

$$\left[\int_{u(t_0,x_0)}^{u(\tau,z(\tau))+g(\tau)} \eta(s)ds + \int_{t_0}^t e^{\lambda g(s)}\bar{\beta}(\dot{z}(s))ds\right]_\tau$$

$$= \eta(u+g)\,(u_\tau + u_j\dot{z}_j + f) + e^{\lambda g}\bar{\beta}(\dot{z})$$

$$\geq e^{\lambda g}\left\{\eta(u+g)e^{-\lambda g}u_j\dot{z}_j + \bar{b}\left[\eta(u+g)e^{-\lambda g}\nabla u\right] + \bar{\beta}(\dot{z})\right\} \geq 0.$$

So we get (4.8). Note that the pointwise estimate for smooth u is equivalent to (4.9). ∎

R e m a r k 4.3. We will derive, at least formally the estimate (4.8) in the case of the double powered equation (see Remark 4.1). Here

$$\eta(u) = e^{\lambda u},\ \lambda = const,\ \bar{b}(q) = \frac{r}{p}|q|^p,\ r > 0,\ p > 1,\ k = r + \lambda(p-1)d > 0$$

and if $\partial_j a_j(\eta(u_0)\nabla u_0) \geq -\omega_0$, the regularizing effect estimate (4.6) holds with $f(t) = \dfrac{\omega_0}{1+\omega_0 f_0 t}$, $f_0 = \dfrac{k}{d}$. For $c \in R^d$ we get

$$c_j(x_j - x_0) = \int_{t_0}^t c_j\dot{z}_j(s)ds = \int_{t_0}^t e^{\lambda g(s)}c_j e^{-\lambda g(s)}\dot{z}_j(s)ds$$

$$\leq \int_{t_0}^{t} e^{\lambda g(s)} \bar{\beta}(\dot{z}_j(s))ds + \bar{b}(c) \int_{t_0}^{t} e^{-\lambda(p-1)g(s)} ds.$$

So

$$H_\lambda = \sup_c \left[c_j(x_j - x_0) - \tau \bar{b}(c) \right] = \tau \bar{\beta} \left(\frac{x - x_0}{\tau} \right)$$

with

$$\tau = \tau(t_0, t) = \int_{t_0}^{t} e^{-\lambda(p-1)g(s)} ds = \int_{t_0}^{t} \left(\frac{1 + \omega_0 f_0 s}{1 + \omega_0 f_0 t_0} \right)^{-\frac{\lambda d(p-1)}{k}} ds$$

$$= \frac{d}{r\omega_0} (1 + \omega_0 f_0 t_0)^{\frac{\lambda d(p-1)}{k}} \left[(1 + \omega_0 f_0 t)^{\frac{r}{k}} - (1 + \omega_0 f_0 t_0)^{\frac{r}{k}} \right].$$

For convenience we will write the pointwise estimate (4.7) for $w = e^u$ in the case $\omega_0 = \infty$. Denote $\alpha = \frac{\lambda d}{k}$, $\sigma = \frac{r}{k}$, $p' = \frac{p}{p-1}$, then

$$(4.10) \qquad t_0^\alpha \frac{1}{\lambda} w^\lambda(t_0, x_0) \leq t^\alpha \frac{1}{\lambda} w^\lambda(t, x) + \frac{1}{p} k(t^\sigma - t_0^\sigma) \left[\frac{|x - x_0|}{k(t - t_0)} \right]^{p'} \quad \text{for } \lambda \neq 0$$

and

$$(4.11) \qquad \ln w(t_0, x_0) \leq \ln w(t, x) + \frac{d}{r} \ln \frac{t}{t_0} + \frac{r}{p'} (t - t_0) \left[\frac{|x - x_0|}{r(t - t_0)} \right]^{p'} \quad \text{for } \lambda = 0.$$

It is clear that estimates (4.10), (4.11) correspond to the classical Moser pointwise estimates (see Mozer [14]) in the linear case.

For the Cauchy problem (4.1) in one-dimensional case we will show that the boundness of the first derivatives for bounded solutions is a consequence of the regularizing effect.

Theorem 4.3. *Let $d = 1$ and $u(t, x)$ be a bounded solution $|u(t, x)| \leq M$ of the Cauchy problem (4.1) and let the regularizing effect (4.6) hold for $u(t, x)$ with a function $f(t)$. Then $\eta(u(t, x))u_x(t, x)$ is a bounded function.*

Proof. Denote $F(t, x; y) = a'(\eta(u(t, x))u_x(t, x) + f(t)(x - y)$. Then $F(t, x; y)$ is an increasing function in x because

$$\frac{d}{dx} F(t, x; y) = \frac{d}{dx} [a'(\eta(u(t, x))u_x(t, x)) + f(t)(x - y)] \geq 0$$

from the regularizing effect (4.6). Let $\alpha(q)$ is Young conjugate function of a, so $\alpha'(a'(\sigma)) = \sigma$, $\alpha''(q) > 0$. Then for $z = x + \frac{1}{f(t)} \alpha'(\eta(u(t, x))u_x(t, x))$ it follows

$$\alpha(a'(\eta(u(t,x))u_x(t,x)) - \alpha(0) - f(t)\int_{u(t,x)}^{u(t,z)} \eta(\sigma)d\sigma$$
$$= f(t)\int_x^z [\alpha'(F(t,x;y)) - \alpha'(F(t,y;y))]dy$$
$$= -f(t)\int_x^z \int_x^y \alpha''(F(t,s;y))\frac{d}{ds}F(t,s;y)dsdy \geq 0.$$

So

$$\alpha(a'(\eta(u(t,x))u_x(t,x))) \leq \alpha(0) + f(t)\int_{u(t,x)}^{u(t,z)} \eta(\sigma)d\sigma \tag{4.12}$$

and $\eta(u(t,x))u_x(t,x)$ is bounded. ■

Note that the boundness of $\eta(u(t,x))u_x(t,x)$ if $|u(t,x)| \leq M$ was proved in Esteban, Vazquez [7] using the Bernstein method.

5. Gradient estimates

In this section we will deal with Cauchy problem

$$\left| \begin{array}{l} u_t = \partial_j \bar{a}_j(\eta(u)\nabla u) + a_0(u,\nabla u), \quad (t,x) \in S_T, \\ u(0,x) = u_0(x), \quad x \in R^d, \end{array} \right. \tag{5.1}$$

with the same conditions on the functions and the initial data as in Section 2. We will take for the present as $b(u,q) = \dfrac{1}{\eta(u)}\bar{b}(\eta(u)q)$ an arbitrary smooth, positive, strictly convex function with respect to q. Remark 2.1 applied to the right hand side of Lemma 2.1 (iii) with

$$\sigma = 0,\ D = E - bA^0 - b^0A,\ \delta = \varepsilon - b\alpha^0 - b^0\alpha$$

leads to the inequality

$$Lb \leq a_0^0 - e_0 + \frac{1}{4}Tr\tilde{A}^{-1}(D-S)B^{-1}(D-S)^* + Tr\ B^{-1}Q, \tag{5.2}$$

where L is the differential operator defined in Section 2. Since the computations are the same as in Section 4 we will point out only some of them. In this case $D = -\lambda(bA_1 + b_1A)$ and since $(B^{-1}\beta)_{ik} = \dfrac{\lambda}{\eta^2}u_iu_k$, $Tr\alpha = \lambda(A\nabla u, \nabla u) = Tr\ B^{-1}\beta A$,

$$Tr\alpha^0 = \lambda'(A\nabla u,\nabla u) + \lambda^2(A_1\nabla u,\nabla u) = \lambda'(A\nabla u,\nabla u) + Tr\ B^{-1}\beta A^0,$$

$$Tr\ B^{-1}Q = b(Tr\alpha^0 - TrB^{-1}\beta A^0) + b^0(Tr\alpha - Tr\ B^{-1}\beta A)$$

$$= \lambda'(A\nabla u, \nabla u)b = \lambda'\eta\bar{a}^l_j u_j u_l b$$

and $e_0 = (\lambda a_{01} - a^0_0)b_1$ we get

$$Lb \le a^0_0 b + (a^0_0 - \lambda a_{01})b_1 + \frac{\lambda^2}{4}Tr\tilde{A}^{-1}(bA_1 + b_1 A)B^{-1}(bA_1 + b_1 A^*)$$

$$+\lambda' a^l_j u_j u_l \equiv l(u,q). \tag{5.3}$$

Recalling the comparison Lemma 3.3, we need such a choice of b that the right hand side of (5.3) $l(u,q)$ becomes less than $l(b)$ - a nonnegative smooth function, which depends only on b. This problem - a nonlinear differential equation - is too complicated, but for the examples, we deal with it may be solved. Moreover in these vases $l(b) = l_0 b^2$. So we limit ourselves with the next lemma, which follows directly from Lemma 3.4.

Lemma 5.1. *Let $l(u,q) = l_0 b^2$, $l_0 = const$. Then under the posed above conditions on the problem (5.1) and $b_0 = \sup_x b(u_0(x), \nabla u_0(x))$ holds*

$$b(u, \nabla u) \le \frac{b_0}{1 - l_0 b_0 t} \quad for\ \ 0 \le t < \frac{1}{l_0 b_0}, \quad if\ l_0 > 0. \tag{5.4}$$

The time $T = \dfrac{1}{l_0 b_0}$ is essential, and the example (iii) in Introduction shows that it is exact blow-up time for the Barenblatt-type solutions. Also, let us note that for the double powered equation $l_0 = \lambda k$ if $b = a_1 = (p-1)a$. Indeed $a_0 = ra$, $b_1 = (p-1)b$, $A_1 = (p-1)A$, $B^{-1} = \dfrac{1}{p-1}A^{-1}$. So $a^0_0 = \lambda r a_1 = \lambda r b$, $a^0_0 - \lambda a_{01} = 0$ and the expression with the trace is $\lambda^2(p-1)db^2$.

Remark 5.1. The inequality $\dfrac{b(\eta(u)\nabla u)}{\eta(u)} \le C$ corresponds for smooth u and $\nu = \eta(\sigma) + \lambda(\sigma)\displaystyle\int_\sigma^u \eta(s)ds \ge \eta(u)$ to Lipshitz type estimate for some function of ν, namely:

$$\int_{\nu(x)}^{\nu(y)} \frac{d\tau}{G_b(\tau, \lambda(y-x))} \le 1, \text{ where } G_b(\tau,\xi) = \sup_\eta\{\xi_j\chi_j | b(\chi) < \tau\} \text{ and } \lambda = \lambda(\sigma).$$

If

$$b(\chi) = \bar{b}(|\chi|),\ \chi \in R^d,\ G_b(\tau, \xi) = |\xi| \sup_x\{|\chi|\ |b(\chi) < \tau\} \equiv |\xi| G_b(\tau),$$

so the above inequality becomes

$$\int_{\nu(x)}^{\nu(y)} \frac{d\tau}{G_b(\tau)} \leq |\lambda||x - y|.$$

Indeed, since $b\left(\frac{\nabla\nu}{\lambda}\right) = b(\eta(u)\nabla u) \leq C\eta(u) \leq C\nu$, then

$$\frac{d}{ds}\left[\int_{\nu(x)}^{\nu(x+sz)} \frac{d\tau}{G_b(C\tau, \lambda z)} - s\right] = \frac{z_j\nu_j}{G_b(C\nu, \lambda z)} - 1 \leq 0$$

which gives for $s = 1$ and $z = y - x$ the result.

Our last aim in this section is to derive an estimate for u_t. Althougt the regularizing effect from Section 4 with $b \geq 0$ gives $u_t \geq -f(t)$, this is not the best result. If $\bar{a}_j(\eta(u)\nabla u)$ in (5.1) is homogeneous, then it is possible to bring out $\eta(u)$ from it and we get the next Cauchy problem

$$(5.5)\qquad \left|\begin{array}{l} u_t = \partial_j\psi(u)\bar{a}_j(\nabla u) + a_0(u, \nabla u),\ \ (t, x) \in S_T, \\ u(0, x) = u_0(x),\ x \in R^d. \end{array}\right.$$

According to Lemma 2.1 (ii) with $a_j(u, \nabla u) = \psi(u)\bar{a}_j(\nabla u)$, it holds

$$L(u_t) = u_t\left[\partial_j\psi'(u)\bar{a}_j(\nabla u) + a_0^0\right] = \nu u_t^2 + (\nu' a_j u_j + a_0^0 - \nu a_0)u_t$$

with $\nu = \nu(u) = \dfrac{\psi'(u)}{\psi(u)}$. So by the same comparison procedure as in the proof of Theorem 4.2, we obtain the next lemma.

Lemma 5.2. *Let $a_j(u, \nabla u) = \psi(u)\bar{a}_j(\nabla u)$, $a_0(u, \nabla u)$ and u_0 satisfy the conditions of Lemma 2.3. Let also $\nu' a_j u_j + a_0^0 - \nu a_0 \leq 0$, then*

$$(5.6)\qquad \begin{array}{l} u_t \geq -\dfrac{1}{\nu_0 t} \quad \text{if} \quad \nu(u) \geq \nu_0 > 0 \\ u_t \leq \dfrac{1}{\nu_1 t} \quad \text{if} \ \ \nu(u) \leq \nu_1 < 0. \end{array}$$

The double powered case gives $\nu = \lambda(p-1)$ in (5.6) and for positive solution w of the original equation (1.1) we have in some weak sense,

$$w_t \geq -\frac{w}{\lambda(p-1)t} \quad \text{for } \lambda > 0 \quad \text{and} \quad w_t \leq \frac{w}{\lambda(p-1)t} \quad \text{for} \quad \lambda < 0.$$

Remark 5.2. For all purposes in this study we could use the equation (5.5) instead of (5.1). However the computations would be more complicated.

6. Approximation of the double powered equation

The aim of this section is to construct uniformly parabolic equations, which approximate the double powered equation. In order to obtain smooth solutions basing on the results in [13] we must avoid as the singularities at $0(\infty)$ so the growth of the coefficients at $\infty(0)$ for $p > (<)2$. The standard approximation $\left(|q|^2+\varepsilon\right)^{\frac{p}{2}}$ only at 0 is not always correct for the purpose since it does not satisfy the restrictive structural conditions in Theorem 4.1, Ch.6 in [13]. It may be used if the corresponding solutions are a priory smooth.

All of the functions involved in this case of double powered equation have the form $c(u,\nabla u) = \dfrac{1}{\eta(u)}\bar{c}(s)$ with $\eta(u) = e^{\lambda u}$, $s = \eta(u)|\nabla u|$, $s_j = \eta(u)u_j$, the corresponding vectors $c^j(u,\nabla u) = \bar{c}'(s)\dfrac{s_j}{s}$, the matrices $C(u,\nabla u) = \eta(u)\bar{C}(s)$ and

$$\bar{C}(s) = \frac{\bar{c}'(s)}{s}\left[I + \left(\frac{s\bar{c}''}{\bar{c}'} - 1\right)T\right], \quad T = \left\{\frac{s_j s_k}{s^2}\right\}, \quad I \text{ is identity matrix.}$$

Note that T is a projector, i.e. $0 \leq T \leq I$, $T^2 = T$, $TrT = 1$. Since in the "pure" powered case $\bar{A} = s^{p-2}\left[I + (p-2)T\right]$ we must regularize the function $\dfrac{\bar{a}'(s)}{s} = s^{p-2}$ in order to obtain a strictly positive and bounded matrix $\bar{A}$. Assume that $\dfrac{\bar{a}'_\varepsilon(s)}{s}$ is such a regularization of s^{p-2} and denote $\mu = \mu_\varepsilon(s) = \dfrac{s\bar{a}''_\varepsilon}{\bar{a}'_\varepsilon}$, $\sigma = \sigma_\varepsilon = \dfrac{s\mu'_\varepsilon}{\mu_\varepsilon}$ the approximation of $p-1$ and 0 respectively. Then, omitting the index ε

$$\bar{A} = \frac{\bar{a}'}{s}\left[I + (\mu - 1)T\right],$$

$$\bar{A}_1 = \frac{\bar{a}'}{s}\mu\left[I + (\mu - 1 + \sigma)T\right], \; \bar{A}^{-1} = \frac{s}{\bar{a}'}\left[I + \left(\frac{1}{\mu} - 1\right)T\right].$$

So $\bar{A}_1 = \mu\bar{A} + \bar{a}'\mu' T$, $\frac{\bar{a}'}{s}\mu T \le \bar{A}$ and

$$(6.1)\quad |\bar{A}_1 - \mu\bar{A}| \le |\sigma|\bar{A},\ (\bar{A}_1 - \mu\bar{A})\bar{A}^{-1}(\bar{A}_1 - \mu\bar{A}) = \sigma(\bar{A}_1 - \mu\bar{A}) \in (0, \sigma^2\bar{A}).$$

(i) Assume that after the regularization we get $\min(1, p-1) \le \mu \le \max(1, p-1)$, $|\sigma| \le \varepsilon$ and there exists $\Lambda_\varepsilon > 0$ such that $\Lambda_\varepsilon \le \frac{\bar{a}'}{s} \le \frac{1}{\Lambda_\varepsilon}$. Let us choose $\bar{B}$ (it will be a choice of $\bar{b}$ as well) as

$$\frac{1}{d}\bar{B} = \begin{cases} \left(\frac{k}{d} + |\lambda|\varepsilon + \frac{d\lambda^2\varepsilon^2}{4r}\right)\bar{A} - \lambda\bar{A}_1, & \lambda(p-2) \ge 0 \\ \left(\frac{r}{d} + |\lambda|\varepsilon + \frac{d\lambda^2\varepsilon^2}{4r}\right)\bar{A}, & \lambda(p-2) < 0 \end{cases}$$

with $k = r + \lambda(p-1)d > 0$, $r > 0$. Under the above assumptions and the inequalities (6.1), we obtain that $\bar{B}$ is strictly positive.

Let $\delta = \frac{r}{d} + [\lambda(p-1-\mu)]_+$ so $\delta \ge \frac{r}{d} > 0$, $\delta + \lambda\mu \ge \frac{r}{d} + \lambda(p-1) = \frac{k}{d}$ and $\operatorname{sgn}[\lambda(p-1-\mu)] = \operatorname{sgn}[\lambda(p-2)]$. According to the regularizing effect – Theorem 4.1, if $M = (\delta+\lambda\mu)\bar{A} - \lambda\bar{A}_1 + \frac{\lambda^2}{4\delta}(\bar{A}_1 - \mu\bar{A})\bar{A}^{-1}(\bar{A}_1 - \mu\bar{A})$ is less than $\frac{1}{d}B$, then $f \ge \delta + \lambda\mu \ge \frac{k}{d}$. But $M \le \left(\delta + \lambda\mu + \frac{d\lambda^2\varepsilon^2}{4r}\right)\bar{A} - \lambda\bar{A}_1 \le \left(\delta + |\lambda|\varepsilon + \frac{d\lambda^2\varepsilon^2}{4r}\right)\bar{A}$. The first expression on the right hand side is less than $\frac{1}{d}B$ for $\lambda(p-2) \ge 0$, since $\delta + \lambda\mu = \frac{k}{d}$, and the second expression is equal to $\frac{1}{d}B$ for $\lambda(p-2) < 0$ since $\delta = \frac{r}{d}$. So the regularizing effect holds with exact constant $f_0 = \frac{k}{d}$.

(ii) Let us estimate now the function $l(u, q)$ in the right hand side of (5.3) after a suitable choice of $b = \frac{1}{\eta(u)}\bar{b}(\eta(u)q)$. Assume also that there exists $c_0 < \infty$ and $(\mu+1)a_0^0 - \lambda\mu a_{01} \le c_0 b$. From (6.1) we have

$$\begin{aligned} &(bA_1 + b_1 A)\bar{A}^{-1}(bA_1 + b_1 A) \\ &= (b_1 + \mu b)^2\bar{A} + 2b(b_1 + \mu b)(\bar{A}_1 - \mu\bar{A}) + b^2(A_1 - \mu A)A^{-1}(A_1 - \mu A) \\ &= (b_1 + \mu b)^2\bar{A} + b[2b_1 + (2\mu + \sigma)b](\bar{A}_1 - \mu\bar{A}) \\ &\le \{(b_1 + \mu b)^2 + b[2b_1 + (2\mu+\sigma)b]|\sigma|\}\bar{A}. \end{aligned}$$

Let $b_1 = \mu b$, i.e. $s\bar{b}' = (\mu+1)\bar{b} = \left(\frac{s\bar{a}''}{\bar{a}'} + 1\right)\bar{b}$, so $\bar{b}(s) = s\bar{a}'(s)$, $\bar{b}' = \bar{a}'(\mu+1)$, $s\bar{b}'' = \left(\frac{\sigma}{\mu+1} + 1\right)\mu\bar{b}'$, $\bar{B}^{-1}A = \frac{\bar{a}'}{\bar{b}'}\left[I + \left(\frac{\mu\bar{b}'}{s\bar{b}''} - 1\right)T\right] =$

$\frac{1}{\mu+1}\left(I-\frac{\sigma}{\sigma+\mu+1}T\right)$ and (5.3) becomes $Lb \le \left(c_0+\frac{\lambda^2 d\mu^2}{\mu+1}c_1\right)b^2$, $c_1 = 1+O(\varepsilon)$ and $\frac{\mu^2}{\mu+1}$ is bounded.

Finally, we give a regularization of the quantities in question that satisfies a priory imposed conditions.

Let $g(t)$ be C^2 function on R^1, such that $|g(t)| \le C$, $g(0)=0$, $0 \le g' \le g'(0)=1$, $|g''| \le C$ with a sufficiently large C. Then $g_\varepsilon(t) \le \frac{1}{\varepsilon}g(\varepsilon t)$ is uniform approximation of t on compact sets of R^1 and $|g_\varepsilon(t)-t| \le \frac{1}{2}C\varepsilon t^2$, $|g'_\varepsilon(t)-1| \le C\varepsilon|t|$, $|g''_\varepsilon(t)| \le C\varepsilon$. So for $s \in R^+$ we can choose $\frac{a'_\varepsilon}{s} = e^{(p-2)g_\varepsilon(\ln s)}$, then $\mu_\varepsilon(s) = \frac{sa''_\varepsilon}{a'_\varepsilon} = 1+(p-2)g'_\varepsilon(\ln s) \in (\min(1,p-1),\max(1,p-1))$, $\left|\frac{s\mu'}{\mu}\right| = \sigma_\varepsilon(s) \le \frac{\varepsilon|p-2||g''_\varepsilon|}{\min(1,p-1)} \le C_p\varepsilon$ and $a'_\varepsilon \to s^{p-1}$ uniformly on compact sets of $\bar{R}^+$.

Indeed, $s^{\mu_2} \le a'_\varepsilon(s) \le s^{\mu_1}$ for $s \le 1$ with $\mu_1 = \min(1,p-1)$, $\mu_2 = \max(1,p-1)$. So for $\delta \in (0,1)$, $0 \le s \le \frac{1}{\delta}$:

$$\begin{aligned}\left|a'_\varepsilon - s^{p-1}\right| &\le \max\left(\max_{(0,\delta)}\left|a'_\varepsilon - s^{p-1}\right|, \max_{(\delta,\frac{1}{\delta})}\left|a'_\varepsilon - s^{p-1}\right|\right)\\ &\le \max\left(\delta^{\mu_1}, \delta^{1-p}\left(e^{|p-2|\frac{\varepsilon}{2}\varepsilon\ln^2\delta}-1\right)\right),\end{aligned}$$

and

$$\limsup_{\varepsilon\to 0}\sup_{s\in(0,k)}\left|a'_\varepsilon(s)-s^{p-1}\right| \le \delta^{\mu_1}, \quad \delta < \frac{1}{k}.$$

Such convergence takes place for $a_\varepsilon(s) \to \frac{1}{p}s^p$, $a_{1\varepsilon}(s) \to \frac{p-1}{p}s^p$ and so for $b_\varepsilon(s) \to \frac{r}{p}s^p$.

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