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Asymptotic Expansion of a Unified Elliptic-Type Integral

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In this paper the authors present a unification and generalization of certain families of elliptic-type integrals, which were studied in a number of earlier works on the subject due to their importance for possible applications in some problems arising in radiation physics. The results recently given by Kalla and Tuan [*Computers Math. Appl.*, **32**, 1996, 49-55] follow, as special cases.

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1. Introduction and preliminaries

Elliptic-type integrals occur in certain radiation field problems [4, 8].

Epstein and Hubbell [5], and subsequently Weiss [17], discussed the following family of elliptic-type integrals

$$(1.1)\qquad \Omega_j(k) := \int_0^\pi (1-k^2\cos\theta)^{-j-\frac{1}{2}}\,d\theta, \quad j=0,1,2,\ldots, \quad 0\le k<1.$$

Generalizations of (1.1) were given by a number of authors, notably by Kalla [9, 10], Kalla, Conde and Hubbell [12], Kalla and Al-Saqabi [11] and Srivastava et al [16].

The following generalization of the integral (1.1),

$$(1.2)\qquad R_\mu(k,\alpha,\gamma) := \int_0^\pi \frac{\cos^{2\alpha-1}(\frac{\theta}{2})\sin^{2\gamma-2\alpha-1}(\frac{\theta}{2})}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\,d\theta,$$

where $0\le k<1$, $Re(\gamma)>Re(\alpha)>0$, $Re(\mu)>-\frac{1}{2}$, was studied by Kalla et al [11, 13] and Glasser and Kalla [7].

Al-Saqabi [2] considered an interesting generalized family of elliptic-type integrals in the form

$$B_\mu(k,m,\nu) := \int_0^\pi \frac{\cos^{2m}(\theta)\sin^{2\nu}(\theta)}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}\,d\theta, \tag{1.3}$$

where $0 \le k < 1;\ \ m \in N_0\ ;\ \ \mu \in C;\ \ Re(\nu) > -\frac{1}{2}$.

On the other hand, Siddiqi [15] discussed yet another generalization of (1.1) in the form

$$\Lambda_\nu(\alpha,k) := \int_0^\pi \frac{exp[\alpha\sin^2(\frac{\theta}{2})]}{(1-k^2\cos\theta)^{\nu+\frac{1}{2}}}\,d\theta, \tag{1.4}$$

where $0 \le k < 1;\ \ \alpha,\nu \in \mathbb{R}$.

An interesting unification of the families of elliptic-type integrals (1.2), (1.3) and (1.4) was given by Srivastava and Siddiqi [16] in the form

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\mu)}(\rho;k) := \int_0^\pi \cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)$$

$$\times\frac{\left[1-\rho\sin^2(\theta/2)\right]^{-\lambda}}{\left[1-k^2\cos\theta\right]^{\mu+\frac{1}{2}}}d\theta, \tag{1.5}$$

where $0 \le k < 1,\ Re(\alpha) > 0,\ Re(\beta) > 0;\ \lambda,\mu \in C:\ |\rho| < 1$.

Recently Kalla and Tuan [14] further extended (1.5) by introducting the following generalization:

$$\Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho;\delta;k) := \int_0^\pi \cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)$$

$$\times\frac{\left[1+\delta\cos^2\theta/2\right]^{-\gamma}\left[1-\rho\sin^2\theta/2\right]^{-\lambda}}{\left[1-k^2\cos\theta\right]^{\mu+\frac{1}{2}}}\,d\theta, \tag{1.6}$$

where $0 \le k < 1,\ Re(\alpha) > 0,\ Re(\beta) > 0;\ \lambda,\mu,\gamma \in C$; either $|\rho|,|\delta| < 1$ or $\rho(\text{or }\delta) \in C$, whenever $\lambda = -m$ (or$\gamma = -m$), $m \in N_0$.

They also discussed the asymptotic expansion of (1.6).

In a recent paper Al-Zamel et al [3] studied a generalized family of elliptic-type integrals in the form:

$$Z^{(\alpha,\beta)}_{(\gamma)}(k) := Z^{(\alpha,\beta)}_{(\gamma_1,\cdots,\gamma_n)}(k_1,\cdots,k_n)$$

$$= \int_0^\pi \cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)\prod_{j=1}^n(1-k_j^2\cos\theta)^{-\gamma_j}\,d\theta \tag{1.7}$$

$$= \prod_{j=1}^{n}(1-k_j^2)^{-\gamma_j} B(\alpha,\beta)$$

$$\times F_D^{(n)}\left(\beta;\gamma_1,\cdots,\gamma_n:\alpha+\beta;\frac{2k_1^2}{k_1^2-1},\cdots,\frac{2k_n^2}{k_n^2-1}\right), \tag{1.8}$$

where $Re(\alpha)$, $Re(\beta) > 0$, $|k_j| < 1$, $\gamma_j \in C$; $j = 1,\cdots,n$; $F_D^{(n)}(.)$ is the Lauricella function of n variables [6, p. 41, Equation 2.1.4].

The object of the paper is to introduce another unification and generalization of the aforementioned family of elliptic-type integrals (1.6) in the following general form, which is different from (1.7):

$$\Omega^{(\alpha,\beta)}_{(\sigma_1,\cdots,\sigma_{n-2};\gamma,\mu)}(\rho_1,\cdots,\rho_{n-2},\delta;k)$$

$$:= \int_0^{\pi} \cos^{2\alpha-1}(\theta/2)\sin^{2\beta-1}(\theta/2)\prod_{j=1}^{n-2}\left[1-\rho_j\sin^2(\theta/2)\right]^{-\sigma_j}$$

$$\times \left[1+\delta\cos^2(\theta/2)\right]^{-\gamma} d\theta, \tag{1.9}$$

where $0 \le k < 1$, $Re(\alpha)$, $Re(\beta) > 0$; $\sigma_j(j = 1,\cdots,n-2)$, $\gamma,\mu \in C$,

$$\max\left\{|\rho_j|, |\frac{\delta}{1+\delta}|, |\frac{2k^2}{k^2-1}|\right\} < 1.$$

$1 \le j \le n-2$.

For $n = 3$, (1.9) reduces to (1.6).

2. Relations with other families of elliptic-type integrals

On comparing (1.9) with the definitions (1.2), (1.3), (1.4), (1.5) and (1.6), we obtain the following relationships:

$$\Omega^{(\alpha,\gamma-\alpha)}_{(0,\cdots,0;0,\mu)}(\rho,\rho_2,\cdots,\rho_{n-2},0;k)$$

$$= \Omega^{(\alpha,\gamma-\alpha)}_{(\sigma_1,\cdots,\sigma_{n-2},0,\mu)}(0,\cdots,0;k) = R_\mu(k,\alpha,\gamma), \tag{2.1}$$

where $0 \le k < 1$; $Re(\gamma)$, $Re(\alpha) > 0$; $\mu \in C$;

$$\Omega^{(\nu+\frac{1}{2},\nu+\frac{1}{2})}_{(-2m,0,\cdots,0;0,\mu)}(2,\rho_2,\cdots,\rho_{n-2},0;k) = \Omega^{(\nu+\frac{1}{2},\nu+\frac{1}{2})}_{(-2m,\rho_2,\cdots,\rho_{n-2};0,\mu)}(2,0,\cdots,0;k)$$

$$= 2^{-2\nu} B_\mu(k, m, \nu), \tag{2.2}$$

where $0 \le k < 1$; $m \in N_0$; $\mu \in C$; $Re(\nu) > -\frac{1}{2}$;

$$\lim_{\sigma\to\infty} \left\{ \Omega^{(\frac{1}{2},\frac{1}{2})}_{(\sigma,\mu)}, (\rho/\sigma, 0, \cdots, 0; k) \right\} = \Lambda_\mu(\rho, k). \tag{2.3}$$

where $0 \le k < 1$ and $\sigma, \mu \in C$;

$$\Omega^{(\alpha,\beta)}_{(\lambda,0,\cdots,0;0,\mu)}(\rho, \rho_2, \cdots, \rho_{n-2}, 0; k)$$

$$= \Omega^{(\alpha,\beta)}_{(\lambda,\sigma_1,\cdots,\sigma_{n-2};0,\mu)}(\rho, 0, \cdots, 0, 0; k) = \Lambda^{(\alpha,\beta)}_{(\lambda,\mu)}(\rho; k), \tag{2.4}$$

where $0 \le k < 1$, $Re(\alpha)$, $Re(\beta) > 0$; $\lambda, \mu \in C$; $|\rho| < 1$;

$$\Omega^{(\alpha,\beta)}_{(\lambda,0,\cdots,0;\gamma,\mu)}(\rho, \rho_2, \cdots, \rho_{n-2}, \delta; k)$$

$$= \Omega^{(\alpha,\beta)}_{(\lambda,\sigma_2,\cdots,\sigma_{n-2};\gamma,\mu)}(\rho, 0, \cdots, 0; \delta, k) = \Lambda^{(\alpha,\beta)}_{(\lambda,\gamma,\mu)}(\rho, \delta; k), \tag{2.5}$$

where $0 \le k < 1$, $Re(\alpha)$, $Re(\beta) > 0$; $\lambda, \mu, \gamma \in c$; either $|\rho|, |\delta| < 1$ or ρ (or δ) $\in c$ whenever $\lambda = -m$(or $\gamma = -m$); $m \in N_0$.

3. Explicit representation and asymptotic expansion

From (1.9), we infer that

$$\Omega^{(\alpha,\beta)}_{(\sigma_1,\cdots,\sigma_{n-2};\gamma,\mu)}(\rho_1, \cdots, \rho_{n-2}, \delta; k) = (1+\delta)^{-\gamma}(1-k^2)^{-\mu-\frac{1}{2}} \tag{3.1}$$

$$\times \int_0^1 t^{\beta-1}(1-t)^{\alpha-1} \left[\prod_{j=1}^{n-2}(1-\rho_j t)^{-\sigma_j}\right] \left[1 - \frac{\delta t}{1+\delta}\right]^{-\gamma} \left[1 - \frac{2k^2 t}{k^2-1}\right]^{-\mu-\frac{1}{2}} dt.$$

On making use of the integral [6, p.49, Equation 2.3.6], (3.1) becomes

$$\Omega^{(\alpha,\beta)}_{(\sigma_1,\cdots,\sigma_{n-2};\gamma,\mu)}(\rho_1, \cdots, \rho_{n-2}, \delta; k)$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(1+\delta)^{-\gamma}(1-k^2)^{-\mu-\frac{1}{2}} \tag{3.2}$$

$$\times F_D^{(n)}\left(\beta, \sigma_1, \cdots, \sigma_{n-2}, \gamma, \mu + \frac{1}{2}; \alpha+\beta; \rho_1, \cdots, \rho_{n-2}, \frac{\delta}{1+\delta}, \frac{2k^2}{k^2-1}\right).$$

It is interesting to observe that for $n = 3$, (3.2) reduces to the result given by Kalla and Tuan [14, p.50, Equation 9] which itself is a generalization of the result given by Srivastava and Siddiqi [16, p.305, Equation (1.36)].

We will now investigate the asymptotic expansion of generalized elliptic-type integral (1.9) as $k^2 \to 1$. Expressing the Lauricella hypergeometric function $F_D^{(n)}$ in (3.2) in terms of Gauss hypergeometric function ${}_2F_1(.)$, it is seen that

$$F_D^{(n)}\left(\beta, \sigma_1, \cdots, \sigma_{n-2}, \gamma, \mu + \frac{1}{2}; \alpha + \beta; \rho_1, \cdots, \rho_{n-2}, \frac{\delta}{1+\delta}, \frac{2k^2}{k^2-1}\right)$$

$$= \sum_{m_1,\cdots,m_{n-1}=0}^{\infty} \frac{(\beta)_{m_1+,\cdots+m_{n-1}} \left\{\prod_{j=1}^{n-2}\left[(\rho_j)^{m_j}(\rho_j)_{m_j}\right]\right\}}{(\alpha+\beta)_{m_1+\cdots+m_{n-1}}(m_1)!\cdots(m_{n-1})!}$$

$$(\gamma)_{m_{n-1}}\left(\frac{\delta}{1+\delta}\right)^{m_{n-1}} \tag{3.3}$$

$$\times\ {}_2F_1\left(\beta + m_1 + \cdots + m_{n-1}, \mu + \frac{1}{2}; \alpha + \beta + m_1 + \cdots + m_{n-1}; \frac{2k^2}{k^2-1}\right).$$

Using the analytic continuation formula for the Gauss hypergeometric function [1, p.559, Equation 15.3.7], it gives

$$F_D^{(n)}\left(\beta, \sigma_1, \cdots, \sigma_{n-2}, \gamma, \mu + \frac{1}{2}; \alpha + \beta; \rho_1, \cdots, \rho_{n-2}, \frac{\delta}{1+\delta}, \frac{2k^2}{k^2-1}\right)$$

$$= \frac{\Gamma(\alpha+\beta)\Gamma(\mu-\beta+\frac{1}{2})}{\Gamma(\alpha)\Gamma(\mu+\frac{1}{2})}\left(\frac{1-k^2}{2k^2}\right)^{\beta} \sum_{m_1,\cdots,m_{n-1}=0}^{\infty} \frac{(\beta)_{m_1+\cdots+m_{n-1}}}{(\frac{1}{2}-\mu+\beta)_{m_1+\cdots+m_{n-1}}}$$

$$\frac{\prod_{j=1}^{n-2}\left[\rho_j\left(\frac{k^2-1}{2k^2}\right)^{m_j}\right]\left[\frac{\delta(k^2-1)}{2k^2(1+\delta)}\right]^{m_{n-1}}\prod_{j=1}^{n-2}\left[(\sigma_j)_{m_j}\right](\gamma)_{m_{n-1}}}{(m_1)!\cdots(m_{n-1})!} \tag{3.4}$$

$$\times\ {}_2F_1\left(\beta + m_1 + \cdots + m_{n-1}, 1-\alpha; \frac{1}{2} - \mu + \beta + m_1 + \cdots + m_{n-1}; \frac{k^2-1}{2k^2}\right)$$

$$+\frac{\Gamma(\alpha+\beta)\Gamma(\beta-\mu-\frac{1}{2})}{\Gamma(\beta)\Gamma(\alpha+\beta+-\mu-\frac{1}{2})}\left(\frac{1-k^2}{2k^2}\right)^{\mu+\frac{1}{2}} \sum_{m_1,\cdots,m_{n-1}=0}^{\infty} \frac{(\beta-\mu-\frac{1}{2})_{m_1+\cdots+m_{n-1}}}{(\alpha+\beta-\mu-\frac{1}{2})_{m_1+\cdots+m_{n-1}}}$$

$$\frac{\prod_{j=1}^{n-2}\left[(\rho_j)^{m_j}(\sigma_j)_{m_j}\right](\gamma)_{m_{n-1}}}{(m_1)!\cdots(m_{n-1})!}\left(\frac{\delta}{1+\delta}\right)^{m_{n-1}}$$

$$\times\ {}_2F_1\left(\mu+\frac{1}{2},\ \mu-\alpha-\beta-m_1-\cdots-m_{n-1}+\frac{3}{2};\right.$$
$$\left.\mu-\beta-m_1-\cdots-m_{n-1}+\frac{3}{2};\frac{1-k^2}{2k^2}\right),$$

provided $\beta-\mu-\frac{1}{2}$ is not an integer. Hence after slight rearrangement of the series, it is found that

$$\Omega^{(\alpha,\beta)}_{(\sigma_1,\cdots,\sigma_{n-2};\gamma,\mu)}(\rho_1,\cdots,\rho_{n-2},\delta;k)=\frac{\Gamma(\beta)\Gamma(\mu-\beta+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})}\alpha^{-\beta}k^{-2\beta}(1+\delta)^{-\gamma}$$

$$\times\ (1-k^2)^{\beta-\mu-\frac{1}{2}}F_D^{(n)}\left[\beta,\sigma_1,\cdots,\sigma_{n-2},\gamma,1-\alpha;\beta-\mu+\frac{1}{2};\frac{\rho_1(k^2-1)}{2k^2},\cdots,\right.$$

$$\left.\frac{\rho_{n-2}(k^2-1)}{2k^2},\frac{\delta(k^2-1)}{2k^2(1+\delta)},\frac{k^2-1}{2k^2}\right] \tag{3.5}$$

$$+\frac{\Gamma(\alpha)\Gamma(\beta-\mu-\frac{1}{2})}{\Gamma(\alpha+\beta-\mu-\frac{1}{2})}2^{-\mu-\frac{1}{2}}k^{-2\mu-1}(1+\delta)^{-\gamma}\sum_{n=0}^{\infty}\frac{(\mu+\frac{1}{2})_n(\mu-\alpha-\beta+\frac{3}{2})_n}{(\mu-\beta+\frac{3}{2})_n\quad n!}\left(\frac{1-k^2}{2k^2}\right)^n$$
$$\times\ F_D^{(n-1)}\left[\beta-\mu-n-\frac{1}{2},\sigma_1,\cdots,\sigma_{n-2},\gamma;\right.$$
$$\left.\alpha+\beta-\mu-n-\frac{1}{2};\rho_1,\cdots,\rho_{n-2},\frac{\delta}{1+\delta}\right].$$

Representation (3.5) may be regarded as the asymptotic series for

$$\Omega^{(\alpha,\beta)}_{(\sigma_1,\cdots,\sigma_{n-2};\gamma,\mu)}(\rho_1,\cdots,\rho_{n-2},\delta;k)\qquad\text{as}\quad k^2,\longrightarrow 1$$

provided $\mu-\beta+\frac{1}{2}$ is not an integer.

For $n=3$, it is interesting to see that $F_D^{(n)}(.)$ and $F_D^{(n-1)}(.)$ respectively, reduce to a Lauricella function of three variables $F_D^{(3)}(.)$ and an Appell hypergeometric function of two variables F_1 [5, p.224, Equation 5.7.1(6)] and consequently, we arrive at the result recently given by Kalla and Tuan [14, p.51, Equation (15)].

Next we consider the expansion of

$$(3.6)\quad {}_2F_1\left(\beta+m_1+\cdots+m_{n-1},\mu+\frac{1}{2};\alpha+\beta+m_1+\cdots+m_{n-1};\frac{2k^2}{k^2-1}\right),$$

occurring in (3.3), when its upper parameters differ by integers.

Two cases arise:

(i) when $\mu+\frac{1}{2}=\beta-\tau,\quad \tau=0,1,2,\cdots,$

and

(ii) when $\mu+\frac{1}{2}=\beta+\tau,\quad \tau=1,2,\cdots.$

In both these cases, the results are derived by Al-Zamel et al [3, Equation (4.4) and (4.7)]. By employing the results of Al-Zamel et al [3], we can easily derive the asymptotic expansion in above two cases. The result in the first case is enumerated below. The second case can be treated similarly.

Using the result given by Al-Zamel et al [3, Equation (4.4)], we find that

$$\Omega^{(\alpha,\beta)}_{(\sigma_1,\cdots,\sigma_{n-2};\gamma,\beta-\tau-\frac{1}{2})}(\rho_1,\cdots,\rho_{n-2},\delta;k)$$

$$=(1-\beta)_\tau(1+\delta)^{-\gamma}(1-k^2)^\tau 2^{-\beta}k^{-2\beta}$$

$$\times\sum_{m_1,\cdots,m_{n-1}=0}^{\infty}\frac{\left[\prod_{j=1}^{n-2}\{(\rho_j)^{m_j}(\sigma_j)_{m_j}\}\right](\gamma)_{m_{n-1}}\left(\frac{\delta}{1+\delta}\right)^{m_{n-1}}\left(\frac{k^2-1}{2k^2}\right)^{m_1+\cdots+m_{n-1}}}{(m_1)!\cdots(m_{n-1})!}$$

$$\times\sum_{m=0}^{\infty}\frac{(\beta)_{m+m_1+\cdots+m_{n-1}}(1-\alpha)_m}{(m)!(\tau+m+m_1+\cdots+m_{n-1})!}\left(\frac{k^2-1}{2k^2}\right)^m$$

$$\left[\ell n(2k^2)-\ell n(1-k^2)+\Psi(1+\tau+m+m_1+\cdots+m_{n-1})+\Psi(1+m)\right.$$

$$\left.-\Psi(\beta+m+m_1+\cdots+m_{n-1})-\Psi(\alpha-m)\right]+(1+\delta)^{-\tau}(2k^2)^{\tau-\beta}$$

$$(3.7)\quad\times\sum_{m_1,\cdots,m_{n-1}=0}^{\infty}\frac{\left[\prod_{j=1}^{n-2}\{(\rho_j)^{m_j}(\sigma_j)_{m_j}\}\right](\gamma)_{m_{n-1}}(\delta/1+\delta)^{m_{n-1}}}{(m_1)!\cdots(m_{n-1})!}$$

$$\times \sum_{m=0}^{\tau+m_1+\cdots+m_{n-1}-1} \frac{(\tau+m_1+\cdots+m_{n-1}-m-1)!(\beta-\tau)_m}{(\alpha)_{m_1}+\cdots m_{n-1}+\tau-m} \left(\frac{k^2-1}{2k^2}\right)^m$$

$$= (1-\beta)_\tau (1+\delta)^{-\tau} (1-k^2)^\tau 2^{-\beta} k^{-2\beta} \sum_{m_1,\cdots,m_{n-1}}^{\infty} \sum_{m=0}^{\infty} \frac{(\beta)_{m+m_1+\cdots+m_{n-1}}}{(m)!}$$

$$\times \frac{\left[\prod_{j=1}^{n-2} \{(\rho_j)^{m_j}(\sigma_j)_{m_j}\}\right] (\gamma)_{m_{n-1}} (1-\alpha)_m (\delta/1+\delta)^{m_{n-1}} \left(\frac{k^2-1}{2k^2}\right)^{m_1+\cdots+m_{n-1}}}{(\tau+m+m_1+\cdots+m_{n-1})!(m_1)!\cdots(m_{n-1})!}$$

$$\times \left[\ell n(2k^2) - \ell n(1-k^2) + \Psi(1+\tau+m+m_1+\cdots m_{n-1}) + \Psi(1+m)\right.$$

$$(3.8) \quad \left. -\Psi(\beta+m+m_1+\cdots+m_{n-1}) - \Psi(\alpha-m)\right] + (1+\delta)^{-\gamma}(2k^2)^{\tau-\beta}$$

$$\sum_{m_1,\cdots,m_{n-1}=0}^{\infty} \sum_{m=0}^{\tau+m_1+\cdots+m_{n-1}-1} \frac{(\tau+m_1+\cdots+m_{n-1}-m-1)!(\beta-\tau)_m}{(\alpha)_{m_1}+\cdots+m_{n-1}+\tau-m}$$

$$\times \left(\frac{k^2-1}{2k^2}\right)^m \frac{\left[\prod_{j=1}^{n-2} \{(\rho_j)_{m_j}(\sigma_j)_{m_j}\}\right] (\gamma)_{m_{n-1}} (\delta/1+\delta)^{m_{n-1}}}{(m)!(m_i)!\cdots(m_{n-1})!}.$$

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