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On the Hermite Method for the Factorization of Algebraic Polynomials

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We find the formulas (9) for the number of distinct zeros with the multiplicities 1,2,3,... of the polynomial (1), respectively.

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Let

$$P_n(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n, \quad a_0 \neq 0, \quad n \geq 1, \tag{1}$$

be a polynomial in z of degree n and coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$ of the complex number field, which is algebraic closed, i.e. all n zeros of (1) belong to this field. Let k_r $(0 \leq k_r \leq n)$ denote the number of the distinct zeros of (1) with the multiplicity r, where $r = 1, 2, \ldots, s$, and where s with $1 \leq s \leq n$ denotes the greatest multiplicity of the zeros of (1). Evidently, $k_s \geq 1$ and, in particular, $k_1 = n$ and $k_n = 1$ $(k_r = 0,\ 1 \leq r \leq n-1,\ n \geq 2)$ for $s = 1$ and $s = n$, respectively. If $k_r \geq 1$ for some r $(1 \leq r \leq s)$, let $z_j^{(r)}$, $j = 1, 2, \ldots, k_r$, be the distinct zeros with the multiplicity r of the polynomial (1). Then we set

$$Z_r = \prod_{j=1}^{k_r} (z - z_j^{(r)}), \quad k_r \geq 1; \ Z_r = 1, \ k_r = 0 \ (1 \leq r \leq s). \tag{2}$$

Hence the polynomial (1) has the factorization

$$P_n(z) = a_0 Z_1^1 Z_2^2 \ldots Z_r^r \ldots Z_s^s, \tag{3}$$

where

(4) $$1.k_1 + 2k_2 + \ldots + sk_s = n.$$

The factors Z_r, $r = 1, 2, \ldots, s$, in (2)-(3) are determined by the Euclidean algorithm and the classical Hermite method (see, for example, [1], pp. 67-70, and [2], pp. 180-181). (It is given an incorrect version of the Hermite method in [3], p. 1144, and [4], p. 332: - for example, the polynomials X_α are defined false since there are repetitions of the binomial factors, and hence, the factorizations (3) in [3] and (40) in [4] are incorrect; also the factorizations (3) in [3] and (40) in [4] are incorrect because the leading coefficient a_0 is ommited as a factor; in addition, the number of the factors of the factorizations (3) in [3] and (40) in [4], and the number of the equations of the triangular system (7) in [3] and (44) in [4] are taken equal to the degree of the polynomial respectively, which unnecessary complicates the calculations, etc.)

The correct formulas of the Hermite method are the following

(5) $$Z_r = \frac{f_r(z)}{f_{r+1}(z)}, \quad r = 1, 2, \ldots, s, \quad f_{s+1}(z) = 1,$$

where

(6) $$f_r(z) = \frac{D_{r-1}}{D_r} = Z_r Z_{r+1} \ldots Z_s,$$
$$r = 1, 2, \ldots, s, \quad D_0 = P_n(z), \quad D_s = a_0,$$

where D_r is the greatest common divisor of D_{r-1} and its derivative D'_{r-1}, calculated by the algorithm

(7) $$D_r(D_{r-1}, D'_{r-1}) = a_0 Z^1_{r+1} Z^2_{r+2} \ldots Z^{s-r}_s,$$
$$r = 1, 2, \ldots, s, \quad D_0 = P_n(z), \quad D'_0 = P'_n(z), \quad D_s = a_0.$$

Thus from (7), (6) and (5) we obtain the correct factorization (3)-(4).

Let n_r denote the degree of the polynomials D_r, where $n_r \geq 1$ for $1 \leq r \leq s-1$ $(s \geq 2)$. Then from (4) and (7) we obtain the triangular system of the linear equations

(8) $$1.k_{r+1} + 2k_{r+2} + \ldots + (s-r)k_s = n_r,$$
$$r = 0, 1, \ldots, s-1, \quad s \geq 2, \quad n_0 = n.$$

Now we shall give an explicit solution of the system (8) with respect to $k_1, \ldots, k_s$.

Theorem. *If $s \geq 2$, then the degrees $k_1, k_2, \ldots, k_s$ of the polynomials (2) are expressed by means of the degrees n and $n_1, \ldots, n_{s-1}$ of the polynomial (1) and the greatest common divisors (7) by the formulas*

(9) $$k_r = n_{r-1} - 2n_r + n_{r+1}, \quad r = 1, 2, \ldots, s, \quad n_0 = n, \quad n_s = n_{s+1} = 0.$$

Corollary. *The number of all distinct zeros with multiplicities* $1, 2, \ldots, r$ *of the polynomial (1), where* $1 \le r \le s$, *is equal to*

$$\sum_{j=1}^{r} k_j = n_0 - n_1 - n_r + n_{r+1}, \quad n_0 = n, \; n_s = n_{s+1} = 0. \tag{10}$$

First proof. Let $s \ge 3$ and for some r with $1 \le r \le s-2$ we take the following three successive equations from (8):

$$\sum_{j=1}^{s-r+1} j k_{r-1+j} = n_{r-1}, \tag{11}$$

$$\sum_{j=1}^{s-r} j k_{r+j} = n_r, \tag{12}$$

$$\sum_{j=1}^{s-r-1} j k_{r+1+j} = n_{r+1}. \tag{13}$$

If we add (11), (12), (13), multiplied by 1, -2, 1, respectively, then we shall obtain

$$n_{r-1} - 2n_r + n_{r+1} = \sum_{j=1}^{s-r+1} j k_{r-1+j} - 2\sum_{j=1}^{s-r} j k_{r+j} + \sum_{j=1}^{s-r-1} j k_{r+1+j} \tag{14}$$

$$= \sum_{j=0}^{s-r} (j+1) k_{r+j} - 2\sum_{j=1}^{s-r} j k_{r+j} + \sum_{j=2}^{s-r} (j-1) k_{r+j} = k_r + \sum_{j=2}^{s-r} 0.k_{r+j} = k_r$$

for $1 \le r \le s-2$ $(s \ge 3)$. Thus (14) yields the formulas (9) for $r = 1, 2, \ldots, s-2$ $(s \ge 3)$. The formulas (9) for k_{s-1} and k_s $(s \ge 2)$ directly follow from the equations (8) for $r = s-2$ and $r = s-1$, having in mind that $n_s = n_{s+1} = 0$.

Second proof. The formulas (9) for $s = 2$ and $s = 3$ directly follow from the system (8) for $r = 1, 0$ and $r = 2, 1, 0$, respectively. If $s \ge 4$, then from (8) for $r = s-1, s-2, s-3$ we conclude that the formulas (9) are valid for k_s, k_{s-1}, k_{s-2}. Let we assume that the formulas (9) are valid for $k_s, k_{s-1}, k_{s-2}, \ldots, k_r$, where $r > 1$ and $r \le s-2$. According to what has been assumed, if in (11) we replace r by $r-1$ and j by $j+2$, then we shall obtain

$$k_{r-1} = n_{r-2} - \sum_{j=0}^{s-r} (j+2) k_{r+j} \, . \tag{15}$$

$$= n_{r-2} - \sum_{j=0}^{s-r}(j+2)(n_{r+j-1} - 2n_{r+j} + n_{r+j+1})$$

$$= n_{r-2} - \sum_{j=0}^{s-r}(j+2)n_{r+j-1} + 2\sum_{j=1}^{s-r}(j+1)n_{r+j-1} - \sum_{j=2}^{s-r} jn_{r+j-1}$$

$$= n_{r-2} - 2n_{r-1} + n_r + \sum_{j=2}^{s-r} 0.n_{r+j-1} = n_{r-2} - 2n_{r-1} + n_r,$$

keeping in mind that $n_s = n_{s+1} = 0$. From (15) it follows that for $s \geq 4$ the formulas (9) are also valid for all members of the sequence $k_s, k_{s-1}, k_{s-2}, \ldots, k_2, k_1$.

T h i r d p r o o f. Let $A = |a_{\nu r}|$, $\nu, r = 1, 2, \ldots, s$, denote the determinant of the system (8) with the elements

$$a_{1r} = r, \quad 1 \leq r \leq s, \ (s \geq 1), \quad a_{\nu r} = 0, \quad 1 \leq r \leq \nu - 1; \tag{16}$$

$$a_{\nu r} = r - \nu + 1, \quad \nu \leq r \leq s \ (2 \leq \nu \leq s).$$

Since $A = 1$, according to the Crammer formulas for solution of the systems of linear equations and the Laplace theorem for developement of the determinants, the solution of the system (8) is

$$k_r = \sum_{\nu=1}^{s} A_{\nu r} n_{\nu-1}, \quad r = 1, 2, \ldots, s \ \ (n_0 = n), \tag{17}$$

where $A_{\nu r}$, $\nu, r = 1, 2, \ldots, s$, are the corresponding cofactors of the elements $a_{\nu r}$, $\nu, r = 1, 2, \ldots, s$, of the determinant A. From A and (16) we find that

$$A_{\nu r} = 0, \quad 1 \leq \nu \leq r - 1 \ (2 \leq r \leq s), \tag{18}$$

$$A_{rr} = 1 \ \ (1 \leq r \leq s, \ s \geq 1),$$

$$A_{r+1,r} = -2 \ \ (1 \leq r \leq s - 1, \ s \geq 2),$$

$$A_{r+2,r} = 1 \ \ (1 \leq r \leq s - 2, \ s \geq 3),$$

$$A_{\nu r} = 0, \quad r + 3 \leq \nu \leq s \ \ (1 \leq r \leq s - 3, \ s \geq 4).$$

Hence the matrix of the system (17) with elements (18) is inverse of the matrix of the system (8) with elements (16). Thus from (18) and (17) we again obtain that the formulas (9) are valid, setting $n_0 = n$, $n_s = n_{s+1} = 0$.

Finally, the formulas (10) follow from the formulas (9).

R e m a r k . As we determined the polynomials (7) and their degrees n_r, $0 \le r \le s-1$, we directly can determine the degrees k_r, $1 \le r \le s$, of the polynomials (2) by the formulas (9) without a computer program for solution of the system (8).

E x a m p l e . Let us consider the polynomial

(19) $$P_7(z) = 4z^7 - 8z^6 - 3z^5 + 15z^4 - 6z^3 - 6z^2 + 5z - 1.$$

By the algorithm (7) we obtain the sequence of the greatest common divisors

(20) $$D_1 = 2z^4 - 3z^3 - z^2 + 3z - 1, \quad D_2 = z - 1, \quad D_3 = 1,$$

where the leading coefficient $a_0 = 4$ is omitted as a factor. Hence the greatest multiplicity of the zeros of (19) is $s = 3$. The degrees of the polynomials (19) - (20) are

(21) $$n_0 = 7, \ n_1 = 4, \ n_2 = 1, \ n_3 = 0 \ (n_4 = 0).$$

With the help of (21) the formulas (9) immediately give the number

(22) $$k_1 = n_0 - 2n_1 + n_2 = 0, \quad k_2 = n_1 - 2n_2 + n_3 = 2,$$

$$k_3 = n_2 - 2n_3 + n_4 = 1$$

of the distinct zeros with multiplicities 1, 2, 3 of the polynomial (19), respectively. Since 1 is a simple zero of D_2 from (20), so 1 is a triple zero of (19). If we divide D_1 by D_2 from (20), then we obtain the double zeros -1 and 1/2 of (19). From (22) and (2) we have $Z_1 = 1$, $Z_2 = (z+1)(z-1/2)$, $Z_3 = z - 1$. By (3) the factorization of (19) is $P_7(z) = 4Z_2^2 Z_3^3$, i.e.

$$P_7(z) = 4(z+1)^2(z-1/2)^2(z-1)^3 = (2z^2 + z - 1)^2(z-1)^3.$$

References

[1] L. Ya. O k u n e v, *Vysshaja Algebra*, Moscow (1966) (In Russian).
[2] N. O b r e s h k o v, *Vissha Algebra*, Sofia (1966) (In Bulgarian).
[3] Khr. I. S e m e r d z i e v and S. G. T a b u r o v, Ob odnom methode opredelenija kratnostej nulej algebraicheskih polinomov, *C. R. Acad. Bulg. Sci.* **37**, No 9 (1984), 1143-1145 (In Russian).

[4] Khr. S e m e r d z i e v, Iteration methods for simultaneous finding all roots of generalized polynomial equations, *Math. Balkanica (New Ser.)* **8**, No 4 (1994), 311-335.

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