Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 15, 2001, Fasc. 1-2

Two Curvature Operators in Riemannian and Lorentzian Geometry [1]

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Presented by Bl. Sendov

In the present paper we characterize a four-dimensional Riemannian manifolds (M, g) such that the traces $(R_X)^k$ for $k = 1, 2$ of the Jacobi operator R_X are globally constants on M. In this paper we investigate also the 4-dimensional Einstein Lorentzian manifolds (M, g) for which the characteristic coefficients of the skew-symmetric curvature operator $R(E^2) = R(X, Y, u)$ are pointwise constants at any point p of the manifold M and for any non-degenerated plane $E^2(p; X, Y)$ in the tangent space M_p to M.

AMS Subj. Classification: Primary 53A15; 53C05

Key Words: Jacobi operator, skew-symmetric operator, pointwise eigenvalues, characteristic coefficients

1. Characterization of a four-dimensional globally two-stein Riemannian manifolds

Let (M, g) be an n-dimensional Riemannian manifold with a metric tensor g and a curvature tensor R. *The Jacobi operator* R_X is a symmetric linear operator of the tangent space M_p at a point $p \in M$ defined by $R_X(u) = R(u, X, Y)$, where X belongs to the unit sphere S_pM at p, see [5]. Since X is an eigenvector of R_X with the corresponding eigenvalue 0, then the characteristic equation $\det(R(e_1, X, X, e_1) - cg_{ij}) = 0$ of R_X can be represented in the form $c(c^{n-1} + J_1c^{n-2} + \ldots + J_{n-2}c + J_{n-1}) = 0$ where $c = c(p; X)$ and $J_i = J_i(p; X)$, $i = 1, 2, \ldots, n$.

A manifold (M, g) is called *a pointwise Osserman*, if the eigenvalues of the Jacobi operator R_X are pointwise constants on M, and (M, g) is called a *globally Osserman*, if the eigenvalues of the Jacobi operator R_X are globally constants on M, [1]. Since $J_1(p; X) = \text{trace } R_X = \rho(X)$, where ρ is the Ricci tensor,

[1] Research partially supported under Contract MM 809, NSFI (Bulgaria)

then the trace R_X is a pointwise constant on the manifold (by $\dim M \geq 3$) if and only if (M, g) is an Einstein manifold, [8]. That means that any pointwise (globally) Osserman manifold is an Einstein manifold.

The k-th Ricci curvature ρ^k for all $k = 1, 2, \ldots$ of M is the symmetric covariant tensor field of degree k given by $\rho^k(X, X, \ldots, X) = \mathrm{trace}\,(R_X)^k$, [7]. We say a Riemannian manifold $(M.g)$ is *k-stein space* provided there is a real valued function μ_k on M such that $\rho^k(X) = \mu^{2k}\|X\|$ for all $X \in S_pM$. If this equality does not depend on the point $p \in M$, then we say (M, g) is a *globally k-stein manifold*, else (M, g) is called *a pointwise k-stein manifold*, [7].

Let (M, g) be a four-dimentional pointwise Osserman Riemannian manifold and let p be a fixed point of M. In this case the Jacobi operator R_X has the following characteristic equation: $c(c^3 - J_1c^2 + J_2c - J_3) = 0$. For any tangent vector $X \in S_pM$ for the curvature components with respect to an orthonormal basis $X = X_1$, X_2, X_3, X_4 of eigenvalues of the Jacobi operator R_X, we have [3], [4]:

$$a = K_{12} = K_{34}, \quad b = K_{13} = K_{24}, \quad c = K_{14} = K_{23},$$

$$R_{1223} = R_{1443} = R_{1332} = R_{1442} = R_{1224} = R_{1334} = 0,$$

$$R_{3124} + R_{3214} = K_{13} - K_{14},$$

$$R_{4132} + R_{4312} = K_{14} - K_{12},$$

$$R_{2143} + R_{2431} = K_{12} - K_{13}$$

where $0, a, b, c$ are the corresponding eigenvalues. Following [4], we obtain:

$$R(x, y, z, u) = aR_1(x, y, z, u) + bR_2(x, y, z, u) + cR_3(x, y, z, u), \tag{1.1}$$

where R_1, R_2, R_3 are the Riemannian (1,3) tensors given by

$$\begin{aligned} R_1(x, y, z, u) = \frac{1}{3}(&[g(x,1)g(y,2) - g(x,2)g(y,1)].[g(z,3)g(u,4) - g(z,4)g(u,3)] \\ &-[g(x,1)g(y,3) - g(x,3)g(y,1)].[g(z,2)g(u,4) - g(z,4)g(u,2)] \\ &+[g(x,1)g(y,4) - g(x,4)g(y,1)].[g(z,2)g(u,3) - g(z,3)g(u,2)] \\ &+[g(x,2)g(y,3) - g(x,3)g(y,2)].[g(z,1)g(u,4) - g(z,4)g(u,1)] \\ &-[g(x,2)g(y,4) - g(x,4)g(y,2)].[g(z,1)g(u,3) - g(z,3)g(u,1)] \\ &+[g(x,3)g(y,4) - g(x,4)g(y,3)].[g(z,1)g(u,2) - g(z,2)g(u,1)] \\ &+[g(x,1)g(y,3) - g(x,3)g(y,1) + g(x,4)g(y,2) - g(x,2)g(y,4)] \\ &.[g(z,1)g(u,3) - g(z,3)g(u,1) + g(z,4)g(u,2) - g(z,2)g(u,4)]), \end{aligned}$$

$$R_2(x,y,z,u)=\frac{1}{3}([g(x,1)g(y,2)-g(x,2)g(y,1)].[g(z,3)g(u,4)-g(z,4)g(u,3)]$$
$$-[g(x,1)g(y,3)-g(x,3)g(y,1)].[g(z,2)g(u,4)-g(z,4)g(u,2)]$$
$$+[g(x,1)g(y,4)-g(x,4)g(y,1)].[g(z,2)g(u,3)-g(z,3)g(u,2)]$$
$$+[g(x,2)g(y,3)-g(x,3)g(y,2)].[g(z,1)g(u,4)-g(z,4)g(u,1)]$$
$$-[g(x,2)g(y,4)-g(x,4)g(y,2)].[g(z,1)g(u,3)-g(z,3)g(u,1)]$$
$$+[g(x,3)g(y,4)-g(x,4)g(y,3)].[g(z,1)g(u,2)-g(z,2)g(u,1)]$$
$$.[g(x,1)g(y,4)-g(x,4)g(y,1)+g(x,2)g(y,3)-g(x,3)g(y,2)]),$$

$$R_3(x,y,z,u)=\frac{1}{3}([g(x,1)g(y,2)-g(x,2)g(y,1)].[g(z,3)g(u,4)-g(z,4)g(u,3)]$$
$$-[g(x,1)g(y,3)-g(x,3)g(y,1)].[g(z,2)g(u,4)-g(z,4)g(u,2)]$$
$$+[g(x,1)g(y,4)-g(x,4)g(y,1)].[g(z,2)g(u,3)-g(z,3)g(u,2)]$$
$$+[g(x,2)g(y,3)-g(x,3)g(y,2)].[g(z,1)g(u,4)-g(z,4)g(u,1)]$$
$$-[g(x,2)g(y,4)-g(x,4)g(y,2)].[g(z,1)g(u,3)-g(z,3)g(u,1)]$$
$$+[g(x,3)g(y,4)-g(x,4)g(y,3)].[g(z,1)g(u,2)-g(z,2)g(u,1)]$$
$$+[g(x,3)g(y,4)-g(x,4)g(y,3)+g(x,1)g(y,2)-g(x,2)g(y,1)]$$
$$.[g(z,1)g(u,4)-g(z,4)g(u,1)+g(z,2)g(u,3)-g(z,3)g(u,2)])$$

and where $g(X,i)$ denote $g(X,X_i)$, $i=1,2,3,4$. From this expression we obtain the following theorem.

Theorem 1. *A four-dimensional Osserman manifold (M,g) is a manifold of constant sectional curvature if and only if for the eigenvalues a, b, c of the Jacobi operator R_X we have $a(p)=b(p)=c(p)$ at any point $p\in M$ and they do not depend on p.*

It was proved that *if a four-dimensional Riemannian manifold (M,g) is a pointwise Osserman manifold*, then [3]:

(1.2)
$$Y=\alpha X+\beta X_1+\gamma X_2+\delta X_3,$$
$$Y_1=-\beta X+\alpha X_1-\delta X_2+\gamma X_3,$$
$$Y_2=-\gamma X+\delta X_1+\alpha X_2-\beta X_3,$$
$$Y_3=-\delta X-\gamma X_1+\beta X_2+\gamma X_3,$$

for the eigenvectors X, X_1, X_2, X_3 of the Jacobi operator R_X and for the eigenvectors Y, Y_1, Y_2, Y_3 of the Jacobi operator R_Y, where X, $Y \in S_pM$ and hence

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1. \tag{1.3}$$

It is well known that if (M, g) is a four-dimensional pointwise Osserman manifold, then the eigenvectors of the Jacobi operator R_X are a smooth vector fields suppose defined in a neighborhood U_p at point $p \in M$. Now we denote X_2 by A, X_3 by B and X_4 by C and we denote the corresponding eigenvalues a, b, c which are a smooth functions on U_p.

Theorem 2. ([4]) *Let (M, g) be a pointwise Osserman manifold. Then the eigenvalues a, b, c and the eigenvectors of the Jacobi operator R_X satisfies the following systems:*

$$\begin{aligned} \varphi(c-b) + \psi(c-a) &= 0, \\ \psi(a-c) + \theta(a-b) &= 0, \\ \theta(b-a) + \varphi(b-c) &= 0; \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} X(a) &= (\mu+\nu)a - \nu b - \mu c, \\ X(b) &= -\nu a + (\nu+\lambda)b - \lambda c, \\ X(c) &= -\mu a - \lambda b + (\lambda+\mu)c, \end{aligned} \tag{1.5}$$

where

$$\varphi = g(\nabla_A B, C),\ \psi = g(\nabla_B C, A),\ \theta = g(\nabla_C A, B),$$
$$\lambda = g(\nabla_A A, X),\ \mu = g(\nabla_B B, X),\ \nu = g(\nabla_C C, X).$$

Further we denote by $\alpha(p)$ the matrix of the system (1.5). Evidently, $\mathrm{Rank}\{\alpha(p)\}$ is a pointwise function of M.

Let Ω_1 be the set of all points p on M such that for the eigenvalues of R_X, for any tangent vector $X \in S_p\Omega_1$ we have

$$a(p) = b(p) = c(p). \tag{1.6}$$

According to Theorem 1, (M, g) is a space of constant sectional curvature at p and hence the eigenvalues a, b, c of R_X are globally constants at p. Since trace

$R_X = a$ is a globally constant, then (M, g) is a *globally Osserman at* p. Since R_X is a symmetric linear operator, then Ω_1 is open and dense almost everywhere on M [2] and hence (M, g) is locally a space of constant sectional curvature on Ω_1. Thus we obtain the following lemma.

Lemma 1. *Let (M, g) be a four-dimensional Riemannian manifold and let Ω_1 be the set of all points p on M such that the eigenvalues of Jacobi operator R_X, for any tangent vector $X \in S_p\Omega_1$ are equal at p. Then (M, g) almost everywhere locally in a neigborhood $U_p \subseteq \Omega_1$ is a space of constant sectional curvature on Ω_1 and for the characteristic coefficients of R_X we have*

$$J_1(p; X) = 3a, \quad J_2(p; X) = 3a^2, \quad J_3(p; X) = a^3. \tag{1.7}$$

Let Ω_2 be the set of all points on M with a property at any point $p \in M$ and for any tangent vector $X \in S_p\Omega_2$ the Jacobi operator R_X has two eigenvalues. The set Ω_2 we characterize by a condition $\text{Rank}\{\alpha(p)\} = 2$ and $\alpha_{31}(p)\alpha_{32}(p)\alpha_{33}(p) = 0$. Because R_X is a symmetric linear operator, then Ω_1 is open and dense almost everywhere on M, [2].

Let Ω_3 be the set of all points on M such that at any point $p \in \Omega_3$ and for any tangent vector $X \in S_p\Omega_3$ the Jacobi operator R_X has three different eigenvalues. We have $\text{Rank}\{\alpha(p)\} = 2$ on Ω_3 and also $\alpha_{31}(p)\alpha_{32}(p)\alpha_{33}(p) \neq 0$, $p \in \Omega_3$. This set is also open and dense almost everywhere on M, [2]. Now we transform the system (1.5) defined around a neighborhood U_p at a point $p \in \Omega_3$. Since $\text{Rank}\{\alpha(p)\} = 2$ and $\alpha_{31}(p) = \alpha_{32}(p) = \alpha_{33}(p) \neq 0$, then we can write [6],

$$\frac{\varphi(p; X)}{\alpha_{31}(p)} = \frac{\psi(p; X)}{\alpha_{32}(p)} = \frac{\theta(p; X)}{\alpha_{33}(p)}. \tag{1.8}$$

From the quaternionic transformation (1.2) by condition (1.3) we obtain:

$$\varphi(p; X) = g(\nabla_A B, C), \ \psi(p; X) = g(\nabla_B C, A), \ \theta(p; X) = g(\nabla_C A, B),$$

$$\varphi(p; A) = g(\nabla_X C, B), \ \psi(p; A) = g(\nabla_C B, X), \ \theta(p; A) = g(\nabla_B X, C),$$

$$\varphi(p; B) = g(\nabla_C X, A), \ \psi(p; B) = g(\nabla_X A, C), \ \theta(p; B) = g(\nabla_X C, B),$$

$$\varphi(p; C) = g(\nabla_B A, X), \ \psi(p; C) = g(\nabla_A X, B), \ \theta(p; C) = g(\nabla_X B, A),$$

and

$$\begin{aligned} \varphi(p; aX + bA) &= -a^3 g(\nabla_A B, C) + a^2 b g(\nabla_X B, C) \\ &+ab^2(g(\nabla_A B, C) - g(\nabla_X B, C)) - b^3 g(\nabla_X B, C), \end{aligned} \tag{1.9}$$

$$\psi(p;aX+bY)=a^3g(\nabla_B C,A)+b^3g(\nabla_C B,X)$$
$$+a^2b(g(\nabla_B B,A)-g(\nabla_B C,X)-g(\nabla_C C,A))$$
$$+ab^2(g(\nabla_C C,X)-g(\nabla_C B,A)-g(\nabla_B B,X)),$$
$$\theta(p;aX+bA)=a^3g(\nabla_C,A)+a^2b(g(\nabla_B A,B)-g(\nabla_C X,B)+g(\nabla_C C,A))$$
$$+ab^2((g(\nabla_B A,B)-g(\nabla_B A,C)-g(\nabla_C C,X))+b^3g(\nabla_B X,C).$$

Using (1.8) after a substitutions of X by A, B, C and having in mind (1.9) we obtain

(1.10)
$$\frac{g(\nabla_A B,C)}{\alpha_{31}(p)}=\frac{g(\nabla_B C,A)}{\alpha_{32}(p)}=\frac{g(\nabla_C A,B)}{\alpha_{33}(p)},$$
$$\frac{g(\nabla_X C,B)}{\alpha_{31}(p)}=\frac{g(\nabla_C B,X)}{\alpha_{32}(p)}=\frac{g(\nabla_B X,C)}{\alpha_{33}(p)},$$
$$\frac{g(\nabla_C X,A)}{\alpha_{31}(p)}=\frac{g(\nabla_X A,C)}{\alpha_{32}(p)}=\frac{g(\nabla_A C,X)}{\alpha_{33}(p)},$$
$$\frac{g(\nabla_B A,X)}{\alpha_{31}(p)}=\frac{g(\nabla_A X,B)}{\alpha_{32}(p)}=\frac{g(\nabla_X B,A)}{\alpha_{33}(p)},$$

where $\alpha_{ij}(p)(i,j=1,2,3,4)$ are the minors of $\alpha(p)$. Further we apply (1.8) for a tangent vector $aX+bY$ where a and b are arbitrary real numbers such that $a^2+b^2=1$. According to (1.9) and (1.10) we obtain

$$a^2b(\alpha_{32}(p)g(\nabla_X C,B)-\alpha_{31}(p)(g(\nabla_B B,A)-g(\nabla_C C,A)-g(\nabla_B C,X))$$
$$+ab^2(\alpha_{32}(p)g(\nabla_A B,C)-\alpha_{31}(p)(g(\nabla_C C,X)-g(\nabla_B B,X)-g(\nabla_C A,B))=0.$$

From this equality and (1.8) according to the remarks above we have:

$$\alpha_{31}(p)(\psi-\theta-\nu+\mu)=0.$$

Applying (1.8) for the tangent vector $aX+bB$ and $aX+bC$ we obtain

$$\alpha_{32}(p)(\psi-\varphi+\nu-\lambda)=0,$$
$$\alpha_{33}(p)(\theta-\psi+\lambda-\mu)=0$$

and hence we have the system:

$$\alpha_{31}(p)(\varphi-\theta-\nu+\mu)=0,$$
$$\alpha_{32}(p)(\psi-\varphi+\nu-\lambda)=0,$$

$$\alpha_{33}(p)(\theta - \psi + \lambda - \mu) = 0.$$

Since the minors $\alpha_{31}(p)$, $\alpha_{32}(p)$, $\alpha_{33}(p)$ are different from zero, then we have the following system:

$$\varphi - \theta - \nu + \mu = 0, \tag{1.11}$$

$$\psi - \varphi + \nu - \lambda = 0,$$

$$\theta - \psi + \lambda - \mu = 0.$$

Let us consider at first the equality

$$\varphi - \theta - \nu + \mu = 0,$$

or

$$g(\nabla_A B, C) - g(\nabla_C A, B) = g(\nabla_C C, X) - g(\nabla_B B, X).$$

Changing in this equality X by $aX + bA$ and using the transformation (1.2) we obtain:

$$a^3(g(\nabla_A B, C) - g(\nabla_C X, B) - g(\nabla_C C, X) - g(\nabla_B B, X))$$

$$+b^3(g(\nabla_X C, B) - g(\nabla_B A, C) - g(\nabla_B B, A) - g(\nabla_C C, A))$$

$$+a^2 b(-g(\nabla_X B, C) - g(\nabla_B X, B) + g(\nabla_C X, C) - g(\nabla_C A, B)$$

$$-g(\nabla_C C, X) - g(\nabla_C B, X) - g(\nabla_C C, A) \tag{1.12}$$

$$+g(\nabla_B B, A) - g(\nabla_B C, X) - g(\nabla_B B, X))$$

$$+ab^2(-g(\nabla_A C, B) + g(\nabla_B X, C) - g(\nabla_B A, B) - g(\nabla_C A, C)$$

$$-g(\nabla_B B, X) - g(\nabla_B C, A) - g(\nabla_C B, A) - g(\nabla_B C, B)$$

$$g(\nabla_C B, A) - g(\nabla_C C, X)) = 0.$$

From here we get

$$g(\nabla_A B, C) - g(\nabla_C A, B) - g(\nabla_C C, X) + g(\nabla_B B, X) = 0, \tag{1.13}$$

$$g(\nabla_X C, B) - g(\nabla_B A, C) - g(\nabla_B B, A) - g(\nabla_C C, A) = 0.$$

Using (1.2) we can check that these equalities are equivalent. From (1.12) we have also

$$g(\nabla_X B, C) + g(\nabla_B B, X) - g(\nabla_C C, X) + g(\nabla_C A, B) \tag{1.14}$$

$$-2g(\nabla_B C, X) - 2g(\nabla_C B, X) - g(\nabla_C C, A) + g(\nabla_B B, A) = 0,$$

$$g(\nabla_X C, B) - g(\nabla_B X, C) + g(\nabla_B B, A) - g(\nabla_C C, A)$$

$$-2g(\nabla_B C, A) - 2g(\nabla_C B, A) - g(\nabla_C C, X) - g(\nabla_B B, X) = 0,$$

and using (1.2), we can see also that these equalities are equivalent. Thus from (1.13) and (1.14) we obtain the system

$$g(\nabla_A B, C) - g(\nabla_C X, B) - g(\nabla_C C, X) - g(\nabla_B B, X) = 0. \tag{1.15}$$

$$-g(\nabla_X B, C) + g(\nabla_B B, X) - g(\nabla_C C, X) - g(\nabla_C A, B)$$

$$-2g(\nabla_B C, X) - 2g(\nabla_c B, X) - g(\nabla_C C, A) + g(\nabla_B B, A) = 0,$$

and from here we have

$$g(\nabla_C A, B) - g(\nabla_C X, B) = 0. \tag{1.16}$$

From this system, after replacing X by $aX + bA$ and using (1.2), we obtain:

$$a^3(g(\nabla_C X, B) - g(\nabla_C B, A)) + b^3(g(\nabla_B X, C) + g(\nabla_B A, C))$$

$$+a^2b(-g(\nabla_B B, X) + g(\nabla_B B, A) + g(\nabla_C C, A) + g(\nabla_C A, B)$$

$$+g(\nabla_C X, B) - g(\nabla_C C, A)) + ab^2(-g(\nabla_B B, X) - g(\nabla_B X, C)$$

$$-g(\nabla_B B, A) + g(\nabla_B A, C) + g(\nabla_C X, X) - g(\nabla_C C, A)) = 0.$$

We sum the coefficients before a^2b and ab^2 which are vanishing and obtain the equality:

$$-2g(\nabla_B B, X) + g(\nabla_C C, X) + g(\nabla_C A, B)$$

$$+g(\nabla_C X, B) - g(\nabla_B X, C) + g(\nabla_B A, C) = 0.$$

Since the coefficients before a^3 and b^3 are vanishing also, then

$$g(\nabla_C X, B) + g(\nabla_C A, B) = 0,$$

$$g(\nabla_B X, C) + g(\nabla_B A, C) = 0$$

and from here we have

$$-\mu + \nu + \theta - \varphi = 0.$$

From the results above, we have

$$-\mu + \nu + \theta - \psi = 0,$$

then after summing of the last two equalities we obtain $\varphi = \psi$. Analogously, changing in (1.16) X by $aX + bB$ and having in mind (1.2), we obtain $\varphi = \theta$. Finally we have $\varphi = \psi = \theta$ and then the system (1.4) has the form:

$$\begin{aligned} \varphi(2a - b - c) &= 0, \\ \psi(2b - c - a) &= 0, \\ \theta(2c - a - b) &= 0. \end{aligned} \tag{1.17}$$

If $\varphi(p; X) \neq 0$, then we obtain (1.6) which is not possible when $p \in \Omega_3$, hence $\varphi(p; X) = 0$. Then $\varphi = \psi = \theta$ and the system (1.5) has the form:

$$\begin{aligned} X(a) &= \lambda(2a - b - c), \\ X(b) &= \lambda(2b - c - a), \\ X(c) &= \lambda(2c - a - b), \end{aligned} \tag{1.18}$$

for any tangent vector $X \in S_p\Omega_3$.

Now we can prove the main result in the present paper.

Theorem 3. *Let (M, g) be a four-dimensional Riemannian manifold. Then (M, g) almost everywhere locally is a globally Osserman manifold if and only if (M, g) is a globally two-stein manifold.*

Proof. Let (M, g) be a globally Osserman Riemannian manifold. Then all characteristic coefficients J_k $(k = 1, 2, 3)$ of Jacobi operator R_X are globally constants on M and because of $\text{trace}(R_X)^2 = J_1^2 - 2J_2$, then (M, g) is a globally two-stein manifold.

If (M, g) is a globally two-stein, then trace $R_X = J_1(p; X)$ is a globally constant on M and (M, g) is an Einstein. Also $J_2(p; X)$ is a globally constant on M either and then from (1.5) we get

$$X(J_2) = \nu(a - b)^2 + \mu(a - c)^2 + \lambda(b - c)^2, \tag{1.19}$$

which holds at any point $p \in M$.

Let $p \in \Omega_1$. Then according to Lemma 1 and (1.7) we have that (M, g) is a globally Osserman manifold.

If $p \in \Omega_2$, then two of the eigenvalues of the Jacobi operator are equal, say $b(p) = c(p)$, and

$$a(p) \neq b(p) = c(p). \tag{1.20}$$

Then the system (1.5) has the form

$$X(a) = (\mu + \nu)(a - b), \tag{1.21}$$

$$X(b) = \mu(b - a) = \nu(b - a),$$

and from here according to (1.20) we obtain

$$\mu(p; X) = \nu(p; X). \tag{1.22}$$

Suppose $J_2(p; X)$ is globally constant on the manifold. Then $X(J_2) = 0$ and from (1.19), and (1.20) it follows that:

$$(\mu + \nu)(a - b)^2 = 0.$$

Since $a(p) \neq b(p)$, then from the last equality we have:

$$\nu(p; X) + \mu(p; X) = 0. \tag{1.23}$$

Now from (1.22) and (1.23) we have

$$\nu(p; X) = \mu(p; X) = 0,$$

for any tangent vector $X \in S_p\Omega_2$. From here and (1.21) we obtain $X(a) = X(b) = 0$, which means that a and b are globally constants on M.

If $p \in \Omega_3$, then the eigenvalues of Jacobi operator R_X are different at a point p, i.e. $a(p) \neq b(p) \neq c(p)$ and then the system (1.5) has the form (1.18). If J_2 is a globally constant then $X(J_2) = 0$ and from (1.19) we obtain

$$\lambda((a - b)^2 + (a - c)^2 + (b - c)^2) = 0. \tag{1.24}$$

In the case $\lambda(p; X) = \mu(p; X) = \nu(p; X) = 0$ from (1.18) we have $X(a) = X(b) = X(c) = 0$ which means a, b, c are globally constants on Ω_3.

If $\lambda(p; X) = \mu(p; X) = \nu(p; X) \neq 0$, then from (1.24) we have $a(p) = b(p) = c(p)$ which is not possible when $p \in \Omega_3$.

Thus we prove that (M, g) is a globally Osserman manifold and hence that (M, g) is a locally rank-one symmetric space [8]. ■

2. Characterization of a four-dimensional Einstein Lorentzian manifolds using a skew-symmetric curvature operator

An n-dimensional Riemannian manifold (M, g) is called a Lorentzian manifold, if at any point $p \in M$ the tangent space M_p to M is an n-dimensional vector space of signature $(-,+,\ldots,+)$ or $(+,-,\ldots,-)$. The set of all unit spacelike (time-like) tangent vectors in M_p we denote by ${}^{+}S_pM({}^{-}S_pM)$. A manifold (M,g) is called an Einstein manifold, if $\rho = \lambda g$, where λ is a constant and ρ is the Ricci tensor. For this class of manifolds we have the following result of A.Z. Petrov [9].

Theorem A. ([9]) *If (M,g) is a four-dimensional Einstein Lorentzian manifold, then at any point $p \in M$ there exist an orthonormal Lorentzian basis e_1, e_2, e_3, e_4 ($e_4 \in {}^{-}S_pM$) with respect to which all curvature components are defined by one of the following formulas:*

(2.1)
$$R_{1212} = -R_{3434} = \alpha_1,$$
$$R_{1313} = -R_{2424} = \alpha_2,$$
$$R_{2323} = -R_{1414} = \alpha_3;$$

(2.2)
$$R_{1212} = -R_{3434} = \alpha_1,$$
$$R_{1313} = -R_{2424} = \alpha_2 + 1,$$
$$R_{2323} = -R_{1414} = \alpha_2 - 1,$$
$$R_{3114} = -R_{3224} = 1;$$

(2.3)
$$R_{1212} = -R_{3434} = R_{1313} = -R_{2424} = R_{2323} = -R_{1414} = \alpha,$$
$$R_{3114} = -R_{3224} = 1,$$
$$R_{2443} = -R_{2113} = 1.$$

The skew symmetric curvature of the tangent space M_p is defined by $R(E^2)(u) = R(X,Y,u)$, where $E^2 = E^2(p;X,Y)$ is an arbitrary plane of M_p spanned from the tangent vectors X, Y. It is easy to see that $R(E^2)$ does not depend on the orthonormal basis X, Y in the plane $E^2(p;X,Y)$. This operator has been defined by G. Stanilov [4] and the problem of a pointwise constancy was stated at the begining by him. At first, this problem was resolved in the Einstein Riemannian setting by G. Stanilov and R. Ivanova [10] who proved that the *curvature operator $R(E^2)$ has a pointwise constants eigenvalues on the four-dimensional Riemannian manifold (M,g) if and only if (M,g) is a space of constant sectional curvature*, [10]. The manifolds where the skew-symmetric curvature operator $R(E^2)$ has a pointwise constant eigenvalues were referred to by P.B. Gilkey as *IP (Ivanov, Petrova) manifolds* because at first S. Ivanov and I. Petrova proved in [11] the following theorem.

Theorem A. *Let (M,g) be a four-dimensional Riemannian manifold such that the eigenvalues of the skew-symmetric curvature operator $R(E^2)$ are pointwise constants at any point p of manifold. Then (M,g) is locally (almost everywhere) isometric to one of the following spaces:*

a) a real space form;

b) a wraped product $B \times_f N$, where B is 1-dimensional space, N is 3-dimensional space form of constant sectional curvature K and f is a smooth function on B given by

$$f(x) = \sqrt{Kx^2 + Cx + D},$$

where K, C, D are constants such that $C^2 - 4KD \neq 0$.

This classification was extended by Gilkey, Leahy and Sadofsky [12] to the dimension $m = 5$, $m = 6$, and $m \geq 9$; subsequently the case $m = 8$ was dealt with in [14]. We summarize those results as follows.

Theorem B. *Let (M,g) be a Riemannian manifold of dimension $m \geq 5$ and $m \neq 7$ such that (M,g) is IP. Then either (M,g) has constant sectional curvature, or (M,g) is locally isometric to a wraped product of the form*

$$ds^2 = dt^2 + f(t)ds_N^2 \qquad on \qquad (t_0, t_1) \times N,$$

where $f(t) := (Kt^2 + At + B)/2 > 0$ and where ds_N^2 has a constant sectional curvature K.

The case $\dim M = 7$ was resolved from P. Gilkey and U. Simelman in [13].

In the present note we continue this investigation on the four-dimensional Einstein Lorentzian setting. We recall IP-manifolds as *a pointwise Stanilov manifolds* and we will generalize this definition in the end of paper. Since in this setting the curvature operator is not always diagonalizable, then we say a Lorentzian manifold (M,g) is *pointwise Stanilov manifold, if the characteristic coefficients of a skew-symmetric curvature operator $R(E^2)$ are a pointwise constants at any point p of M* (the same definition can be used in the Riemannian geometry).

Theorem C. *Let (M,g) be a four-dimensional Einstein Lorentzian manifold. Then the following conditions are equivalent:*

a) (M,g) is a pointwise Stanilov manifold,

b) (M,g) is a space of constant sectional curvature (in particular, (M,g) is flat) or (M,g) is a Petrov's space of maximal mobility with a metric given in a special coordinate system by [9]:

$$ds^2 = dx_1^2 + \sinh^2(x_1 - x_4)dx_2^2 + \sin^2(x_1 - x_4)dx_3^2 - dx_4^2.$$

Proof. We prove only the implication $a) \Rightarrow b)$ because the implication $b) \Rightarrow a)$ is trivial. Let (M,g) be a four-dimensional Einstein Lorentzian manifold, then at any point $p \in M$ for the curvature components with respect to the invariants of the Petrov basis $e_1, e_2, e_3, e_4, (e_4 \in {}^-S_pM)$ in the tangent space M_p we have one of the following possibilities:

$$\alpha_1 = \alpha_2 = \alpha_3, \quad \beta_1 = \beta_2 = \beta_3 = 0; \tag{2.4}$$

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0; \tag{2.5}$$

$$\alpha_1 = \varepsilon\frac{1}{2}, \quad \alpha_2 = \lambda - \varepsilon\frac{1}{2}, \quad \varepsilon = \pm 1, \quad \beta_1 = \mp\sqrt{\frac{1}{12}}, \quad \beta_2 = \pm\sqrt{\frac{1}{12}}; \tag{2.6}$$

$$\alpha = 0. \tag{2.7}$$

It is easy to check that the characteristic equation of the skew-symmetric curvature operator $R(E^2)$ with respect to any orthonormal Lorentzian basis $e_1, e_2, e_3, e_4, (e_4 \in {}^-S_pM)$ in M_p has the form:

$$\mu^4 - I_2\mu - I_4 = 0,$$

where for the characteristically coefficients of $R(E^2)$ we have:

$$I_2(p;X,Y) = R^2_{XY12} + R^2_{XY13} + R^2_{XY23} - R^2_{XY14} - R^2_{XY24} - R^2_{XY34}, \tag{2.8}$$

$$I_4(p;X,Y) = -(R_{XY12}.R_{XY34} - R_{XY13}.R_{XY24} + R_{XY23}.R_{XY14})^2.$$

Assume that e_1, e_2, e_3, e_4, ($e_4 \in {}^{-}S_pM$) is Petrov basis in M_p at a point $p \in M$ and suppose we have (2.1). Then from (2.8) we get:

$$I_2(p;e_1,e_2) = -I_2(p;e_3,e_4) = \alpha_1^2 - \beta_1^2, \tag{2.9}$$

$$I_2(p;e_1,e_3) = -I_2(p;e_2,e_4) = \alpha_2^2 - \beta_2^2,$$

$$I_2(p;e_2,e_3) = -I_2(p;e_1,e_4) = \alpha_3^2 - \beta_3^2;$$

$$I_4(p;e_1,e_2) = -I_4(p;e_3,e_4) = -\alpha_1^2\beta_1^2, \tag{2.10}$$

$$I_4(p;e_1,e_3) = -I_4(p;e_2,e_4) = -\alpha_2^2\beta_2^2.$$

$$I_4(p;e_2,e_3) = -I_4(p;e_1,e_4) = -\alpha_3^2\beta_3^2,$$

Hence the conditions

$$I_k(p;e_1,e_2) = I_k(p;e_1,e_3) = I_k(p;e_2,e_3), \tag{2.11}$$

and

$$I_k(p;e_1,e_4) = I_k(p;e_2,e_4) = I_k(p;e_3,e_4), \qquad (k=2,4) \tag{2.12}$$

coincide.

Let $a = \alpha_1^2$, $b = \beta_1^2$ be the roots of the equation $x^2 + px + q$ and let $c = \alpha_2^2$ and $d = \beta_2^2$ be the roots of the equation $y^2 + ty + f = 0$. From (2.11) in the case $k = 4$ we obtain:

$$\alpha_1^2\beta_1^2 = \alpha_2^2\beta_2^2 = \alpha_3^2\beta_3^2,$$

hence $q = f$ and

$$a - b = \alpha_1^2 - \beta_1^2 = -\sqrt{p^2 - 4q},$$

$$c - d = \alpha_2^2 - \beta_2^2 = -\sqrt{t^2 - 4q}.$$

From these equalities and (2.11) by $k = 2$ we get $p = \pm t$ and hence

$$\alpha_1^2 + \beta_1^2 = \varepsilon_1(\alpha_2^2 + \beta_2^2),$$

analogously we obtain

$$\alpha_1^2 + \beta_1^2 = \varepsilon_2(\alpha_3^2 + \beta_3^2),$$

where $\varepsilon_s = \pm 1$ $(s = 1, 2)$. From this relation in the case $\varepsilon_1 = -1$ we get

$$\alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2 = 0$$

and then

$$\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0.$$

In this case $\alpha_3^2 + \beta_3^2 = 0$ hence $\alpha_3 = \beta_3 = 0$ which means that (M, g) is flat. The same results can be obtained in the case $\varepsilon_2 = -1$. Let $\varepsilon_1 = \varepsilon_2 = 1$. Then

$$\alpha_1^2 + \beta_1^2 = \alpha_2^2 + \beta_2^2 = \alpha_3^2 + \beta_3^2.$$

Now from (2.11) when $k = 2$ it follows that

$$\alpha_1^2 - \beta_1^2 = \alpha_2^2 - \beta_2^2 = \alpha_3^2 - \beta_3^2.$$

From the last two relations we have

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2, \qquad \beta_1^2 = \beta_2^2 = \beta_3^2.$$

Now using the first Bianci identity, we get $\beta_i = 0$ $(i = 1, 2, 3)$. From here and $\alpha_1^2 = \alpha_2^2 = \alpha_3^2$ we have that (2.4) holds or we have the relations:

$$\alpha_1 = \alpha_2 = -\alpha_3,$$

$$\beta_1 = \beta_2 = \beta_3 = 0.$$

If (2.4) holds, then (M, g) is a space of constant sectional curvature at a point p. In the case

$$\alpha_1 = \alpha_2 = -\alpha_3, \quad \beta_1 = \beta_2 = \beta_3 = 0,$$

for the curvature tensor R we have a representation:

$$R(x, y, z, u) = ((g_{x3}g_{y4} - g_{x4}g_{y3}) - (g_{z4}g_{u3} - g_{z3}g_{u4})$$

$$-(g_{x1}g_{y2} - g_{x2}g_{y1})(g_{z2}g_{u1} - g_{z1}g_{u2}) + (g_{x2}g_{y4} - g_{x4}g_{y2})(g_{z4}g_{u2} - g_{z2}g_{u4})$$

$$-(g_{x1}g_{y3} - g_{x3}g_{y1})(g_{z3}g_{u1} - g_{z1}g_{u3}) + (g_{x1}g_{y4} - g_{x4}g_{y1})(g_{z4}g_{u1} - g_{z1}g_{u4})$$

$$-(g_{x2}g_{y3} - g_{x3}g_{y2})(g_{z3}g_{u2} - g_{z2}g_{u3}))\alpha_1.$$

From here it follows that $K_{14} = 9\alpha_1$. Since from (2.1) we have $K_{14} = \alpha_1$, then $\alpha_1 = 0$ and hence (M, g) is flat at p.

Suppose we have (2.2). Then from (2.2) and (2.11) it follows that

$$I_2(p; e_1, e_2) = -I_2(p; e_3, e_4) = \alpha_1^2 - \beta_1^2, \tag{2.13}$$

$$I_4(p; e_1, e_2) = -I_4(p; e_3, e_4) = -\alpha_1^2\beta_1^2;$$

$$I_2(p;e_1,e_3) = -I_2(p;e_2,e_4) = (\alpha_2+1)^2 - \beta_2^2 - 1,$$
$$I_4(p;e_1,e_3) = -I_4(p;e_2,e_4) = -\beta_2^2(\alpha_2+1)^2;$$

$$I_2(p;e_2,e_3) = -I_2(p;e_1,e_4) = (\alpha_2-1)^2 - \beta_2^2 - 1,$$
$$I_4(p;e_2,e_3) = -I_4(p;e_1,e_4) = -\beta_2^2(\alpha_2-1)^2,$$

which means that the conditions (2.11) and (2.12) coincide again. Now from $I_2(p;e_1,e_3) = I_2(p;e_2,e_3)$ we obtain $(\alpha_2+1)^2 = (\alpha_2-1)^2$ and hence $\alpha_2 = 0$. Then from the condition $I_4(p;e_1,e_2) = I_4(p;e_1,e_3)$ we get $\alpha_1^2\beta_1^2 = \beta_2^2$. Since $\beta_1 + 2\beta_2 = 0$, then $(4\alpha_1^2 - 1).\beta_2^2 = 0$.

If $\beta_2 = 0$, then $\beta_1 = 0$ and from the assumption $I_2(p;e_1,e_2) = I_2(p;e_1,e_3)$ it follows that $\alpha_1 = 0$. Hence we have (2.5) which means that all invariants of Petrov basis are independent from p. In the case

$$\alpha_1 = \varepsilon\frac{1}{2}, \quad \alpha_2 = \lambda - \varepsilon\frac{1}{2}, \quad \beta_1 = -\varepsilon\sqrt{\frac{1}{12}}, \quad \beta_2 = \varepsilon\sqrt{\frac{1}{12}}, \qquad \varepsilon = \pm 1;$$

we have that all invariants of Petrov basis are independent from p either.

Let us assume that (2.3) are satisfied. From these formulas and (2.11) we can obtain the following characteristically coefficients:

$$I_2(p;e_1,e_2) = -I_2(p;e_3,e_4) = \alpha^2 + 1;$$
$$I_2(p;e_1,e_3) = -I_2(p;e_2,e_4) = \alpha^2 - 1;$$
$$I_2(p;e_2,e_3) = -I_2(p;e_1,e_4) = \alpha^2.$$

From here and the condition (2.11) in the case $k = 2$ we get (2.7). ∎

Thus, if (M,g) is a four-dimensional Einstein Lorentzian *pointwise Stanilov* manifold, then at any point $p \in M$ for the curvature components with respect to Petrov basis e_1, e_2, e_3, e_4 ($e_4 \in {}^-S_pM$) in the tangent space M_p we have one of the possibilities (2.4) - (2.7). Obviously, the curvature operator $R(E^2)$ has non-zero characteristic coefficients in the cases (2.4), (2.6) and (2.7). The results of A.Z. Petrov in [9] show that the cases (2.6) and (2.7) are not possible and hence we have only the case (2.4). In this case, (M,g) is a space of constant sectional curvature, or (M,g) is a reducible space - the second case is not possible when (M,g) is non-flat.

In the case (2.5) the characteristic coefficients of $R(E^2)$ are vanishing on the manifold. In this case

$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 0,$$

which is possible if and only if (M, g) is a symmetric Einstein Lorentzian manifold such that at any point $p \in M$ for the invariants of Petrov basis we have (2.2). A.Z. Petrov proved that in this case (M, g) is a space of maximal mobility (with a metric given above) and these manifolds are founded as a solutions of the Einstein equation in Relativity, [15].

Finally, we remark that the term *pointwise Stanilov maniflolds* in the Riemannian setting can be generalized by the following definition:

Let (M, g) be an n-dimensional Riemannian manifold, let E^k be an arbitrary k-dimensional subspace of the tangent space M_p, let e_1, e_2, ..., e_k be an arbitrary orthonormal basis in E^k and let $S(E^k)$ be the linear symmetric operator of the tangent space M_p defined by

$$S(E^k)(u) = S(e_1, e_2, \dots, e_k)(u) = \sum_{i<j} R(e_i, e_j, R(e_i, e_j)u).$$

We say that (M, g) is a pointwise Stanilov manifold of order k, if the eigenvalues of the curvature operator $S(E^k)$ are pointwise constants at any point $p \in M$.

Evidently in the case $\dim M = 4$ it is possible to exist only pointwise Stanilov manifolds of order 3 and a pointwise Stanilov manifolds of order 2; the last class of manifolds was described above, by Theorem A. In a next paper we will prove the following (see [16]):

Theorem D. *Let (M, g) be a four-dimensional Riemannian manifold. Then (M, g) is a pointwise Stanilov manifold of order 3 if and only if (M, g) is a pointwise Stanilov manifold of order 2.*

Acknowledgment. The author would like to thank Professor G. Stanilov for the direction and for the help during the preparation of this paper.

References

[1] P. G i l k e y, A. S w a n n, L. v a n H e c k e, Isoparametric geodesic spheres and a conjecture of Osserman regarding the Jacobi operator, *Jour. Quart. Math.* **46** (1995), 299-320.

[2] T. K a t o, *Perturbation Theory for Linear Operators.* Springer-Verlag, Berlin (1996).

[3] K. S e k i g a w a, L. V a n h e c k e, Volume preserving geodesic symmetries of four-dimensional 2-stein spaces, *Kodai Math. J.* **9** (1986), 215-224.

[4] Gr. S t a n i l o v, On the geometry of the Jacobi operators on 4-dimensional Riemannian manifolds. *Tensor, N. S.* **51** (1992), 9-15.

[5] R. O s s e r m a n, Curvature in the eighties, *Jour. Amer. Math. Monthly* **97** (1990), 731-756.

[6] A. G. K u r o s h, *Course of Higher Algebra*, Nauka, Moscow (1959) (In Russian).

[7] T. W i l l m o r e, P. C a r p e n t e r, A. G r a y, The curvature of Einstein spaces, *Quart. J. Math. Oxford* **(20), 33** (1982), 45-64.

[8] C h i - Q - S h i, Curvature characterisation and classification of the rank-one symmetric spaces, *Pacific J. Math.* **150** (1991), 31-42.

[9] A. Z. P e t r o v, *Einstein Spaces*, GIFML, Moscow (1961).

[10] R. I v a n o v a, Gr. S t a n i l o v, Skew-symmetric curvature operator in Riemannian geometry, In: *Second Gauss-Symposium*, Munchen (1993).

[11] S. I v a n o v, I. P e t r o v a, Riemannian manifold in which the skew-symmetric curvature operator has a pointwise constant eigenvalues, *Geometrie Dedicata* **70** (1998), 262-282.

[12] P. G i l k e y, J. L e a h y, H. S a d o f s k y, Riemannian manifolds whose skew-symmetric curvature operator has constant eigenvalues, *Preprint* (1997).

[13] P. G i l k e y, U. S i m e l m a n, Spinors, self-duality, and IP algebraic curvature tensors, *Preprint* (1998).

[14] P. G i l k e y, *Riemannian manifolds whose skew-symmetric curvature operator has constant eigenvalue: II, Preprint* (1998).

[15] A. E i n s t e i n, *Zur allgemeinen Relativitaetstheorie*, Berlin Berichte (1915).

[16] V. V i d e v, J. C a n k o v, *A generalization of IP-comjecture in the four-dimensional Riemannian geometry, Preprint* (1999).

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Received: 17.06.1999