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The Order of Approximation by Certain Linear Positive Operators

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The order of approximation of continuous functions with polynomial growth at infinity by the Szasz operators and its generalizations are established.

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1. Introduction

Approximations of continuous functions by Szasz operators were investigated by many authors. We refer to papers [1]-[5]. This paper is also devoted to this problem. Our main result connects with the problem of estimate of difference $S_n(f;x) - f(x)$ by the modulus of continuity of function f, which is continuous on $[0,\infty)$ and has a polynomial grows at infinity.

Let m be a fixed natural number and $C_{x^{2m}}(0,\infty)$ be the space of all continuous functions on the real semi-axis for which the estimate

$$(1) \qquad |f(x)| \leq M_f(1+x^{2m}), \quad x \in [0,\infty)$$

holds, where M_f is constant, depending on every function f only. For any positive d we denote by $\omega_d(f,\delta)$ the modulus of continuity of function f on closed interval $[0,d]$, that is

$$\omega_d(f,\delta) = \sup\left\{|f(t)-f(x)| : \ t,x \in [0,d], \ \ |t-x| \leq \delta\right\}.$$

Obviously, for any d the modulus of continuity $\omega_d(f,\delta)$ of functions $f \in C_{x^{2m}}(0,\infty)$ tends to zero as $\delta \to 0$, since such a function is uniformly continuous on $[0,d]$.

For the Szasz operator S_n, defined by

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right) \quad x \geq 0, \tag{2}$$

M. Müller [5] and T. Hermann [3], respectively, obtained the following inequalities

$$|S_n(f;x) - f(x)| \leq (1+\sqrt{a})\,\omega_\infty\left(f, \frac{1}{\sqrt{n}}\right), \qquad 0 \leq x \leq a, \tag{3}$$

$$|S_n(f;x) - f(x)| \leq (1+\sqrt{x})\,\omega_\infty\left(f, \frac{1}{\sqrt{n}}\right), \qquad 0 \leq x < \infty, \tag{4}$$

where $\omega_\infty(f,\delta_n)$ is the modulus of continuity of function f on infinite interval $[0,\infty)$.

It is well-known that the modulus of continuity $\omega(f,\delta)$ of function f on some interval tends to zero as $\delta \to 0$ if and only if f is an uniformly continuous function on this interval. Therefore, modulus of continuity $\omega_\infty\left(f, \frac{1}{\sqrt{n}}\right)$ in (3) and (4) must not tends to zero as $n \to \infty$ if f is not uniformly continuous on all positive semi axis. Moreover, this modulus of continuity may be even unbounded. Therefore (3) and (4), in general, not gave any information about an order of approximation of functions by Szasz operators.

The order of approximation for Szasz operator by the modulus of continuity was firstly established in [4] as

$$\|S_n(f;x) - f(x)\|_{C[0,a]} \leq C_f(1+a)^2\,\omega_{a+1}\left(f, \sqrt{\tfrac{a}{n}}\right), \tag{5}$$

where $f \in C_{x^2}(0,\infty)$ and C_f is constant depending on f.

The aim of this paper is the generalization of this estimate for functions, belonging to $C_{x^{2m}}(0,\infty)$. Moreover, we shall consider also some generalizations of operators (2) in Lipschitz classes $Lip_M\gamma$.

Lemma 1. *For any natural ν,*

$$S_n(t^\nu;x) = x^\nu + \frac{b_{1,\nu}}{n}x^{\nu-1} + \frac{b_{2,\nu}}{n^2}x^{\nu-2} + \ldots + \frac{b_{\nu-1,\nu}}{n^{\nu-1}}x, \tag{6}$$

where $b_{k,\nu}$, $k = 1,2,\ldots$ are the constants, such that

$$b_{k,\nu} = \begin{cases} 0\,, & k \geq \nu \\ \neq 0\,, & k < \nu \end{cases}. \tag{7}$$

Proof. Since for $\nu = 1$ (6) is true, we can use the method of induction.

Suppose that (6) holds for any natural $\nu \leq m$.

Choosing the constants $a_1, a_2, ..., a_m$ from the equality

$$\left(\frac{k}{n}\right)^{m+1} = \frac{k(k-1)...(k-m)}{n^{m+1}} + \sum_{j=1}^{m} \left(\frac{k}{n}\right)^j a_j \frac{1}{n^{m+1-j}}$$

and using (6), we can found $S_n(t^{m+1}; x)$ by the following formulas:

$$S_n(t^{m+1}; x) = \frac{e^{-nx}}{n^{m+1}} \sum_{k=m+1}^{\infty} \frac{(nx)^k}{(k-m-1)!} + \sum_{j=1}^{m} a_j \frac{1}{n^{m-j+1}} S_n(t^j; x)$$
$$= \frac{n^{m+1}x^{m+1}}{n^{m+1}} e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} + \sum_{j=1}^{m} a_j \frac{1}{n^{m-j+1}} S_n(t^j; x)$$
$$= x^{m+1} + \sum_{j=1}^{m} a_j \frac{1}{n^{m-j+1}} \left\{ x^j + \frac{b_{1,j}}{n} x^{j-1} + \frac{b_{2,j}}{n^2} x^{j-2} + ... + \frac{b_{j-1,j}}{n^{j-1}} x \right\}.$$

By expanding the sums in the last equality and by making the rearrangements, we arrive to the form

$$\begin{aligned} S_n(t^{m+1}; x) &= x^{m+1} + \frac{a_m}{n} x^m + \frac{a_{m-1} + a_m b_{1,m}}{n^2} x^{m-1} \\ &+ ... + \frac{(a_1 + a_2 b_{1,2} + ... + a_m b_{m-1,m})}{n^m} x. \end{aligned}$$

Denoting

$$\begin{aligned} b_{1,m+1} &= a_m \\ b_{2,m+1} &= a_{m-1} + a_m b_{1,m} \\ &... \\ b_{m,m+1} &= a_1 + a_2 b_{1,2} + ... + a_m b_{m-1,m}, \end{aligned}$$

we can write

$$S_n(t^{m+1}; x) = x^{m+1} + \frac{b_{1,m+1}}{n} x^m + \frac{b_{2,m+1}}{n^2} x^{m-1} + ... + \frac{b_{m,m+1}}{n^m} x,$$

which gives the desired result. ■

Now we can prove the following main theorem.

Theorem 2. *Let $f \in C_{x^{2m}}(0, \infty)$ and $\omega_{a+1}(f, \delta)$ be its modulus of continuity on the finite interval $[0, a+1]$, where $a > 0$. Then for the sequence of the positive linear Szasz operators $\{S_n\}$ the inequality*

(8) $$\|S_n(f; x) - f(x)\|_{C[0,a]} \leq C_f (1 + a^m)^2 \omega_{a+1}\left(f, \sqrt[2m]{\frac{a^{2m-1}}{n}}\right)$$

holds for a sufficiently large n.

Proof. Obviously, for $x \in [0,a]$ and $t \in [0,\infty)$ we can divide the line in two point sets

$$E_1 = \{(x,t):\ x \in [0,a],\ t > a+1\},$$
$$E_2 = \{(x,t):\ x \in [0,a],\ t \le a+1\}.$$

Let $(x,t) \in E_1$. In this case $t^m - x^m > 1$ and we easy obtain the inequality

$$|f(t) - f(x)| \le 3M_f(1+a^m)^2(t^m - x^m)^2. \tag{9}$$

If $(x,t) \in E_2$, we have the inequality

$$|f(t) - f(x)| \le \omega_{a+1}(f, |t-x|) \le \omega_{a+1}(f,\delta_n)\left(1 + \frac{|t-x|}{\delta_n}\right) \tag{10}$$

with the some positive δ_n.

(9) and (10) gives the inequality

$$|f(t) - f(x)| \le 3M_f(1+a^m)^2(t^m - x^m)^2 + \omega_{a+1}(f,\delta_n)\left(1 + \frac{|t-x|}{\delta_n}\right) \tag{11}$$

for any $t \ge 0$ and $x \in [0,a]$. Applying S_n to both sides of (11) and using the Hölder inequality, we obtain

$$\begin{aligned} S_n(|f(t) - f(x)|;x) \le\ & 3M_f(1+a^m)^2 S_n\left((t^m - x^m)^2;x\right) \\ & + \omega_{a+1}(f,\delta_n)\left(1 + \frac{1}{\delta_n}\sqrt[2m]{S_n\left((t-x)^{2m};x\right)}\right). \end{aligned} \tag{12}$$

On the other hand, by using the monotonity of the positive operators and the inequality $(t-x)^{2m} \le (t^m - x^m)^2$, we get

$$\begin{aligned} S_n(|f(t) - f(x)|;x) \le\ & 3M_f(1+a^m)^2 S_n\left((t^m - x^m)^2;x\right) \\ & + \omega_{a+1}(f,\delta_n)\left(1 + \frac{1}{\delta_n}\sqrt[2m]{S_n\left((t^m - x^m)^2;x\right)}\right). \end{aligned} \tag{13}$$

Now, we calculate $S_n\left((t^m - x^m)^2;x\right)$ in (13).

From (6) and (7), we obtain for $0 \le x \le a$:

$$\begin{aligned} S_n\left((t^m - x^m)^2;x\right) &= S_n(t^{2m};x) - 2x^m S_n(t^m;x) + x^{2m}S_n(1;x) \\ &= \left(x^{2m} + \frac{b_{1,2m}}{n}x^{2m-1} + \frac{b_{2,2m}}{n^2}x^{2m-2}\right. \\ &\quad \left. + \ldots + \frac{b_{m-1,2m}}{n^{m-1}}x^{m+1} + \frac{b_{m,2m}}{n^m}x^m + \ldots + \frac{b_{2m-1,2m}}{n^{2m-1}}x\right) \\ &\quad -2x^m\left(x^m + \frac{b_{1,m}}{n}x^{m-1} + \frac{b_{2,m}}{n^2}x^{m-2} + \ldots + \frac{b_{m-1,m}}{n^{m-1}}x\right) + x^{2m} \\ &\le \frac{1}{n}\sum_{k=1}^{2m-1}|b_{k,2m} - 2b_{k,m}|\,x^{2m-k} \le \frac{a^{2m-1}}{n}\sum_{k=1}^{2m-1}|b_{k,2m} - 2b_{k,m}|. \end{aligned} \tag{14}$$

Hence, we obtain

$$(15)\ \left(\max_{0\le x\le a} S_n\left((t^m-x^m)^2;x\right)\right)^{\frac{1}{2m}} \le \sqrt[2m]{\frac{a^{2m-1}}{n}}\left(\sum_{k=1}^{2m-1}|b_{k,2m}-2b_{k,m}|\right)^{\frac{1}{2m}}.$$

Using (15) in (13), we get

$$S_n\left(|f(t)-f(x)|;x\right) \le 3M_f(1+a^m)^2\frac{a^{2m-1}}{n}\left(\sum_{k=1}^{2m-1}|b_{k,2m}-2b_{k,m}|\right)$$
$$+\omega_{a+1}(f,\delta_n)\left[1+\frac{1}{\delta_n}\sqrt[2m]{\frac{a^{2m-1}}{n}}\left(\sum_{k=1}^{2m-1}|b_{k,2m}-2b_{k,m}|\right)^{\frac{1}{2m}}\right].$$

Taking $\delta_n = \sqrt[2m]{\frac{a^{2m-1}}{n}}$ in this inequality, we can write

$$S_n\left(|f(t)-f(x)|;x\right) \le 3M_f(1+a^m)^2\frac{a^{2m-1}}{n}\left(\sum_{k=1}^{2m-1}|b_{k,2m}-2b_{k,m}|\right)$$
$$+\omega_{a+1}\left(f,\sqrt[2m]{\frac{a^{2m-1}}{n}}\right)\left[1+\left(\sum_{k=1}^{2m-1}|b_{k,2m}-2b_{k,m}|\right)^{\frac{1}{2m}}\right].$$

Since $S_n(1;x)=1$ and for a sufficiently large n, $\frac{a^{2m-1}}{n} \le \sqrt[2m]{\frac{a^{2m-1}}{n}}$, we obtain

$$\|S_n(f;x)-f(x)\|_{C[0,a]} \le 3M_f(1+a^m)^2\sqrt[2m]{\frac{a^{2m-1}}{n}}\left(\sum_{k=1}^{2m-1}|b_{k,2m}-2b_{k,m}|\right)$$
$$+\omega_{a+1}\left(f,\sqrt[2m]{\frac{a^{2m-1}}{n}}\right)\left[1+\left(\sum_{k=1}^{2m-1}|b_{k,2m}-2b_{k,m}|\right)^{\frac{1}{2m}}\right].$$

By the properties of the modulus of continuity, there exists a constant K_f, depending on function f, such that $\omega_{a+1}(f,\delta_n)\ge K_f\delta_n$. Therefore we can write

$$\begin{aligned}\|S_n(f;x)-f(x)\|_{C[0,a]} \le &\left\{3\frac{M_f}{K_f}\left(\sum_{k=1}^{2m-1}|b_{k,2m}-2b_{k,m}|\right)\right.\\ &\left.+\left[1+\left(\sum_{k=1}^{2m-1}|b_{k,2m}-2b_{k,m}|\right)^{\frac{1}{2m}}\right]\right\}\\ &\times(1+a^m)^2\omega_{a+1}\left(f,\sqrt[2m]{\frac{a^{2m-1}}{n}}\right).\end{aligned}$$

Denoting

$$C_f = 3\frac{M_f}{K_f}\left(\sum_{k=1}^{2m-1}|b_{k,2m} - 2b_{k,m}|\right) + \left[1 + \left(\sum_{k=1}^{2m-1}|b_{k,2m} - 2b_{k,m}|\right)^{\frac{1}{2m}}\right],$$

we can find

$$\|S_n(f;x) - f(x)\|_{C[0,a]} \le C_f(1+a^m)^2\omega_{a+1}\left(f, \sqrt[2m]{\frac{a^{2m-1}}{n}}\right)$$

which gives the proof. ■

Corollary 3. *If $f \in Lip_M\alpha$ on $[0, a+1]$, then*

$$\|S_n(f;x) - f(x)\|_{C[0,a]} \le C_f(1+a^m)^2 M\left(\frac{a^{2m-1}}{n}\right)^{\frac{\alpha}{2m}}$$

holds, for a large n.

Consider now some generalization of operators S_n.

Let β_n be a sequence of nonnegative numbers such that

$$\lim_{n\to\infty}\beta_n = \infty \qquad \text{and} \qquad \lim_{n\to\infty}\frac{\beta_n}{n} = 1\,.$$

Consider the sequence of the linear positive operators S_n^* defined by

$$S_n^*(f;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k+\alpha}{\beta_n}\right), \quad (\alpha > 0). \tag{16}$$

This type generalization with $\beta_n = n + \beta$ was considered first by D.D. Stancu [6]. Now, using Corollary 3, we give the following more general theorem on approximation in Lipschitz classes.

Theorem 4. *For $0 < \gamma \le 1$, let $f \in Lip_M\gamma$. Then for the sequence of the positive linear operators $\{S_n^*\}$ the inequality*

$$\|S_n^*(f;x) - f(x)\|_{C[0,a]} \le C.\max\left\{\frac{1}{n^{\gamma/2m}}, \left(1 - \frac{n}{\beta_n}\right)^{\gamma}\right\} \tag{17}$$

holds for a large n, where C is constant, defined as

$$C = \left(M\alpha^\gamma + MC_f(1+a^m)^2 a^{(2m-1)\gamma/2m} + Ma^\gamma\right).$$

Proof. Clearly,

$$
\begin{aligned}
|S_n^*(f;x) - f(x)| \le\ & \sum_{k=0}^{\infty} \left| f\left(\frac{k+\alpha}{\beta_n}\right) - f\left(\frac{k}{\beta_n}\right) \right| \frac{(nx)^k}{k!} e^{-nx} \\
& + |S_n(f;x) - f(x)| \\
& + \sum_{k=0}^{\infty} \left| f\left(\frac{k}{\beta_n}\right) - f\left(\frac{k}{n}\right) \right| \frac{(nx)^k}{k!} e^{-nx}.
\end{aligned}
$$

Using Corollary 3 and the condition $f \in Lip_M\gamma$, we can write

$$
\begin{aligned}
|S_n^*(f;x) - f(x)| \le\ & M \sum_{k=0}^{\infty} \left| \frac{k+\alpha}{\beta_n} - \frac{k}{\beta_n} \right|^{\gamma} \frac{(nx)^k}{k!} e^{-nx} \\
& + C_f (1+a^m)^2 M \left(\frac{a^{2m-1}}{n} \right)^{\gamma/2m} \\
& + M \sum_{k=0}^{\infty} \left| \frac{k}{\beta_n} - \frac{k}{n} \right|^{\gamma} \frac{(nx)^k}{k!} e^{-nx}
\end{aligned}
$$

$$
\begin{aligned}
\text{(18)} \qquad \le\ & M \left(\frac{\alpha}{\beta_n} \right)^{\gamma} + C_f(1+a^m)^2 M \left(\frac{a^{2m-1}}{n} \right)^{\gamma/2m} \\
& + M \left(1 - \frac{n}{\beta_n} \right)^{\gamma} \sum_{k=0}^{\infty} \left(\frac{k}{n} \right)^{\gamma} \frac{(nx)^k}{k!} e^{-nx}.
\end{aligned}
$$

Using the Hölder inequality with $p = \frac{1}{\gamma}$ to sum on the right-hand side of (18), we obtain

$$
\begin{aligned}
\le\ & M \left(\frac{n}{\beta_n} \right)^{\gamma} \frac{\alpha^{\gamma}}{n^{\gamma}} + C_f(1+a^m)^2 M \left(\frac{a^{2m-1}}{n} \right)^{\gamma/2m} \\
& + M \left(1 - \frac{n}{\beta_n} \right)^{\gamma} \left(\sum_{k=0}^{\infty} \left(\frac{k}{n} \right) \frac{(nx)^k}{k!} e^{-nx} \right)^{1/\gamma}
\end{aligned}
$$

and therefore, for $x \in [0, a]$,

$$
\begin{aligned}
|S_n^*(f;x) - f(x)| \le\ & M \left(\frac{n}{\beta_n} \right)^{\gamma} \frac{\alpha^{\gamma}}{n^{\gamma}} + C_f(1+a^m)^2 M \frac{a^{(2m-1)\gamma/2m}}{n^{\gamma/2m}} \\
& + M \left(1 - \frac{n}{\beta_n} \right)^{\gamma} a^{\gamma}.
\end{aligned}
$$

Now, since $S_n^*(1;x)=1$ and $\frac{1}{n^\gamma}<\frac{1}{n^{\gamma/2m}}$, we can find

$$\|S_n^*(f;x)-f(x)\|_{C[0,a]}\leq C.\max\left\{\frac{1}{n^{\gamma/2m}},\left(1-\frac{n}{\beta_n}\right)^\gamma\right\},$$

which completes the proof. ■

Let $\beta_n=n+\beta$, $\beta>0$. Then we get a generalization of the type

$$S_n^{(\alpha,\beta)}(f;x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k+\alpha}{n+\beta}\right)\quad,\qquad(\alpha,\beta>0). \tag{19}$$

Corollary 5. *For $0<\gamma\leq 1$, let $f\in Lip_M\gamma$. Then for the sequence of the positive linear Szasz-Stancu operators $\left\{S_n^{(\alpha,\beta)}\right\}$ the inequality*

$$\left\|S_n^{(\alpha,\beta)}(f;x)-f(x)\right\|_{C[0,a]}\leq C\frac{1}{n^{\gamma/2m}}$$

holds, where

$$C=M(\alpha^\gamma+\beta^\gamma a^\gamma+C_f(1+a^m)^2a^{(2m-1)\gamma/2m}).$$

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