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Generalizations of the Condition Number

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Generalizations of the condition number $K(A) = \|A\|\|A^{\dagger}\|$, based on $\{1\}$-inverses, are introduced. A few properties of the generalized condition number, which represent extensions of known results concerning the condition number $K(A)$, are investigated. The introduced condition number by means of $\{1\}$-inverses can be used in the error analysis in computation of these inverses as well as a measure in construction of adequate test matrices.

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1. Introduction

Let $\mathbf{C}^{m\times n}$ be the set of $m \times n$ complex matrices, and $\mathbf{C}_r^{m\times n} = \{X \in \mathbf{C}^{m\times n} : \ \text{rank}(X) = r\}$. By $\mathcal{R}(A)$ we denote the range of A, and by $\mathcal{N}(A)$ the kernel of A. For any matrix $A \in \mathbf{C}^{m\times n}$ consider the following equations in X:

$$(1)\quad AXA = A, \quad (2)\quad XAX = X, \quad (3)\quad (AX)^* = AX, \quad (4)\quad (XA)^* = XA$$

and if $m = n$, also

$$(1^k)\quad A^{k+1}G = A^k, \qquad (5)\qquad AX = XA.$$

For a sequence $\mathcal{S}$ of elements from the set $\{1,2,3,4,5\}$, the set of matrices obeying the equations represented in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$.

For the sake of completeness we restate briefly a few known facts about the condition numbers.

Definition 1.1. The *condition number* of a given regular matrix A, denoted by $\text{cond}(A)$, is defined by

$$\text{cond}(A) = \|A\|\|A^{-1}\|.$$

The condition number provides a global measure for the sensitivity of A with regard to the numerical solution of the system of linear equations $Ax = b$, or to the numerical computation of the inverse A^{-1} [8]. The condition number is used as a magnification factor in determination of the bound of the relative error in solving the system of linear equations $Ax = b$.

Theorem 1.1. *For a given regular matrix A of the order $n \times n$ the following statement is valid:*

$$\frac{\|\delta x\|}{\|x\|} \leq cond(A)\frac{\|\delta b\|}{\|b\|}.$$

The notion of the condition number of an arbitrary matrix (expect the zero matrix) can be defined as follows (see [5], p.140).

Definition 1.2. The *generalized condition number* (sometimes also called pseudo condition number) $K(A)$ of a rectangular or singular matrix $A \in \mathbf{C}_r^{m\times n}$, with respect to spectral norm, is defined as

$$K(A) = \|A\| \, \|A^{\dagger}\| \ .$$

Pseudo-condition number of any matrix A is used as an important tool for measuring of the sensitivity of the solution of problems to small perturbations of given data [9]. For example, the pseudo-condition number is important as a measure of sensitivity in the numerical computation of the system of linear equations,

$$Ax = b, \quad A \in \mathbf{C}^{m\times n}, \quad x \in \mathbf{C}^n, \quad b \in \mathbf{C}^n, \tag{1.1}$$

when one actually uses $A^{\dagger}$to "solve" the system. Also, generalized condition number is used as a measure of the inaccuracy expected in the numerical computation of the Moore-Penrose inverse. For this purpose, it is used as a main criterion in the construction of test matrices [8].

In the paper [2], Ben-Israel uses generalized condition number of the matrix A in the derivation of some estimates of the relative error of the best approximate solution $x = A^{\dagger}b$. Besides of small perturbations in the vector x, Ben-Israel investigated small perturbations in the matrix A as well as in the vector b. For the sake of completeness we restate known results from [2].

Theorem 1.2. [2] *If matrices $A \in \mathbf{C}^{m\times n}$ and $B \in \mathbf{C}^{n\times m}$ satisfy*

$$AA^{\dagger}B = B, \tag{1.2}$$

$$A^{\dagger}AB^* = B^*, \tag{1.3}$$

$$\|A^{\dagger}B\| < 1, \tag{1.4}$$

then the following statements are valid:

$$\text{(1.5)}\qquad \begin{aligned} &(i)\quad (A+B)^\dagger = (I + A^\dagger B)^{-1}A^\dagger,\\ &(ii)\quad (A+B)^\dagger - A^\dagger = \sum_{k=1}^{\infty}(-1)^k (A^\dagger B)^k A^\dagger\\ &(iii)\quad \|(A+B)^\dagger - A^\dagger\| \le \frac{\|A^\dagger B\|\|A^\dagger\|}{1-\|A^\dagger B\|}. \end{aligned}$$

Also, in [2] is studied the sensitivity of the best approximate solution $x = A^\dagger b$ to variations in the data A, b.

Theorem 1.3. [2] *Let the system* $Ax = b$ *is solvable, and* $x = A^\dagger b$. *Then the following statements are valid:*

(i) If

$$(x+\delta x) = A^\dagger(b+\delta b), \tag{1.6}$$

then

$$\frac{\|\delta x\|}{\|x\|} \le K(A)\frac{\|\delta b\|}{\|b\|}. \tag{1.7}$$

(ii) If

$$(x+\delta x) = (A+\delta A)^\dagger b, \tag{1.8}$$

and the matrix δA *satisfies conditions:*

$$AA^\dagger \delta A = \delta A, \tag{1.9}$$

$$A^\dagger A(\delta A)^* = (\delta A)^*, \tag{1.10}$$

$$\|A^\dagger\|\|\delta A\| < 1, \tag{1.11}$$

then

$$\frac{\|\delta x\|}{\|x\|} \le \frac{\|A^\dagger\|\|\delta A\|}{1-\|A^\dagger\|\|\delta A\|} = K(A)\frac{\|\delta A\|}{\|A\|\,(1-\|A^\dagger\|\|\delta A\|)}. \tag{1.12}$$

Theorem 1.4. [2] *If the system* $Ax = b$ *is consistent,* $x = A^\dagger b$ *and if* δx *is defined by*

$$(x+\delta x) = (A+\delta A)^\dagger(b+\delta b), \tag{1.13}$$

where the matrix δA *satisfies* (1.9), (1.10) *and* (1.11), *then*

$$\|\delta x\| \le \frac{\|A^\dagger\|\,(\|\delta A\|\|A^\dagger\|\|b\| + \|\delta b\|)}{1-\|A^\dagger\|\|\delta A\|} = K(A)\frac{\|\delta A\|\|A^\dagger\|\|b\| + \|\delta b\|}{\|A\|\,(1-\|A^\dagger\|\|\delta A\|)}. \tag{1.14}$$

Stewart in [7] (Lemma 3.1) proves the part (i) of Theorem 1.2, but assuming instead of the condition (1.3) the following condition:

$$AA^{\dagger}BA^{\dagger}A = B.$$

Generalized condition number is investigated in [2], [7], [8], [9]. In [8] the generalized condition number is used in the construction of test matrices for the Moore-Penrose inverse. Also, in [9] the following generalized condition numbers with respect to used matrix norms are investigated:

$$K_p(A) = \|A\|_p \|A^{\dagger}\|_p, \quad p \in \{1, 2, \infty, F, M, G\}.$$

Matrix norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$, $\|\cdot\|_F$, $\|\cdot\|_M$ and $\|\cdot\|_G$ are defined in [9].

The following theorem of the equivalence of condition numbers is introduced in [9].

Theorem 1.5. [9] *For any pair of condition numbers*

$$K_p(A) = \|A\|_p \|A^{\dagger}\|_p, \quad K_q(A) = \|A\|_q \|A^{\dagger}\|_q,$$

where $p, q \in \{1, 2, \infty, F, M, G\}$, *there exist positive constants* $c_1 = c_1(p, q)$ *and* $c_2 = c_2(p, q)$ *such that*

$$c_1 K_p(A) \le K_q(A) \le c_2 K_p(A).$$

The constants c_1 *and* c_2 *are arranged in the appropriate Table 2 in [9].*

The main aim of this paper is as follows. In the parers [2], [5] are introduced generalizations of the condition number of regular matrix, by means of the Moore-Penrose inverse. Following this idea, a more general condition number by means of $\{1\}$-inverses is introduced in [4]:

$$\mathcal{K}_p(A) = \|A\|_p \inf\left\{\|A^{(1)}\|_p, \quad A^{(1)} \in A\{1\}\right\}. \tag{1.15}$$

In this paper we define condition numbers with respect to any selected $\{\mathcal{S}\}$-inverse of A, $1 \in \mathcal{S}$. We investigate properties of this condition number which represent generalizations of known results concerning the condition number $K(A)$ from [2], [7]. Also, we investigate properties of the generalized condition number with respect to various matrix norms. A generalization of known result from [9] is derived. Introduced condition numbers by means of $\{\mathcal{S}\}$-inverses may be used in the error analysis in computation of $\{\mathcal{S}\}$-inverses as well as a test in construction of adequate test matrices.

2. Generalized condition numbers

We use a more general definition of the condition number as follows.

Definition 2.1. Generalized condition number of a matrix $A \in \mathbf{C}_r^{m\times n}$, corresponding to any generalized inverse $A^{(\mathcal{S})}$, is equal to

$$K_{(p,\mathcal{S})}(A) = \|A\|_p \|A^{(\mathcal{S})}\|_p \ ,$$

where $\|\cdot\|_p$ is one of the matrix norms and $A^{(\mathcal{S})}$ and $1 \in \mathcal{S} \subseteq \{1,2,3,4,5\}$.

Lemma 2.1. *For an arbitrary matrix $A \in \mathbf{C}_r^{m\times n}$, for any matrix norm and an arbitrary generalized inverse $A^{(\mathcal{S})} \in A\{1\}$, the generalized condition numbers satisfy*

$$K_{(p,\mathcal{S})}(A) \geq 1.$$

Proof. Since $A^{(\mathcal{S})}$ is at least $\{1\}$-inverse of A, we get $\|AA^{(\mathcal{S})}\|_p \leq \|AA^{(\mathcal{S})}\|_p^2$. Therefore, $\|AA^{(\mathcal{S})}\|_p \geq 1$, and the proof can be completed using

$$\|A\|_p \|A^{(\mathcal{S})}\|_p \geq \|AA^{(\mathcal{S})}\|_p.$$

■

Since the introduced condition numbers satisfy $K_{(p,\mathcal{S})}(A) \geq 1$, they can be used as a magnification factors in some error estimates. In this way, these condition numbers may be used in the error analysis in computation of generalized inverses $A^{(\mathcal{S})}$, as a criterion in construction of adequate test matrices, as well as a measure in numerical solution of the linear system $Ax = b$ when we use $A^{(\mathcal{S})}$ to solve the system.

Theorem 2.1. *If the matrices $A \in \mathbf{C}^{m\times n}$ and $B \in \mathbf{C}^{n\times m}$ satisfy the following conditions:*

(2.1) $$AA^{(\mathcal{S})}B = B,$$

(2.2) $$\left(A(I + A^{(\mathcal{S})}B)\right)^{(\mathcal{S})} = (I + A^{(\mathcal{S})}B)^{(\mathcal{S})}A^{(\mathcal{S})},$$

(2.3) $$\|A^{(\mathcal{S})}B\|_p < 1,$$

then, for arbitrary S-inverse $A^{(S)}$ of A, the following statements are true:

(2.4) $$(i)\quad (A+B)^{(S)} = (I + A^{(S)}B)^{-1}A^{(S)},$$

$$(ii)\quad (A+B)^{(S)} - A^{(S)} = \sum_{k=1}^{\infty}(-1)^k \left(A^{(S)}B\right)^k A^{(S)}$$

(2.5) $$(iii)\quad \|(A+B)^{(S)} - A^{(S)}\|_p \le \frac{\|A^{(S)}B\|_p\|A^{(S)}\|_p}{1-\|A^{(S)}B\|_p}.$$

Proof.

(i) According to conditions (2.1) and (2.2) we get

$$(A+B)^{(S)} = (A + AA^{(S)}B)^{(S)} = (I + A^{(S)}B)^{(S)}A^{(S)}.$$

Since $\|A\| < 1$ implies that $I+A$ is nonsingular [2] (Corollary 1.4), in view of (2.3) it is easy to verify that $I + A^{(S)}B$ is nonsingular.

(ii) An application of part (i) implies

(2.6) $$(A+B)^{(S)} - A^{(S)} = (I + A^{(S)}B)^{-1}A^{(S)} - A^{(S)}.$$

Since (2.3) is assumed, we get

$$(I + A^{(S)}B)^{-1} = \sum_{k=0}^{\infty}(-1)^k \left(A^{(S)}B\right)^k,$$

and

$$\begin{aligned}(A+B)^{(S)} - A^{(S)} &= \sum_{k=0}^{\infty}(-1)^k \left(A^{(S)}B\right)^k A^{(S)}b - A^{(S)}\\ &= \sum_{k=1}^{\infty}(-1)^k \left(A^{(S)}B\right)^k A^{(S)}.\end{aligned}$$

(iii) Starting from (i) and using (2.6), we get the following inequality:

$$\|(A+B)^{(S)} - A^{(S)}\|_p \le \|(I + A^{(S)}B)^{-1} - I\|_p\|A^{(S)}\|_p.$$

Now, using the following Corollary 2 from [2]:

$$\|A\| < 1 \;\Rightarrow\; \|(I+A)^{-1} - I\| \le \frac{\|A\|}{1-\|A\|},$$

in view of (2.3), we conclude

$$\|(A+B)^{(S)} - A^{(S)}\|_p \le \frac{\|A^{(S)}B\|_p}{1 - \|A^{(S)}B\|_p} \cdot \|A^{(S)}\|_p.$$

■

From the statement (iii) of Theorem 2.1 we obtain the following corollary.

Corollary 2.1. *If the matrices A and B satisfy* (2.1) *and* (2.2), *and if the condition* (2.3) *is replaced by the condition*

$$\|A^{(S)}\|_p \|B\|_p < 1,$$

then

$$\|(A+B)^{(S)} - A^{(S)}\|_p \le \frac{\|A^{(S)}\|_p^2 \|B\|_p}{1 - \|A^{(S)}\|_p \|B\|_p}.$$

Conditions (2.1) and (2.2) are derived as generalizations of the corresponding conditions (1.2) and (1.4), respectively. In the following lemma we show that in the partial case $A^{(S)} = A^{\dagger}$, the conjunction of conditions (1.2), (1.3) and (1.4) can be derived from the conjunction of conditions (2.1), (2.2) and (2.3).

Lemma 2.2. *Conditions* (1.2), (1.3) *and* (1.4) *are equivalent to the conjunction of the conditions* (1.2), (1.4) *with the following condition:*

$$\left(A(I + A^{\dagger}B)\right)^{\dagger} = (I + A^{\dagger}B)^{\dagger}A^{\dagger}. \tag{2.7}$$

Proof. Using the results from [2] (Lemma 1) one can verify that the conditions (1.2) and (1.3) imply (2.7).

On the other hand, assume the conditions (1.2), (1.4) and (2.7). We must verify the condition (1.3). It is well-known equivalence of the equation $(AC)^{\dagger} = C^{\dagger}A^{\dagger}$ with both of the following conditions (see [1], [3]):

$$\begin{aligned} A^{\dagger}ACC^*A^* &= CC^*A^*, \\ CC^{\dagger}A^*AC &= A^*AC. \end{aligned}$$

Since (2.7) is satisfied, we get

$$A^{\dagger}A(I + A^{\dagger}B)(I + A^{\dagger}B)^*A^* = (I + A^{\dagger}B)(I + A^{\dagger}B)^*A^*, \tag{2.8}$$

$$(I + A^{\dagger}B)(I + A^{\dagger}B)^{\dagger}A^*A(I + A^{\dagger}B) = A^*A(I + A^{\dagger}B). \tag{2.9}$$

In view of (1.4) the matrix $I + A^\dagger B$ is invertible, so that (2.9) is clear. Also, from (2.8) we have

$$A^\dagger A(I + A^\dagger B)(I + B^* A^{\dagger^*})A^* = (I + A^\dagger B)(I + B^* A^{\dagger^*})A^*.$$

Using basic properties of the Moore-Penrose inverse and (1.2) we obtain

$$\begin{aligned}(I + A^\dagger B)(I + B^* A^{\dagger^*})A^* &= (I + A^\dagger B)(A^* + (AA^\dagger B)^*)\\ &= A^* + B^* + A^\dagger B(A^* + B^*)\end{aligned} \tag{2.10}$$

and

$$A^\dagger A(I + A^\dagger B)(I + B^* A^{\dagger^*})A^* = A^* + A^\dagger AB^* + A^\dagger B(A^* + B^*). \tag{2.11}$$

From (2.10) and (2.11) we immediately obtain the condition (1.3). ∎

We now investigate sensitivity of the solution $x = A^{(S)}b$ varying the data contained in A and b. In this way, we generalize the results of Theorem 1.2 and Theorem 1.3.

Theorem 2.2. *Assume $A \in \mathbf{C}^{m\times n}$ and $b \in \mathbf{C}^m$. Also, let the system $Ax = b$ is solvable, and the vector $x \in \mathbf{C}^n$ is determined by $x = A^{(S)}b$. If the condition*

$$(x + \delta x) = A^{(S)}(b + \delta b)$$

is satisfied, then

$$\frac{\|\delta x\|_p}{\|x\|_p} \le K_{(p,S)}(A)\frac{\|\delta b\|_p}{\|b\|_p}, \quad \textit{where} \quad p \in \{1, 2, \infty, F, M, G\}.$$

P r o o f. According to the assumptions we conclude $\delta x = A^{(S)}\delta b$, which implies

$$\|\delta x\|_p \le \|A^{(S)}\|_p\|\delta b\|_p.$$

Since the system $Ax = b$ is solvable and $\|b\|_p \le \|A\|_p\|x\|_p$, we get

$$\frac{\|\delta x\|_p}{\|x\|_p} \le \|A^{(S)}\|_p\|A\|_p\frac{\|\delta b\|_p}{\|b\|_p} = K_{(p,S)}(A)\frac{\|\delta b\|_p}{\|b\|_p}.$$

∎

Theorem 2.3. *If the equation $Ax = b$ is solvable, x satisfies the condition $x = A^{(S)}b$, the vector δx is defined by*

$$x + \delta x = (A + \delta A)^{(S)}b,$$

and the matrices $A \in \mathbf{C}^{m\times n}$ *and* $\delta A \in \mathbf{C}^{n\times m}$ *satisfy*

$$AA^{(S)}\delta A = \delta A, \tag{2.12}$$

$$\left(A(I + A^{(S)}\delta A)\right)^{(S)} = (I + A^{(S)}\delta A)^{(S)}A^{(S)}, \tag{2.13}$$

$$\|A^{(S)}\|_p\|\delta A\|_p < 1, \tag{2.14}$$

then

$$\begin{aligned}\frac{\|\delta x\|_p}{\|x\|_p} &\leq \frac{\|A^{(S)}\|_p\|\delta A\|_p}{1-\|A^{(S)}\|_p\|\delta A\|_p}\cdot\|A^{(S)}A\|_p \\ &\leq K_{(p,(S))}(A)\frac{\|A^{(S)}\|_p\|\delta A\|_p}{1-\|A^{(S)}\|_p\|\delta A\|_p} \\ &= \left(K_{(p,(S))}(A)\right)^2\frac{\|\delta A\|_p}{\|A\|_p(1-\|A^{(S)}\|_p\|\delta A\|_p)}\end{aligned} \tag{2.15}$$

Proof. Using $x = A^{(S)}b$ and $x + \delta x = (A + \delta A)^{(S)}b$, we get

$$\delta x = \left[(A+\delta A)^{(S)} - A^{(S)}\right] b.$$

Now, applying part (i) of Theorem 2.1, we get

$$\delta x = \sum_{k=1}^{\infty}(-1)^k\left(A^{(S)}\delta A\right)^k A^{(S)}b = \sum_{k=1}^{\infty}(-1)^k\left(A^{(S)}\delta A\right)^k A^{(S)}Ax.$$

Now, it is not difficult to verify

$$\|\delta x\|_p \leq \sum_{k=1}^{\infty}\left(\|A^{(S)}\|_p\|\delta A\|_p\right)^k \|A^{(S)}A\|_p\|x\|_p$$

and

$$\begin{aligned}\frac{\|\delta x\|_p}{\|x\|_p} &\leq \|A^{(S)}A\|_p\cdot\|A^{(S)}\|_p\|\delta A\|_p\cdot\sum_{k=0}^{\infty}\left(\|A^{(S)}\|_p\|\delta A\|_p\right)^k \\ &= \frac{\|A^{(S)}\|_p\|\delta A\|_p}{1-\|A^{(S)}\|_p\|\delta A\|_p}\cdot\|A^{(S)}A\|_p.\end{aligned}$$

Finally, using $\|A^{(S)}A\|_p \leq \|A^{(S)}\|_p\|A\|_p = K_{(p,S)}(A)$, we get the following upper bounds for $\frac{\|\delta x\|_p}{\|x\|_p}$:

$$\frac{\|\delta x\|_p}{\|x\|_p} \leq K_{(p,S)}(A)\frac{\|A^{(S)}\|_p\|\delta A\|_p}{1-\|A^{(S)}\|_p\|\delta A\|_p}$$

$$
\begin{aligned}
&= K_{(p,\mathcal{S})}(A)\frac{\|A\|_p\|A^{(\mathcal{S})}\|_p\|\delta A\|_p}{\|A\|_p\left(1-\|A^{(\mathcal{S})}\|_p\|\delta A\|_p\right)}\\
&= \left(K_{(p,\mathcal{S})}(A)\right)^2\frac{\delta A\|_p}{\|A\|_p\left(1-\|A^{(\mathcal{S})}\|_p\|\delta A\|_p\right)}
\end{aligned}
$$

which completes the proof.

Remark 2.1. In the case $2\in\mathcal{S}$, $4\in\mathcal{S}$ the matrix $A^{(\mathcal{S})}A$ is an idempotent and Hermitian, which implies $\|A^{(\mathcal{S})}A\|_p=1$. Then the identity (2.15) can be stated in the following form

$$
\frac{\|\delta x\|_p}{\|x\|_p}\le K_{(p,(\mathcal{S}))}(A)\cdot\frac{\|\delta A\|_p}{\|A\|_p\left(1-\|A^{(\mathcal{S})}\|_p\|\delta A\|_p\right)}.
$$

Theorem 2.4. *If the equation $Ax=b$ is solvable, x satisfies the condition $x=A^{(\mathcal{S})}b$ and δx is defined by*

(2.16) $$x+\delta x=(A+\delta A)^{(\mathcal{S})}(b+\delta b),$$

where $A\in\mathbf{C}^{m\times n}$ and $\delta A\in\mathbf{C}^{n\times m}$ satisfy the conditions (2.12), (2.13) *and* (2.14), *then*

(2.17) $$\|\delta x\|_p\le K_{(p,\mathcal{S})}(A)\cdot\frac{\|A^{(\mathcal{S})}\|_p\|\delta A\|_p\|b\|_p+\|\delta b\|_p}{\|A\|_p\left(1-\|A^{(\mathcal{S})}\|_p\|\delta A\|_p\right)}.$$

Proof. Using $x=A^{(\mathcal{S})}b$ and $x+\delta x=(A+\delta A)^{(\mathcal{S})}(b+\delta b)$, we derive

(2.18) $$\delta x=\left[(A+\delta A)^{(\mathcal{S})}-A^{(\mathcal{S})}\right]b+(A+\delta A)^{(\mathcal{S})}\delta b.$$

Applying Corollary 2.1, in a partial case $B=\delta A$, we get

(2.19) $$\|\left[(A+\delta A)^{(\mathcal{S})}-A^{(\mathcal{S})}\right]b\|_p\le\frac{\|A^{(\mathcal{S})}\|_p^2\|\delta A\|_p\|b\|_p}{1-\|A^{(\mathcal{S})}\|_p\|\delta A\|_p}.$$

An application of part (i) of Theorem 2.1 leads to

$$
\begin{aligned}
\|(A+\delta A)^{(\mathcal{S})}\delta b\|_p &= \|\left(I+A^{(\mathcal{S})}\delta A\right)^{-1}A^{(\mathcal{S})}\delta b\|_p\\
&\le \|\left(I+A^{(\mathcal{S})}\delta A\right)^{-1}\|_p\|A^{(\mathcal{S})}\|_p\|\delta b\|_p.
\end{aligned}
$$

Now, using again the result [2] (Corollary 2), one can verify the following

$$\|(A+\delta A)^{(\mathcal{S})}\delta b\|_p \le \frac{\|A^{(\mathcal{S})}\|_p\|\delta b\|_p}{1-\|A^{(\mathcal{S})}\delta A\|_p} \le \frac{\|A^{(\mathcal{S})}\|_p\|\delta b\|_p}{1-\|A^{(\mathcal{S})}\|_p\|\delta A\|_p}. \tag{2.20}$$

From (2.18), (2.19) and (2.20) we derive the following transformations:

$$\begin{aligned}\|\delta x\|_p &\le \frac{\|A^{(\mathcal{S})}\|_p^2\|\delta A\|_p\|b\|_p}{1-\|A^{(\mathcal{S})}\|_p\|\delta A\|_p} + \frac{\|A^{(\mathcal{S})}\|_p\|\delta b\|_p}{1-\|A^{(\mathcal{S})}\|_p\|\delta A\|_p} \\ &= \frac{\|A^{(\mathcal{S})}\|_p\left(\|A^{(\mathcal{S})}\|_p\|\delta A\|_p\|b\|_p+\|\delta b\|_p\right)}{1-\|A^{(\mathcal{S})}\|_p\|\delta A\|_p} \\ &= K_{(p,\mathcal{S})}(A)\cdot\frac{\|A^{(\mathcal{S})}\|_p\|\delta A\|_p\|b\|_p+\|\delta b\|_p}{\|A\|_p\left(1-\|A^{(\mathcal{S})}\|_p\|\delta A\|_p\right)}.\end{aligned}$$

■

Remark 2.2. It is not difficult to verify the following:

(i) Theorem 2.1 represents a generalization of the statement (i) from Theorem 1.1.

(ii) Theorem 2.2 represents a generalization of the statement (i) from Theorem 1.2.

(iii) Theorem 2.3 is a generalization of Theorem 1.3.

In the case $2 \in \mathcal{S}$ we obtain the following result.

Corollary 2.2. *For arbitrary matrices $A \in \mathbf{C}^{m\times n}$ and $B \in \mathbf{C}^{n\times m}$ the following statements are valid:*

(a) If the matrices A and B satisfy the following conditions:

$$\begin{aligned} AA^{(1,2)}B &= B, \\ \|A^{(1,2)}B\|_p &< 1, \end{aligned}$$

then the following statements are true:

(i) $(A+B)^{(1,2)} = (I+A^{(1,2)}B)^{-1}A^{(1,2)}$,

(ii) $(A+B)^{(1,2)} - A^{(1,2)} = \sum_{k=1}^{\infty}(-1)^k\left(A^{(1,2)}B\right)^k A^{(1,2)}$

(iii) $\|(A+B)^{(1,2)} - A^{(1,2)}\|_p \le \dfrac{\|A^{(1,2)}B\|_p\|A^{(1,2)}\|_p}{1-\|A^{(1,2)}B\|_p}.$

(b) If the equation $Ax = b$ is solvable, x satisfies the condition $x = A^{(1,2)}b$, the vector δx is defined by

$$x + \delta x = (A + \delta A)^{(S)} b,$$

and the matrices $A \in \mathbf{C}^{m\times n}$ and $\delta A \in \mathbf{C}^{n\times m}$ satisfy

$$\begin{aligned} AA^{(1,2)}\delta A &= \delta A, \\ \|A^{(1,2)}\|_p \|\delta A\|_p &< 1, \end{aligned}$$

then

$$\begin{aligned} \frac{\|\delta x\|_p}{\|x\|_p} &\leq \frac{\|A^{(1,2)}\|_p\|\delta A\|_p}{1 - \|A^{(1,2)}\|_p\|\delta A\|_p} \cdot \|A^{(1,2)}A\|_p \\ &\leq K_{(p,(1,2))}(A) \cdot \frac{\|A^{(1,2)}\|_p\|\delta A\|_p}{1 - \|A^{(1,2)}\|_p\|\delta A\|_p} \\ &= \left(K_{(p,(1,2))}(A)\right)^2 \cdot \frac{\|\delta A\|_p}{\|A\|_p\left(1 - \|A^{(1,2)}\|_p\|\delta A\|_p\right)}. \end{aligned}$$

P r o o f. Follows from Theorem 2.1, Theorem 2.3 and the following known result [6] (Theorem 3.1): For arbitrary $C \in \mathbf{C}^{m\times n}$ and $D \in \mathbf{C}^{n\times p}$ the inclusion

$$(CD)\{1,2\} \subseteq B\{1,2\}A\{1,2\}$$

is always true. ∎

As a generalization of Theorem 1.4, where the equivalence of the condition numbers $K_p(A)$ is introduced, in the following theorem we prove the equivalence of the condition numbers

$$K_{(p,S)}(A) = \|A\|_p\|A^{(S)}\|_p.$$

Theorem 2.5. *The condition numbers*

$$\begin{aligned} K_{(1,S)}(A) &= \|A\|_1\|A^{(S)}\|_1, \quad K_{(2,S)}(A), \quad K_{(\infty,S)}(A), \\ & K_{(F,S)}(A), \quad K_{(M,S)}(A), \quad K_{(G,S)}(A) \end{aligned}$$

of a given complex matrix $A \in \mathbf{C}_r^{m\times n}$, satisfy the following inequalities, where $r = rank(A)$ and $\max = \max(m,n)$:

$$\frac{1}{\sqrt{mn}}K_{(1,\mathcal{S})} \le K_{(2,\mathcal{S})} \le \sqrt{mn}K_{(1,\mathcal{S})} \qquad \frac{1}{mn}K_{(1,\mathcal{S})} \le K_{(\infty,\mathcal{S})} \le mnK_{(1,\mathcal{S})}$$

$$\frac{1}{\sqrt{mn}}K_{(1,\mathcal{S})} \le K_{(F,\mathcal{S})} \le \sqrt{mn}K_{(1,\mathcal{S})} \qquad \frac{\max^2}{mn}K_{(1,\mathcal{S})} \le K_{(M,\mathcal{S})} \le \max^2 K_{(1,\mathcal{S})}$$

$$K_{(1,\mathcal{S})} \le K_{(G,\mathcal{S})} \le mnK_{(1,\mathcal{S})} \qquad \frac{1}{\sqrt{mn}}K_{(2,\mathcal{S})} \le K_{(1,\mathcal{S})} \le \sqrt{mn}K_{(2,\mathcal{S},\sqcup)}$$

$$\frac{1}{\sqrt{mn}}K_{(2,\mathcal{S})} \le K_{(\infty,\mathcal{S})} \le \sqrt{mn}K_{(2,\mathcal{S})} \qquad K_{(2,\mathcal{S})} \le K_{(F,\mathcal{S})} \le rK_{(1,\mathcal{S})}$$

$$\frac{\max^2}{mn}K_{(2,\mathcal{S})} \le K_{(M,\mathcal{S})} \le \max^2 K_{(2,\mathcal{S})} \qquad K_{(2,\mathcal{S})} \le K_{(G,\mathcal{S})} mnK_{(2,\mathcal{S})}$$

$$\frac{1}{mn}K_{(\infty,\mathcal{S})} \le K_{(1,\mathcal{S})} \le mnK_{(\infty,\mathcal{S})} \qquad \frac{1}{\sqrt{mn}}K_{(\infty,\mathcal{S})} \le K_{(2,\mathcal{S})} \le \sqrt{mn}K_{(\infty,\mathcal{S})}$$

$$\frac{1}{\sqrt{mn}}K_{(\infty,\mathcal{S})} \le K_{(F,\mathcal{S})} \le \sqrt{mn}K_{(\infty,\mathcal{S})} \qquad \frac{\max^2}{mn}K_{(\infty,\mathcal{S})} \le K_{(M,\mathcal{S})} \le \max^2 K_{(\infty,\mathcal{S})}$$

$$K_{(\infty,\mathcal{S})} \le K_{(G,\mathcal{S})} mnK_{(\infty,\mathcal{S})} \qquad \frac{1}{\sqrt{mn}}K_{(F,\mathcal{S})} \le K_{(1,\mathcal{S})} \le \sqrt{mn}K_{(F,\mathcal{S})} \tag{2.21}$$

$$\frac{1}{r}K_{(F,\mathcal{S})} \le K_{(2,\mathcal{S})} \le K_{(F,\mathcal{S})} \qquad \frac{1}{\sqrt{mn}}K_{(F,\mathcal{S})} \le K_{(\infty,\mathcal{S})} \le \sqrt{mn}K_{(F,\mathcal{S})}$$

$$\frac{\max^2}{mn}K_{(F,\mathcal{S})} \le K_{(M,\mathcal{S})} \le \max^2 K_{(F,\mathcal{S})} \qquad K_{(F,\mathcal{S})} \le K_{(G,\mathcal{S})} mnK_{(F,\mathcal{S})}$$

$$\frac{1}{\max^2}K_{(M,\mathcal{S})} \le K_{(1,\mathcal{S})} \le \frac{mn}{\max^2}K_{(M,\mathcal{S})} \qquad \frac{1}{\max^2}K_{(M,\mathcal{S})} \le K_{(2,\mathcal{S})} \le \frac{mn}{\max^2}K_{(M,\mathcal{S})}$$

$$\frac{1}{\max^2}K_{(M,\mathcal{S})} \le K_{(\infty,\mathcal{S})} \le \frac{mn}{\max^2}K_{(M,\mathcal{S})} \qquad \frac{1}{\max^2}K_{(M,\mathcal{S})} \le K_{(F,\mathcal{S})} \le \frac{mn}{\max^2}K_{(M,\mathcal{S})}$$

$$\frac{mn}{\max^2}K_{(M,\mathcal{S})} \le K_{(G,\mathcal{S})} \le \frac{mn}{\max^2}K_{(M,\mathcal{S})} \qquad \frac{1}{mn}K_{(G,\mathcal{S})} \le K_{(1,\mathcal{S})} \le K_{(G,\mathcal{S})}$$

$$\frac{1}{mn}K_{(G,\mathcal{S})} \le K_{(2,\mathcal{S})} \le K_{(G,\mathcal{S})} \qquad \frac{1}{mn}K_{(G,\mathcal{S})} \le K_{(\infty,\mathcal{S})} \le K_{(G,\mathcal{S})}$$

$$\frac{1}{mn}K_{(G,\mathcal{S})} \le K_{(F,\mathcal{S})} \le K_{(G,\mathcal{S})} \qquad \frac{\max^2}{mn}K_{(G,\mathcal{S})} \le K_{(M,\mathcal{S})} \le \frac{\max^2}{mn}K_{(G,\mathcal{S})}\,.$$

P r o o f. The constants presented in the table can be generated using definitions of the condition numbers, equivalence of matrix norms (see [9]), and that for $A \in \mathbf{C}^{m\times n}$ any generalized inverse $A^{\mathcal{S}}$ is an $n \times m$ matrix. ■

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