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# On Difference Mappings Associated with Hadamard's Inequality

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*Presented by P. Kenderov*

Recently striking properties have been found for some mappings connected with Hadamard's inequality. We derive further properties and applications and introduce some natural related mappings.

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## 1. Introduction

Let $I \subset \mathbf{R}$ denote an interval, $I^o$ its interior and $a, b \in I^o$ with $a < b$. Throughout the paper, $f : I \to \mathbf{R}$ is a (measurable) convex function.

The well-known Hadamard inequality for convex functions, which more accurately might be termed the Hermite–Hadamard inequality (see [3], [4]), states that

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(s)ds \le \frac{f(a)+f(b)}{2}.$$

*Inter alia*, this gives upper and lower point estimates for $\int_a^b f(s)ds$, useful in numerical analysis. The associated "difference mappings" $L, P : [a, b] \to \mathbf{R}$ defined by

$$L(t) := \frac{f(t)+f(a)}{2}(t-a) - \int_a^t f(s)ds,$$

$$P(t) := \int_a^t f(s)ds - (t-a)f\left(\frac{t+a}{2}\right)$$

are relevant for studying the errors in these estimates. Dragomir and Agarwal [2] have derived some sharp properties for these mappings.

**Theorem A.** *The mappings $L$, $P$ satisfy the following:*

*(i) $L$ is nonnegative, nondecreasing and convex on $[a,b]$;*

*(ii) $P$ is nonnegative and nondecreasing on $[a,b]$. If $f$ is twice differentiable and $f$ and $f'$ convex on $I^o$, then $P$ is also convex on $I^o$;*

*(iii) $P(t) \leq L(t)$ holds for all $t \in [a,b]$.*

The nonnegativity of $L$ and $P$ is a direct consequence of Hadamard's inequality. The proofs of the other properties are less immediate. In fact, there is a blemish in the demonstration of the monotonicity and convexity of $L$ in [2]. The authors use the second part of Hadamard's inequality to show that

$$L(x) - L(y) \geq (x-y)L'_+(y)$$

for $x > y$. It is stated that the proof for $y > x$ is similar. However, because of the asymmetry between the two parts of Hadamard's inequality, the strategy used does not cover that case.

In Section 2 we note simpler and more direct derivations of these monotonicity and convexity properties and some consequent results. We derive also analogous properties for the further mappings $R, S : [a,b] \to \mathbf{R}$ given by

$$R(t) := \frac{f(t)+f(b)}{2}(b-t) - \int_t^b f(s)ds,$$

$$S(t) := \int_t^b f(s)ds - (b-t)f\left(\frac{b+t}{2}\right).$$

Section 3 is devoted to higher–order properties of these mappings, which appear to be new. In Section 4 we introduce some symmetric difference mappings and derive their basic properties. We conclude in Section 5 with higher–order properties of these mappings.

## 2. Basic results

First we remark that the monotonicity of $L$ follows from

$$L'_+(y) = \frac{1}{2}\left[f'_+(y)(y-a) - f(y) + f(a)\right],$$

since for $f$ a convex function, $f(y) - f(a) \leq (y-a)f'_+(y)$ for $a \leq y$. Similarly $P$ is nondecreasing, since $f$ is convex and

$$P'_+(t) = f(t) - f\left(\frac{t+a}{2}\right) - \frac{t-a}{2}f'_+\left(\frac{t+a}{2}\right),$$

which is nonnegative for $a \le t \le b$.

The same motivation provides a proof of the convexity of $L$. For $f$ convex, we have $f'_+(x) \ge f'_+(y)$ whenever $x \ge y$ and so for $a \le y \le x \le b$,

$$\begin{aligned} L'_+(y) &= \tfrac{1}{2}\left[f'_+(y)(y-a) - f(y) + f(a)\right] \\ &\le \tfrac{1}{2}\left[f'_+(x)(y-a) - f(y) + f(a)\right] \\ &\le \tfrac{1}{2}\left[f'_+(x)(y-a) + (x-y)f'_+(x) - f(x) + f(a)\right] \\ &= \tfrac{1}{2}\left[f'_+(x)(x-a) - f(x) + f(a)\right] \\ &= L'_+(x). \end{aligned}$$

Hence $L$ is convex.

The final relation (iii) is due to Bullen [1] (see also [5, pp. 140-141]).

An application is the interpolation

$$\begin{aligned} \tfrac{1}{b-a}\int_a^b f(s)ds &\le \tfrac{1}{b-a}\int_y^b f(s)ds + \tfrac{y-a}{b-a}\cdot\tfrac{f(a)+f(y)}{2} \\ &\le \tfrac{1}{b-a}\int_x^b f(s)ds + \tfrac{x-a}{b-a}\cdot\tfrac{f(a)+f(x)}{2} \\ &\le \tfrac{f(a)+f(b)}{2} \end{aligned}$$

of the second part of Hadamard's inequality for $a \le y \le x \le b$. The successive inequalities are just $L(y) \ge 0$, $L(y) \le L(x)$ and $L(x) \le L(b)$. Similarly, we may interpolate the first part of Hadamard's inequality by

$$\begin{aligned} f\left(\tfrac{a+b}{2}\right) &\le \tfrac{1}{b-a}\left[(b-a)f\left(\tfrac{a+b}{2}\right) - (y-a)f\left(\tfrac{a+y}{2}\right)\right] + \tfrac{1}{b-a}\int_a^y f(s)ds \\ &\le \tfrac{1}{b-a}\left[(b-a)f\left(\tfrac{a+b}{2}\right) - (x-a)f\left(\tfrac{a+x}{2}\right)\right] + \tfrac{1}{b-a}\int_a^x f(s)ds \\ &\le \tfrac{1}{b-a}\int_a^b f(s)ds. \end{aligned}$$

The successive inequalities are $P(y) \ge 0$, $P(y) \le P(x)$ and $P(x) \le P(b)$. This refinement of Hadamard's inequality extends slightly that proved in [2].

We now establish corresponding properties of the difference mappings $R$, $S$ defined at the close of the introduction.

**Theorem 1.** *The mappings $R$ and $S$ have the following properties:*

*(i) $R$ is nonnegative, nonincreasing and convex on $[a,b]$;*

*(ii) $S$ is nonnegative and nonincreasing on $[a,b]$. If $f$ is twice differentiable and $f$ and $f'$ convex on $I^o$, then $S$ is also convex on $[a,b]$;*

*(iii) $S(t) \le R(t)$ for all $t \in [a,b]$.*

Proof. Define $F(t) := f(a+b-t)$. Set

$$\begin{aligned} L_1(t) &:= \tfrac{F(t)+F(a)}{2}(t-a) - \int_a^t F(s)ds \\ &= \tfrac{f(a+b-t)+f(b)}{2}(t-a) - \int_{a+b-t}^b f(u)du, \end{aligned}$$

so that

$$R(t) = L_1(a+b-t), \tag{1}$$

and

$$\begin{aligned} P_1(t) &:= \int_a^t F(s)ds - (t-a)F\left(\tfrac{t+a}{2}\right) \\ &= \int_{a+b-t}^b f(u)du - (t-a)f\left(\tfrac{b-t}{2}\right), \end{aligned}$$

so that

$$S(t) = P_1(a+b-t).$$

Then (i)–(iii) follow immediately from Theorem A.

These properties lead to the interpolations

$$\begin{aligned} \tfrac{1}{b-a}\int_a^b f(s)ds &\le \tfrac{1}{b-a}\int_a^x f(s)ds + \tfrac{b-x}{b-a}\cdot\tfrac{f(x)+f(b)}{2} \\ &\le \tfrac{1}{b-a}\int_a^y f(s)ds + \tfrac{b-y}{b-a}\cdot\tfrac{f(y)+f(b)}{2} \\ &\le \tfrac{f(a)+f(b)}{2}, \end{aligned}$$

$$\begin{aligned} f\left(\tfrac{a+b}{2}\right) &\le \tfrac{1}{b-a}\int_a^y f(s)ds + \tfrac{b-y}{b-a}\cdot f\left(\tfrac{b+y}{2}\right) \\ &\le \tfrac{1}{b-a}\int_a^x f(s)ds + \tfrac{b-x}{b-a}\cdot f\left(\tfrac{b+x}{2}\right) \\ &\le \tfrac{1}{b-a}\int_a^b f(s)ds \end{aligned}$$

of the two parts of Hadamard's inequality.

## 3. Higher–order properties

In this section we obtain some further properties of $L$ and $R$. First we introduce some terminology. A function $f$ is said to be *absolutely monotone of order $n$ on $[a,b]$*, if $f^{(k)}(t) \ge 0$ for $t \in [a,b]$ $(k = 0,1,...,n)$, and *completely monotone of order $n$ on $[a,b]$*, if $(-1)^k f^{(k)}(t) \ge 0$ for $t \in [a,b]$ $(k = 0,1,...,n)$. We say $f$ is *absolutely convex of order $n$*, if $f^{(2k)}(t) \ge 0$ for $t \in [a,b]$ $(k = 0,1,...,n)$.

Further, a function $f \in C^\infty[a,b]$ is *absolutely monotone on $[a,b]$*, if it is absolutely monotone of all orders. Corresponding definitions apply for *completely monotone* and *absolutely convex* functions.

We now derive some new properties of the mapping $L$.

**Theorem 2.**

*(i) Suppose $f \in C^n[a,b]$ $(n \ge 3)$. If $f''$ is absolutely monotone of order $n-2$, then $L$ is absolutely monotone of order $n$.*

*(ii) Suppose $f \in C^\infty[a,b]$. If $f''$ is absolutely monotone, then $L$ is also absolutely monotone.*

Proof. (i) Under the given conditions, $f'' \geq 0$ on $[a,b]$ and so, $f$ is convex. Hence by Theorem A, $L$ is nonnegative and nondecreasing, that is, $L^{(k)}(t) \geq 0$ for $k = 0, 1$. By a simple calculation

$$L^{(k)}(t) = \frac{1}{2}\left[f^{(k)}(t)(t-a) + (k-2)f^{(k-1)}(t)\right] \quad (2 \leq k \leq n),$$

whence $L^{(k)}(t) \geq 0$ $(k = 2, ..., n)$. Therefore $L$ is absolutely monotone of order $n$.

(ii) This is immediate from (i). ■

**Theorem 3.**

*(i) Suppose $f \in C^n[a,b]$ $(n \geq 3)$. If $f''$ is completely monotone of order $n-2$, then $R$ is completely monotone of order $n$.*

*(ii) Suppose $f \in C^\infty[a,b]$. If $f''$ is completely monotone, then $R$ is also completely monotone.*

Proof. These follow from (1) or from

$$R^{(n)}(t) = \frac{1}{2}\left[f^{(n)}(t)(b-t) - (n-2)f^{(n-1)}(t)\right], \quad (n \geq 2).$$

Thus from (1) we have

$$(-1)^n R^{(n)}(t) = L_1^{(n)}(a+b-t) \geq 0$$

for each $t \in [a,b]$, whence we have (i) and (ii). ■

## 4. Symmetric difference mappings

Set $A = (a+b)/2$. We define the "symmetric difference mappings" $T, U, V, W : [0, (b-a)/2] \to \mathbf{R}$ by

$$\begin{aligned}
T(t) &:= t\,[f(A+t) + f(A-t)] - \textstyle\int_{A-t}^{A+t} f(s)ds, \\
U(t) &:= \textstyle\int_{A-t}^{A+t} f(s)ds - 2tf(A), \\
V(t) &:= \tfrac{f(a+t)+f(b-t)}{2}(b-a-2t) - \textstyle\int_{a+t}^{b-t} f(s)ds, \\
W(t) &:= \textstyle\int_{a+t}^{b-t} f(s)ds - (b-a-2t)f(A).
\end{aligned}$$

We develop the properties of these mappings in the spirit of our foregoing work.

**Theorem 4.**
*(i) $T$ is nonnegative, nondecreasing and convex on $[0,(b-a)/2]$.*
*(ii) Hadamard's inequality possesses the refinement*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(s)ds &\leq \frac{1}{b-a}\left[\int_a^b f(s)ds + T(y)\right]\\ &\leq \frac{1}{b-a}\left[\int_a^b f(s)ds + T(x)\right]\\ &\leq \frac{f(a)+f(b)}{2}\end{aligned}$$

*for* $0 \leq y \leq x \leq (b-a)/2$.

Proof. (i) That $T$ is nondecreasing follows from

$$T'_+(x) = \left[f'_+(A+x) - f'_+(A-x)\right] x \geq 0.$$

Nonnegativity now follows from $T(0) = 0$. Further, for $x \geq y$ we have

$$T'_+(x) \geq [f'_+(A+y) - f'_+(A-y)]x \geq [f'_+(A+y) - f'_+(A-y)]y = T'_+(y),$$

so that $T$ is convex.

Part (ii) is immediate, since it is equivalent to

$$0 \leq T(y) \leq T(x) \leq T\left(\frac{b-a}{2}\right).$$

■

**Theorem 5.**
*(i) $U$ is nonnegative, nondecreasing and convex on $[0,(b-a)/2]$.*
*(ii) Hadamard's inequality possesses the interpolation*

$$\begin{aligned}f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_{A-y}^{A+y} f(s)ds + \frac{b-a-2y}{b-a} f\left(\frac{a+b}{2}\right)\\ &\leq \frac{1}{b-a}\int_{A-x}^{A+x} f(s)ds + \frac{b-a-2x}{b-a} f\left(\frac{a+b}{2}\right)\\ &\leq \frac{1}{b-a}\int_a^b f(s)ds\end{aligned}$$

*for* $0 \leq y \leq x \leq (b-a)/2$.

Proof. (i) By Jensen's inequality for convex functions, we obtain

$$U'_+(t) = f(A+t) + f(A-t) - 2f(A) \geq 0,$$

so that $U$ is nondecreasing. Nonnegativity follows from $U(0) = 0$. On the other hand, since convex functions have nondecreasing increments, we have for $x \geq y$ that

$$f(A-y) - f(A-x) \leq f(A+x) - f(A+y).$$

Thus

$$\begin{aligned} U'_+(x) &= f(A+x) + f(A-x) - 2f(A) \\ &\geq f(A+y) + f(A-y) - 2f(A) \\ &= U'_+(y), \end{aligned}$$

whence we have convexity. The proof of (ii) follows in the same way as in the corresponding part of Theorem 4. ■

Analogous results for $V$ and $W$ may be derived from Theorems 4 and 5 *via* the transformation: $t \to (b-a)/2 - t$ as with the proof of Theorem 1 from Theorem A.

**Theorem 6.**
*(i) $V$ is nonnegative, nonincreasing and convex on $[0, (b-a)/2]$.*
*(ii) Hadamard's inequality possesses the refinement*

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(s)ds &\leq \frac{1}{b-a}\left[\int_a^b f(s)ds + V(x)\right] \\ &\leq \frac{1}{b-a}\left[\int_a^b f(s)ds + V(y)\right] \\ &\leq \frac{f(a)+f(b)}{2} \end{aligned}$$

*for* $0 \leq y \leq x \leq (b-a)/2$.

**Theorem 7.**
*(i) $W$ is nonnegative, nonincreasing and convex on $[0, (b-a)/2]$.*
*(ii) Hadamard's inequality may be refined as*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_{a+y}^{b-y} f(s)ds + \frac{2y}{b-a} f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{b-a}\int_{a+x}^{b-x} f(s)ds + \frac{2x}{b-a} f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{b-a}\int_a^b f(s)ds \end{aligned}$$

*for* $0 \leq y \leq x \leq (b-a)/2$.

## 5. Higher–order properties of symmetric difference mappings

To conclude, we present some higher–order results for the symmetric difference mappings.

**Theorem 8.**
*(i) Suppose $f \in C^{2m}[a, b]$. If $f''$ is absolutely convex of order $m-1$, then $T$ is absolutely monotone of order $2m$.*

*(ii) Suppose $f \in C^{\infty}[a,b]$. If $f''$ is absolutely convex, then $T$ is absolutely monotone.*

Proof. (i) $T$ is nonnegative, or $T^{(0)}(t) \geq 0$. Also, we have

$$T'(t) = [f'(A+t) - f'(A-t)]t.$$

Since $f$ is convex, $f'$ is nondecreasing and so $T'$ is nonnegative. By a simple calculation we obtain

$$T^{(2k)}(t) = [f^{(2k)}(A+t) + f^{(2k)}(A-t)]t + (2k-1)[f^{(2k-1)}(A+t) - f^{(2k-1)}(A-t)].$$

By assumption, $f^{(2k)} \geq 0$ $(k \leq m)$ and thus $f^{(2k-1)}$ is nondecreasing. Hence $T^{(2k)}$ is nonnegative, so $T$ is absolutely monotone of order $2m$.

(ii) From the assumptions, we have as in (i) that $T^{(2k)} \geq 0$ for each $k$. Also

$$T^{(2k+1)}(t) = [f^{(2k+1)}(A+t) - f^{(2k+1)}(A-t)]t + 2k[f^{(2k)}(A+t) + f^{(2k)}(A-t)].$$

By assumption, $f''$ is absolutely convex, so $f^{(2k)} \geq 0$ for each $k$. Thus $f^{(2k+2)} \geq 0$ and so $f^{(2k+1)}$ is nondecreasing. Hence $T^{(2k+1)} \geq 0$ for each $k$ and thus $T$ is absolutely monotone. ∎

**Theorem 9.**

*(i) Suppose $f \in C^{2m}[a,b]$. If $f''$ is absolutely convex of order $m-1$, then $U$ is absolutely monotone of order $2m+1$.*

*(ii) Suppose $f \in C^{\infty}[a,b]$. If $f''$ is absolutely convex, then $U$ is absolutely monotone.*

Proof. We argue as in Theorem 7, making use of the identities

$$U^{(2k)}(t) = f^{(2k-1)}(A+t) - f^{(2k-1)}(A-t),$$

$$U^{(2k+1)}(t) = f^{(2k)}(A+t) + f^{(2k)}(A-t).$$

∎

The transformation: $t \to (b-a)/2 - t$ now gives corresponding theorems for $V$ and $W$.

**Theorem 10.**

*(i) Suppose $f \in C^{2m}[a,b]$. If $f''$ is absolutely convex of order $m-1$, then $V$ is completely monotone of order $2m$.*

*(ii) Suppose $f \in C^\infty[a,b]$. If $f''$ is absolutely convex, then $V$ is completely monotone.*

**Theorem 11.**

*(i) Suppose $f \in C^{2m}[a,b]$. If $f''$ is absolutely convex of order $m-1$, then $W$ is completely monotone of order $2m+1$.*

*(ii) Suppose $f \in C^\infty[a,b]$. If $f''$ is absolutely convex, then $W$ is completely monotone.*

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