Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

The attached copy is furnished for non-commercial research and education use only. Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on Mathematica Balkanica visit the website of the journal
http://www.mathbalkanica.info
or contact:
Mathematica Balkanica - Editorial Office;
Acad. G. Bonchev str., Bl. 25A, 1113 Sofia, Bulgaria
Phone: +359-2-979-6311, Fax: +359-2-870-7273,
E-mail: balmat@bas.bg

Mathematica Balkanica

New Series Vol. 15, 2001, Fasc. 1-2

On the (g, φ, ξ, η)-Linear Connections

Nicolae A. Soare

Presented by Bl. Sendov

In this note, we present a method to determine the linear connections compatible with the (g, φ, ξ, η)-structure consisting on a metric tensor g, a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η on a differentiable manifold M wich satisfy $\varphi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$ and $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$.

AMS Subj. Classification: 53A45, 53C05, 53C15

Key Words: vector and tensor fields, C^∞-manifold, linear connections, almost contact structure, almost contact metric manifold

1. Introduction

Let M be a (2n+1)-dimensional C^∞ manifold and let $F(M)$ be the algebra of all differentiable functions on M. We denote by $T_s^r(M)$ the $F(M)$-module of the tensor fields of type (r, s). For $T_0^1(M)$ is used the notation $\chi(M)$. Let $C(M)$ be the affin modul of the linear connections on M.

An almost contact structure on M is defined by a C^∞ $(1,1)$-tensor field φ, a C^∞ vector field ξ and a C^∞ one-form η on M such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

where $\otimes$ denotes the tensor product and I is the identity tensor. This implies $\varphi\xi = 0$ and $\eta \circ \xi = 0$. Manifolds equipped with an almost contact structure are called almost contact manifolds [1].

A Riemannian manifold M with metric tensor g and an almost contact structure (φ, ξ, η) such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2}$$

for all $X, Y \in \chi(M)$ is called an almost contact metric manifold.

The Sasaki form Φ of an almost contact metric manifold $(M, g, \varphi, \xi, \eta)$ is defined by

$$\Phi(X,Y) = g(X, \varphi Y) \quad \text{for all} \quad X, Y \in \chi(M). \tag{3}$$

2. Linear connections on almost contact manifolds

Let M be a differentiable manifold with an almost contact structure (φ, ξ, η). We consider the distributions $H = Ker\ \eta$ and $V = Ker\ \varphi = \{\xi\}$ on M and we denote

$$h = I - \xi \otimes \eta, \quad v = \xi \otimes \eta \tag{4}$$

the projections on H and V respectively. We have (see [2]):

$$\begin{gathered} h^2 = h, \quad v^2 = v, \quad hv = vh = 0, \\ \varphi^2 = -h, \quad h\varphi = \varphi h = \varphi, \quad v\varphi = \varphi v = 0. \end{gathered} \tag{5}$$

Thus h and v are complementary projection operators on M.

Definition 2.1. We call Obata operators associated to an almost contact structure (φ, ξ, η) the applications $A, A^* : T_1^1(M) \to T_1^1(M)$ defined by

$$\begin{gathered} A(w) = \frac{1}{2}(w - v \circ w - w \circ v + 3v \circ w \circ v - \varphi \circ w \circ \varphi), \\ A^*(w) = w - A(w). \end{gathered} \tag{6}$$

Proposition 2.1. *A and A^* are complementary projection operators on $T_1^1(M)$.*

Proposition 2.2. *The tensorial equation*

$$A^*(u) = a, \quad a \in T_1^1(M) \tag{7}$$

has a solution $u \in T_1^1(M)$ if and only if $a \in Ker\ A$. If $a \in Ker\ A$, then the general solution of the equation (6) is

$$u = a + A(w), \quad w \in T_1^1(M). \tag{8}$$

A similar result holds for the equations of the form $A(u) = a$.

In the following, every tensor field $u \in T_1^1(M)$ may be considered as a field of $\chi(M)$-valued differential 1-forms. So, if ∇ is a linear connection on M, then we denote with D and $\tilde{D}$ the associated connections acting on the $\chi(M)$-valued differential 1-forms and respectively on the differential 1-forms

$$(D_X u)(Y) = \nabla_X(uY) - u(\nabla_X Y) \tag{9}$$

$$(\tilde{D}_X \eta)(Y) = X(\eta(Y)) - \eta(\nabla_X Y) \tag{10}$$

for every $u \in T_1^1(M)$ and $X, Y \in \chi(M)$.

Definition 2.2. A linear connection ∇ on M is called a (φ, ξ, η)-linear connection, if

$$D\varphi = 0, \quad \tilde{D}\eta = 0 \quad \nabla\xi = 0. \tag{11}$$

Of course, for every (φ, ξ, η)-linear connection we have

$$\nabla_X v = v\nabla_X, \quad \nabla_X h = h\nabla_X, \quad \forall X \in \chi(M). \tag{12}$$

We see that D and $\tilde{D}$ commute with the operators A and A^*.

We take $\nabla_X = \overset{\circ}{\nabla}_X + V_X$, where $\overset{\circ}{\nabla}$ is a linear connection fixed on M such that $\overset{\circ}{\nabla}\xi = 0$, $\overset{\circ}{\tilde{D}}\eta = 0$ and $V_X Y = V(X, Y)$, $V \in T_2^1(M)$ for every $X, Y \in \chi(M)$ and we find the tensor field V so that it satisfies the conditions (11).

∇ will be a (φ, ξ, η)-linear connection if and only if the field V satisfies the system of the tensorial equations

$$\begin{cases} V_X \circ \varphi - \varphi \circ V_X = -(\overset{\circ}{D}_X \varphi), & \forall X \in \chi(M) \\ \eta \circ V_X = 0, \quad V_X \xi = 0. \end{cases} \tag{13}$$

This system is equivalent with the system

$$\begin{cases} A^*(V_X) = -\frac{1}{2}\{\varphi \circ \overset{\circ}{D}_X \varphi + 2v \circ \overset{\circ}{D}_X h - \overset{\circ}{D}_X h \circ v\} \\ \eta \circ V_X = 0, \quad V_X \xi = 0, \quad \forall X \in \chi(M). \end{cases} \tag{14}$$

By straightforward calculus it is proved that

$$a(X) = -\frac{1}{2}(\varphi \circ \overset{\circ}{D}_X \varphi + 2v \circ \overset{\circ}{D}_X h - \overset{\circ}{D}_X h \circ v) \tag{15}$$

is a nontrivial solution of the system (14) and applying Proposition 2.2 it becomes that the system (14) has the general solution

$$V_X = a(X) + A(W_X), \tag{16}$$

where $W \in T_2^1(M)$ must be verify the conditions

$$\eta \circ A(W_X) = 0, \quad A(W_X)(\xi) = 0, \quad \forall X \in \chi(M). \tag{17}$$

Theorem 2.1. *There are (φ, ξ, η)-linear connections: one of them is*

$$\nabla_X = \overset{\circ}{\nabla}_X - \frac{1}{2}(\varphi \circ \overset{\circ}{D}_X \varphi + 2v \circ \overset{\circ}{D}_X h - \overset{\circ}{D}_X h \circ v), \tag{18}$$

where $\overset{\circ}{\nabla}$ is a linear connection on M such that $\overset{\circ}{\nabla}\xi = 0$, $\tilde{\overset{\circ}{D}}\eta = 0$, $\overset{\circ}{D}$ and $\tilde{\overset{\circ}{D}}$ being its associate connections.

Theorem 2.2. *The set of all (φ, ξ, η)-linear connections is given by*

$$\overline{\nabla}_X = \nabla_X + A(W_X), \tag{19}$$

where ∇ is an (φ, ξ, η)-linear connection and $W \in T_2^1(M)$ satisfies the conditions (17).

Observing that (17) and (19) can be considered as a transformation of (φ, ξ, η)-linear connections we have the following theorem.

Theorem 2.3. *The set of the transformations of (φ, ξ, η)-linear connections together the multiplication of transformations is an abelian group. Furthermore, this group, denoted by $G(\varphi, \xi, \eta)$ - is isomorphic to the additive group of the tensors $W \in T_2^1(M)$ which satisfies the conditions (17) and (19).*

3. Linear connections on almost contact metric manifolds

Let M be a differentiable manifold with an almost contact metric structure (g, φ, ξ, η). The Riemannian structure g on M can be considered as a $T_0^1(M)$-valued differential 1-form meaning that $g : T_0^1(M) \to T_0^1(M)$, $g(X) = g_X$, where $g_X(Y) = g(X, Y)$ for every $X, Y \in T_0^1(M)$. For $\psi \in T_1^1(M)$, ${}^t\psi$ denotes the transpose of ψ, ${}^t\psi : T_1^0(M) \to T_1^0(M)$, ${}^t\psi(\omega) = \omega \circ \psi$, $\forall \omega \in T_1^0(M)$. We remark that ${}^t\varphi(\eta) = 0$.

Proposition 3.1. *For an almost contact metric structure (g, φ, ξ, η) on M and h, v defined by the equations (5) we have*

$$g \circ \varphi = -{}^t\varphi \circ g, \quad \varphi \circ g^{-1} = -g^{-1} \circ {}^t\varphi$$
$$g \circ v = {}^t v \circ g, \quad g \circ h = {}^t h \circ g \tag{20}$$
$$v \circ g^{-1} = g^{-1} \circ {}^t v, \quad h \circ g^{-1} = g^{-1} \circ {}^t h.$$

We also consider the Obata operators associated to g, [4],

$$B(u) = \frac{1}{2}(u - g^{-1t}ug), \quad B^*(u) = u - B(u). \tag{21}$$

Proposition 3.2. *For an almost contact metric structure (g, φ, ξ, η) on M and for A, A^* and B, B^* defined by (6) and (21) we have:*
1) B and B^* are complementary operators on $T_1^1(M)$;
2) B and B^* commute pairwise with A and A^*;
3) $A \circ B$ and $A^* \circ B^*$ are projections on $T_1^1(M)$;
4) $Ker\ A \cap KerB = Im(A \circ B)$.

In fact,

$$(A \circ B - B \circ A)(u) = \frac{1}{4}\{(v \circ g^{-1} \circ {}^t u \circ g$$
$$-g^{-1} \circ {}^t v \circ {}^t u \circ g) + (g^{-1} \circ {}^t u \circ g \circ v - g^{-1} \circ {}^t u \circ {}^t v \circ g)$$
$$-3(v \circ g^{-1} \circ {}^t u \circ g \circ v - g^{-1} \circ {}^t v \circ {}^t u \circ {}^t v \circ g)$$
$$+(\varphi \circ g^{-1} \circ {}^t u \circ g \circ \varphi - g^{-1} \circ {}^t\varphi \circ {}^t u \circ {}^t\varphi \circ g)\} = 0$$

for every $u \in T_1^1(M)$, or

$$A \circ B = B \circ A.$$

Thus we have the relations

$$A \circ B^* = B^* \circ A, \quad A^* \circ B = B \circ A^*.$$

The above relations give us the possibility to formulate the following proposition (see [6]).

Proposition 3.3. *The system of tensorial equations*

$$A(u) = a, \quad B(u) = b \tag{22}$$

has a solution $u \in T_1^1(M)$ *if and only if*

$$A(a) = 0, \quad B(b) = 0, \quad A^*(b) = B^*(a). \tag{23}$$

If the conditions (23) are fulfilled, then the general solution of the system (22) is

$$u = a + A(b) + (A \circ B)(w), \quad \forall w \in T_1^1(M). \tag{24}$$

Definition 3.1. A linear connection ∇ on M is called (g, φ, ξ, η)-connection, if

$$\tilde{D}g = 0, \quad D\varphi = 0, \quad \tilde{D}\eta = 0, \quad \nabla\xi = 0, \tag{25}$$

where the associate connection D acts on the differential 1-forms $g : \chi(M) \to \chi^*(M)$ in the following way:

$$D_X g = {}^t\nabla_X \circ g - g \circ \nabla_X, \quad X \in \chi(M). \tag{26}$$

Of course, for every (g, φ, ξ, η)-linear connection we have

$$D_X h = \nabla_X h - h\nabla_X = 0, \quad D_X v = \nabla_X v - v\nabla_X = 0 \tag{27}$$

and the associate connections D and $\tilde{D}$ commute with the operators A, A^*, B and B^*.

We take $\nabla_X = \overset{\circ}{\nabla}_X + V_X$, where $\overset{\circ}{\nabla}$ is an linear connection fixed on M such that $\overset{\circ}{\nabla}\xi = 0$, $\overset{\circ}{\bar{D}}\eta = 0$, and $V_X Y = V(X, Y)$, $V \in T_2^1(M)$ for every $X, Y \in \chi(M)$ and we find the tensor field V so that it satisfies the conditions (25).

∇ will be a (g, φ, ξ, η)-linear connection if and only if the field V verifies the following system of tensorial equations

$$\begin{cases} V_X \circ \varphi - \varphi \circ V_X = -\overset{\circ}{D}\varphi, & \eta \circ V_X = 0, \\ {}^tV_X \circ g + g \circ V_X = \overset{\circ}{\tilde{D}}_X g, & V_X\xi = 0. \end{cases} \tag{28}$$

This system is equivalent with the system

$$\begin{cases} A^*(V_X) = -\frac{1}{2}(\varphi \circ \overset{\circ}{D}_X\varphi + \overset{\circ}{D}_X v - 3v \circ \overset{\circ}{D}_X v) \\ B^* = \frac{1}{2}g^{-1} \circ \overset{\circ}{\tilde{D}}_X g, \quad \eta \circ V_X = 0, \quad V_X\xi = 0. \end{cases} \tag{29}$$

Applying Proposition 3.3, it becomes evident that the system (28) has solutions and the general solution is

$$V_X = -\frac{1}{2}(\varphi \circ \overset{\circ}{D}_X \varphi + \overset{\circ}{D}_X v - 3v \circ \overset{\circ}{D}_X v) \tag{30}$$

$$+\frac{1}{4}g^{-1}(\overset{\circ}{D}_X g - {}^t\varphi \circ D_X g \circ \varphi - \overset{\circ}{D}_X g \circ v - {}^t v \circ \overset{\tilde{\circ}}{D}_X g + 3{}^t v \circ \overset{\tilde{\circ}}{D}_X g \circ v)$$

$$+(A \circ B)(W_X),$$

where $W \in T_2^1(M)$ must be verify the conditions

$$\eta \circ A \circ B(W_X) = 0, \quad A \circ B(W_X)(\xi) = 0, \quad \forall X \in \chi(M). \tag{31}$$

Thus we have the following theorem.

Theorem 3.1 *There are (g, φ, ξ, η)-linear connections: one of them is*

$$\nabla_X = \overset{\circ}{\nabla}_X + V_X, \tag{32}$$

where $\overset{\circ}{\nabla}$ is a linear connection such that $\overline{\overset{\circ}{D}}\eta = 0$, $\overset{\circ}{\nabla}\xi = 0$, and V_X is given by (30), $W \in T_2^1(M)$ being a tensor field which verifies the conditions (31).

References

[1] D. E. B l a i r, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics **509**, Springer-Verlag, Berlin (1976).

[2] P. M a t z e u, Su una caratterizzazione delle varieta normali, *Bolletino U.M.I., Algebra e Geometria, Serie VI* **IV-D**, No 1 (1985), 153-160.

[3] D. M o n a r, Affine connections on manifolds with almost contact 3-structure, *Riv. Mat. Univ. Parma* **14**, No 4 (1988), 237-248.

[4] R. M i r o n, Gh. A t a n a s i u, Sur les (f, g)-connexions lineaires et l'integrabilité des (f, g)-structures, *Bull. Math. de la Soc. Sci. Math. de la R.S. de Roumanie* **30 (78)**, No 3 (1986), 245-256.

[5] N. S o a r e, On the existence of the linear connections compatible with some (f, g)-structures, *Demonstratio Math.* **XXIX**, No 1 (1996), 149-154.

[6] A. C. W i l d e, Commutative projection operators, *Atti Sem. Math. Fis. Univ. Modena* **XXXV** (1987), 167-172.

Facultatea de Matematică
Str. Academiei 14
70109 București, ROMANIA

Received: 12.12.1999