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On a Nonlocal Boundary Value Problem For an Equation of Mixed Type with Discontinuous Coefficients

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A nonlocal boundary-value problem for a second order differential equation of mixed type with discontinuous coefficients in a bounded multidimensional cylindrical domain is considered. Uniqueness of a generalized solution from an appropriate class of functions is proved.

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1. Introduction

Let G be a bounded domain in the space $\mathbf{R}^{m-1}$, where $m \geq 2$ is an integer number. We suppose that $G = G_1 \cup G_2 \cup \sigma$, where G_1 and G_2 are domains in $\mathbf{R}^{m-1}$. For $m \geq 3$ we suppose that $\bar{G}_1 \subset G$, $G_2 = G \backslash \bar{G}_1$, ∂G and $\sigma = \partial G_1$ are $(m-2)-$ dimensional smooth surfaces (smooth curves for $m = 3$). If $m = 2$, then $G_1 = (A_1, A_2)$, $G_2 = (A_2, A_3)$, $G = (A_1, A_3)$. We denote $D = G \times (0, T)$, $\sigma_T = \sigma \times (0, T)$, $S_T = \partial G \times (0, T)$, $D_r = G_r \times (0, T)$, $r = 1, 2$, where $T = const > 0$. Let $x' = (x_1, \ldots, x_{m-1})$, $x = (x_1, \ldots, x_m)$.

In D_r we consider the operator

$$\mathcal{L}_r u = \sum_{i,j=1}^{m-1} a_{ij}^{(r)}(x) u_{x_i x_j} + k^{(r)}(x) u_{x_m x_m} + \sum_{i=1}^{m} b_i^{(r)}(x) u_{x_i} + c^{(r)}(x) u, \tag{1}$$

where $k^{(r)}, a_{ij}^{(r)} \in C^2(\bar{D}_r)$, $b_i^{(r)}, b_m^{(r)}, c^{(r)} \in C^1(\bar{D}_r)$, $a_{ij}^{(r)} = a_{ji}^{(r)}$, $i, j = 1, \ldots, m-1$; there exists $a_0^{(r)} = const > 0$ such that $\sum_{i,j=1}^{m-1} a_{ij}^{(r)}(x)\xi_i\xi_j \geq a_0^{(r)} \sum_{i=1}^{m-1} \xi_i^2$, $\forall x \in$

$\bar{D}_r$, $\forall\xi' \in \mathbf{R}^{m-1}$; $k^{(r)}(x',T) = k^{(r)}(x',0) \le 0$, $\forall x' \in \bar{G}_r$. All the functions in the present paper are assumed real-valued.

There are no any restrictions on the sign of $k^{(r)}(x)$ for $x \in D_r$. Then the operator (1) is of mixed type in D_r.

Let $f^{(r)}$, defined in D_r, and $\gamma^{(r)} = \gamma^{(r)}(x') > 0$ $\forall x' \in \bar{G}_r$ be given functions, $r = 1, 2$, and $\mu \in \mathbf{R} \setminus \{0\}$ be a given constant.

We consider the following nonlocal boundary-value problem.

Find a function $u(x)$ in $\bar{D}$ which satisfies the equations

$$\mathcal{L}_r u = f^{(r)} \quad \text{in} \quad D_r,\ r = 1, 2, \tag{2}$$

the boundary conditions

$$u = 0 \quad \text{on} \quad S_T,\ u(x',T) = \mu u(x',0) \quad \text{in}\ G, \tag{3}$$

$$u^{(r)}_{x_m}(x',T) = \mu u^{(r)}_{x_m}(x',0) \quad \text{in} \quad G_r^-,\ r = 1, 2, \tag{4}$$

and the conditions

$$u^{(1)}|_{\sigma_T} = u^{(2)}|_{\sigma_T},\ \gamma^{(1)}\frac{\partial u^{(1)}}{\partial N^{(1)}}|_{\sigma_T} = \gamma^{(2)}\frac{\partial u^{(2)}}{\partial N^{(2)}}|_{\sigma_T}\ , \tag{5}$$

where $G_r^- = \{x' \in G_r :\ k^{(r)}(x',0) < 0\}$, $\dfrac{\partial}{\partial N^{(r)}} = \sum\limits_{i,j=1}^{m-1} a_{ij}^{(r)}(x) n_j^{(1)} \dfrac{\partial}{\partial x_i}$, $(n_1^{(r)}, \ldots, n_m^{(r)})$ is the normal unit vector outward to D_r at the point $x \in \sigma_T$, $u^{(r)} = u|_{\bar{D}_r}$, $r = 1, 2$.

In the case $\mu = 0$ problem (2)-(5) is local one and it is investigated in [1]-[4], [6], [7] and other papers.

We denote by $\tilde{C}$ the set of all functions, which are continuous in $\bar{D}$, belong to $C^2(\bar{D}_r)$, $r = 1, 2$, and satisfy (3)-(5). Let $\tilde{W}^1$ be the closure of $\tilde{C}$ with respect to the norm $\|u\|_1 = (\|u\|_0^2 + \sum\limits_{i=1}^{m} \|u_{x_i}\|_0^2)^{1/2}$ of the Sobolev space $W_2^1(D)$, where we used the notation $\|\,.\,\|_0$ for the usual norm of $L_2(D)$. Let W^1 be the closure in the norm $\|\,.\,\|_1$ of the set of all continuous in $\bar{D}$ functions, which belong to $C^2(\bar{D}_r)$, $r = 1, 2$, and satisfy the conditions

$$v = 0 \quad \text{on} \quad S_T,\quad v(x',0) = \mu v(x',T) \quad \text{in} \quad G_1^- \cup G_2^-. \tag{6}$$

Let $f^{(r)} \in L_2(D)$, $\gamma^{(r)} \in C^1(\bar{G}_r)$, $r = 1, 2$.

Definition. A function u(x) is called a generalized solution of problem (2)-(5), if $u \in \tilde{W}^1$ and

$$B[u,\,v] \equiv \sum_{r=1}^{2} \int_{D_r} \{-(k^{(r)}\gamma^{(r)}v)_{x_m} u_{x_m} - \sum_{i,j=1}^{m-1} (a_{ij}^{(r)}\gamma^{(r)}v)_{x_j} u_{x_i} + (c^{(r)}u$$

$$(7)\qquad + \sum_{i=1}^{m} b_i^{(r)} u_{x_i})\gamma^{(r)}v\}\,dx = \sum_{r=1}^{2} \int_{D_r} f^{(r)}\gamma^{(r)}v\,dx, \qquad \forall v \in W_1.$$

In the sequel we assume $0 < |\mu| < 1$ and denote $\nu = -T^{-1}\ln\mu^2$.

It is not difficult to prove the following two lemmas.

Lemma 1. *Let $u \in C(\bar{D})$ and*

$$(8)\qquad V(x) = (\mu - 1)\int_0^{x_m} \exp(\nu\theta)u(x',\theta)\,d\theta - \mu \int_0^{T} \exp(\nu\theta)u(x',\theta)\,d\theta$$

for $x \in \bar{D}$. Then a constant $c(\mu) > 0$, depending only on μ, exists such that $\|V\|_0 \le c(\mu)T\|u\|_0$.

Lemma 2. *Let $u \in \tilde{C}$ and V be the function defined by (8). Then V, $V_{x_m} \in C(\bar{D})$; $V^{(r)}$, $V_{x_m}^{(r)} \in C^2(\bar{D}_r)$, $r = 1,2$; V satisfies the conditions (6) and $V_{x_m} = 0$ on S_T, $V_{x_i}(x',0) = \mu V_{x_i}(x',T)$ on $G_1 \cup G_2$, $i = 1,2,\ldots,m$.*

Lemma 3. *For each $u \in \tilde{W}^1$ a unique element $V \in W^1$ exists with the property: if $\{u_n\}_{n=1}^{\infty} \subset \tilde{C}$ is a sequence convergent to u in $W_2^1(D)$ and*

$$V_n(x) = (\mu - 1)\int_0^{x_m} \exp(\nu\theta)u_n(x',\theta)\,d\theta - \mu \int_0^{T} \exp(\nu\theta)u_n(x',\theta)\,d\theta \quad \text{in } \bar{D}$$

for $n = 1,2,\ldots$, then $V_n \underset{n\to\infty}{\longrightarrow} V$ in $W_2^1(D)$.

Proof. Let $u \in \tilde{W}^1$, $\{u_n\}_{n=1}^{\infty} \subset \tilde{C}$ and $u_n \underset{n\to\infty}{\longrightarrow} u$ in $W_2^1(D)$. From Lemma 1 it follows that $\|V_n - V_s\|_0 \le c(\mu)T\|u_n - u_s\|_0\ \forall n \in \mathbf{N}$, $\forall s \in \mathbf{N}$. Then $V \in L_2(D)$ exists such that $V_n \underset{n\to\infty}{\longrightarrow} V$ in $L_2(D)$. For $1 \le i \le m-1$ we find $\dfrac{\partial V_n}{\partial x_i}$ in $\bar{D} \setminus \bar{\sigma}_T$ differentiating with respect to x_i under the sign of the integrals in the expression for V_n, $n \in \mathbf{N}$. Hence $\dfrac{\partial V_n}{\partial x_i} \underset{n\to\infty}{\longrightarrow} w_i$ in $L_2(D)$. Obviously, $\dfrac{\partial V_n}{\partial x_m} \underset{n\to\infty}{\longrightarrow} (\mu - 1)\exp(\nu x_m)u$ in $L_2(D)$. Then the generalized derivatives of V are

$V_{x_i} = w_i$, $i = 1, \dots, m-1$, $V_{x_m} = (\mu - 1)\exp(\nu x_m)u$ (see [5], Ch. 1, Theorem 4.1). Hence $V_n \underset{n\to\infty}{\longrightarrow} V$ in $W_2^1(D)$ and $V \in W^1$ due to Lemma 2.

Further, if $\{\tilde{u}_n\}_{n=1}^{\infty} \subset \tilde{C}$ is convergent to u in $W_2^1(D)$, then $\tilde{V}_n \underset{n\to\infty}{\longrightarrow} \tilde{V}$ in $W_2^1(D)$, where

$$\tilde{V}_n(x) = (\mu - 1)\int_0^{x_m} \exp(\nu\theta)\tilde{u}_n(x',\theta)\,d\theta - \mu\int_0^T \exp(\nu\theta)\tilde{u}_n(x',\theta)\,d\theta \quad \text{in } \bar{D}.$$

The inequality $\|V - \tilde{V}\|_0 \le \|V - V_n\|_0 + c(\mu)T\|u_n - \tilde{u}_n\|_0 + \|\tilde{V}_n - \tilde{V}\|_0$ implies that $V = \tilde{V}$ in D. ■

We denote $\beta_j^{(r)}(x) = b_j^{(r)}(x)\gamma^{(r)}(x') - \sum_{i=1}^{m-1}[a_{ij}^{(r)}(x)\gamma^{(r)}(x')]_{x_i}$ for $j = 1, \dots, m-1$ and $r = 1, 2$.

Lemma 4. *Let $\gamma^{(r)} \in C^2(\bar{G}_r)$ and the derivatives $b^{(r)}_{mx_mx_m}$, $k^{(r)}_{x_mx_mx_m}$ exist and belong to $C(\bar{D}_r)$, $r = 1, 2$. Let $a_{ij}^{(r)}(x',T) = a_{ij}^{(r)}(x',0)\ \forall x' \in \bar{G}_r$, $i,j = 1,\dots,m-1$, $r = 1,2$, and the following conditions*

(9) $$\gamma^{(r)}(2b_m^{(r)} - 3k_{x_m}^{(r)} - \nu k^{(r)}) \ge 2\alpha_r \quad \text{in } \bar{D}_r, \quad \alpha_r = const > 0,$$

(10) $$\begin{cases} \gamma^{(r)}\sum_{i,j=1}^{m-1}[\nu a_{ij}^{(r)}(x) - a_{ijx_m}^{(r)}(x)]\xi_i\xi_j \ge a_1^{(r)}\sum_{i=1}^{m-1}\xi_i^2, \quad \forall x \in \bar{D}_r, \\ \text{and } \forall \xi' \in \mathbf{R}^{m-1}, \ a_1^{(r)} = const \ge \dfrac{2}{\alpha_r}\max_{\bar{D}_r}\sum_{j=1}^{m-1}[\beta_j^{(r)}(x)]^2, \end{cases}$$

(11) $$\begin{cases} \nu[-c^{(r)} + (b_m^{(r)} - k_{x_m}^{(r)})_{x_m}] + c_{x_m}^{(r)} - (b_m^{(r)} - k_{x_m}^{(r)})_{x_mx_m} \\ \qquad \ge \dfrac{2}{\gamma^{(r)}\alpha_r}(\sum_{j=1}^{m-1}|\beta_{jx_j}^{(r)}|)^2 \quad \text{in } \bar{D}_r, \end{cases}$$

(12) $$(b_m^{(r)} - k_{x_m}^{(r)})(x',T) = (b_m^{(r)} - k_{x_m}^{(r)})(x',0) \quad \text{in } \bar{G}_r,$$

(13) $$\sum_{j=1}^{m-1}\beta_j^{(1)}n_j^{(1)}|_{\sigma_T} = \sum_{j=1}^{m-1}\beta_j^{(2)}n_j^{(1)}|_{\sigma_T},$$

(14) $$[c^{(r)} - (b_m^{(r)} - k_{x_m}^{(r)})_{x_m}](x',T) \le [c^{(r)} - (b_m^{(r)} - k_{x_m}^{(r)})_{x_m}](x',0) \quad \text{in } \bar{G}_r$$

hold for $r = 1,2$. Then a constant $c_0 > 0$ exists such that for every $u \in \tilde{W}^1$ and for its corresponding element V from Lemma 3 the estimate

(15) $$B[u, V] \ge c_0(1-\mu)\int_D \exp(\nu x_m)u^2\,dx$$

is valid.

Proof. Let $u \in \tilde{C}$ and V be given by (8). Using the connection $u(x) = (\mu - 1)^{-1} \exp(-\nu x_m) V_{x_m}(x)$ we express the first order derivatives of u by those of V up to the second order and put them in $B[u, V]$. Then, integrating by parts, we find

$$B[u, V] = \frac{1}{\mu - 1} \sum_{r=1}^{2} \{ \int_{D_r} \exp(-\nu x_m) \gamma^{(r)} (-b_m^{(r)} + \frac{3}{2} k_{x_m}^{(r)} + \frac{\nu}{2} k^{(r)}) V_{x_m}^2 \, dx$$

$$+ \frac{1}{2} \int_{D_r} \exp(-\nu x_m) \gamma^{(r)} \sum_{i,j=1}^{m-1} (a_{ijx_m}^{(r)} - \nu a_{ij}^{(r)}) V_{x_i} V_{x_j} \, dx$$

$$- \int_{D_r} \exp(-\nu x_m) V_{x_m} \sum_{j=1}^{m-1} \beta_j^{(r)} V_{x_j} \, dx - \int_{D_r} \exp(-\nu x_m) V V_{x_m} \sum_{j=1}^{m-1} \beta_{jx_j}^{(r)} \, dx$$

$$- \frac{1}{2} \int_{D_r} \exp(-\nu x_m) \gamma^{(r)} [\nu(-c^{(r)} + (b_m^{(r)} - k_{x_m}^{(r)})_{x_m}) + c_{x_m}^{(r)}$$

$$- (b_m^{(r)} - k_{x_m}^{(r)})_{x_m x_m}] V^2 \, dx + \frac{1}{2} \int_{\partial D_r} \exp(-\nu x_m) \gamma^{(r)} [c^{(r)}$$

$$- (b_m^{(r)} - k_{x_m}^{(r)})_{x_m}] V^2 n_m^{(r)} \, ds \} = \frac{1}{\mu - 1} \sum_{r=1}^{2} \sum_{j=1}^{6} I_j^{(r)} .$$

The other integrals on ∂D_r, $r = 1, 2$, are equal to zero due to (13), (14), the properties of V and the choice $\nu = -T^{-1} \ln \mu^2$.

Using the Hölder inequality for sums and the inequality $|ab| \le \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2$ for a, $b \in \mathbf{R}$ and $\varepsilon > 0$, we obtain the estimate

$$|I_3^{(r)} + I_4^{(r)}| \le \frac{1}{\alpha_r} \max_{\bar{D}_r} \sum_{j=1}^{m-1} [\beta_j^{(r)}(x)]^2 \int_{D_r} \exp(-\nu x_m) \sum_{j=1}^{m-1} V_{x_j}^2 \, dx$$

$$+ \frac{\alpha_r}{2} \int_{D_r} \exp(-\nu x_m) V_{x_m}^2 \, dx + \frac{1}{\alpha_r} \int_{D_r} \exp(-\nu x_m) V^2 (\sum_{j=1}^{m-1} |\beta_{jx_j}^{(r)}|)^2 \, dx.$$

Then from (9)-(12) it follows that

$$B[u, V] \ge \frac{1}{1 - \mu} \sum_{r=1}^{2} \frac{\alpha_r}{2} \int_{D_r} \exp(-\nu x_m) V_{x_m}^2 \, dx.$$

Hence (15) holds with $u \in \tilde{C}$ and V from (8). The general case of Lemma 4 is a consequence of the considered case and Lemma 3. ∎

Theorem 1. *Let all the assumptions of Lemma 4 hold. Then problem (2)-(5) can have no more than one generalized solution.*

P r o o f. If u_1 and u_2 are two generalized solutions of problem (2)-(5), then $u = u_1 - u_2$ is a generalized solution of that problem for $f^{(r)} = 0$, $r = 1, 2$. We apply Lemma 4 with V, corresponding to u according to Lemma 3, and from (7) we find that $0 \geq c_0(1-\mu)\int_D \exp(\nu x_m) u^2\,dx$. Hence $u = 0$ in D, i.e. $u_1 = u_2$ in D. ∎

Example. Let $G_1 = \{x' \in \mathbf{R}^2 : x_1^2 + x_2^2 < 1\}$, $G_2 = \{x' \in \mathbf{R}^2 : 1 < x_1^2 + x_2^2 < 4\}$, $\sigma = \{x' \in \mathbf{R}^2 : x_1^2 + x_2^2 = 1\}$, $G = G_1 \cup G_2 \cup \sigma$, $T = 2$, $\gamma^{(1)} = \gamma^{(2)} = 1$. We consider the equations

$$(16) \qquad \begin{cases} u_{x_1x_1} + u_{x_2x_2} + k^{(1)}(x_3)u_{x_3x_3} + b_3^{(1)}u_{x_3} - M^{(1)}u = f^{(1)} & \text{in} D_1 \\ u_{x_1x_1} + u_{x_2x_2} - u_{x_3x_3} - M^{(2)}u = f^{(2)} & \text{in} D_2, \end{cases}$$

where $k^{(1)}(x_3) = x_3^2(x_3-2)^2(1-x_3)$ and $b_3^{(1)}$, $M^{(1)}$, $M^{(2)}$ are constants. Let $\mu \neq 0$, $-1 < \mu < 1$ and $\nu = -\ln|\mu|$. If we take $b_3^{(1)} > \frac{3}{2}k_{x_3}^{(1)} + \frac{\nu}{2}k^{(1)}$ and $M^{(1)} \geq k_{x_3x_3}^{(1)} - \frac{1}{\nu}k_{x_3x_3x_3}^{(1)}$ in $\bar{D}_1$, $M^{(2)} \geq 0$, then the assumptions of Theorem 1 are satisfied. Hence problem (16), (3)-(5) can have at most one generalized solution.

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