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On a Class of Hyperbolic Equations with Order Degeneration

Rossitza Semerdjieva

Presented by V. Kiryakova

Consider the equation

$$L(u) := k(y)u_{xx} - \partial_y(\ell(y)u_y) + a(x,y)u_x + r(x,y)u = f(x,y),$$

with $k(y) > 0$, $\ell(y) > 0$ for $y > 0$ and $k(0) = \ell(0) = 0$; it is strictly hyperbolic for $y > 0$ and its order degenerates on the line $y = 0$. We study the boundary value problem $Lu = f(x,y)$ in G, $u|_{AC} = 0$, where G is a simply connected domain in R^2 with piecewise smooth boundary $\partial G = AB \cup AC \cup BC$; $AB = \{(x,0): 0 \le x \le 1\}$, $AC: x = F(y) = \int_0^y (k(t)/\ell(t))^{1/2} dt$ and $BC: x = 1 - F(y)$ are characteristic curves. Under the assumptions $k, r \in C(\overline{G})$, $\ell, a \in C^1(\overline{G})$ and $a(x,0) > 0$ for $x \in [0,1]$, it is proved that for each $f \in L^2(G)$ the boundary value problem has a unique strong solution in an appropriate weighted Sobolev space.

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1. Introduction

Degenerated hyperbolic equations in the plane are mainly studied in the case where the type of equation degenerates to parabolic one on the line of degeneration (see [1, 3, 8, 9] and the bibliography there). Bitsadze [1] observed that the case of order degeneration (i.e., where the entire principal part vanishes on the line of degeneration) deserves a special attention and requires a special treatment. Moreover, in that case the coefficients of lower order terms determine whether some boundary value problems are well posed (see [2, 4, 5, 6, 7] and the literature cited therein).

Consider the equation

$$L(u) := k(y)u_{xx} - \partial_y(\ell(y)u_y) + a(x,y)u_x + r(x,y)u = f(x,y), \tag{1}$$

where $k(y) > 0$, $\ell(y) > 0$ for $y > 0$, $k(0) = \ell(0) = 0$ and $\lim_{y\to 0} k(y)/\ell(y)$ exists. The equation (1) is strictly hyperbolic for $y > 0$ and its order degenerates on the line $y = 0$.

Let G be a simply connected domain on the (x, y) plane with piecewise smooth boundary $\partial G = AB \cup AC \cup BC$, where $AB = \{(x, 0) : 0 \le x \le 1\}$, and $AC : x = F(y) = \int_0^y (k(t)/\ell(t))^{1/2} dt$ and $BC : x = 1 - F(y)$ are characteristics of (1) issued from the point $C(1/2, Y)$, where the constant $Y > 0$ is determined by $F(Y) = 1/2$. Our aim here is to study the following boundary value problem.

Problem B. Find in the domain G a solution of (1) satisfying the boundary condition $u = 0$ on AC.

We set

$$(u, v)_0 = \int_G u(x, y)v(x, y)dx\, dy, \qquad \|u\|_0 = (u, u)_0^{1/2}$$

and

$$(u, v)_1 = \int_G [u_x(x, y)v_x(x, y) + \ell(y)u_y(x, y)v_y(x, y) + u(x, y)v(x, y)]dx\, dy.$$

Let $C^p_{AC}(\overline{G})$ and $C^p_{BC}(\overline{G})$, $p = 1, 2, \ldots, \infty$, be the sets of functions $u, v \in C^p(\overline{G})$ such that, respectively, $u|_{AC} = 0$ or $v|_{BC} = 0$. Denote, respectively, by H^1, H^1_{AC}, H^1_{BC} the corresponding weighted Sobolev spaces as completions of the spaces $C^\infty(\overline{G})$, $C^\infty_{AC}(\overline{G})$ and $C^\infty_{BC}(\overline{G})$ with respect to the norm

$$\|u\|_1 = (u, u)_1^{1/2} = \left(\int_G (u_x^2 + \ell u_y^2 + u^2)dx\, dy\right)^{1/2}.$$

If $u \in C^\infty_{AC}(\overline{G})$ and $v \in C^\infty_{BC}(\overline{G})$, then by Green's formula

$$\int_G \{\partial_x(ku_xv) - \partial_y(\ell u_y v)\}dxdy = \int_{\partial G} (ku_x v dy + \ell u_y v dx) = 0.$$

Indeed, $\int_{\partial G} = \int_{AB} + \int_{BC} + \int_{CA}$, where $\int_{AB} = 0$ because $k(0) = 0$ and $\ell(0) = 0$; $\int_{BC} = 0$ because $v \equiv 0$ on BC; finally

$$\int_{CA} = -\int_0^Y \sqrt{k\ell}\left(u_x\sqrt{k/\ell} + u_y\right) v dy = 0$$

since on $CA : x = F(y)$ we have $dx = F'(y)dy = \sqrt{k/\ell}dy$ and $u \equiv 0$ on CA implies $u_x\sqrt{k/\ell} + u_y = 0$.

Therefore, for $u \in C^\infty_{AC}(\overline{G})$ and $v \in C^\infty_{BC}(\overline{G})$

$$(2) \qquad (Lu, v)_0 = \int_G \{Lu \cdot v\} dx dy = B[u, v],$$

where

$$B[u, v] = \int_G \{-k u_x v_x + \ell u_y v_y + a u_x v + r u v\} dx dy.$$

Suppose $f(x, y) \in L^2(G)$. In view of identity (2) we state the following definition:

Definition 1. A function $u \in H^1_{AC}$ is called a generalized solution of Problem B, if the identity

$$(3) \qquad B[u, v] = \int_G f(x, y) v dx\, dy$$

is satisfied for any function $v \in H^1_{BC}$.

We consider also the notion of strong solution.

Definition 2. A function $u \in H^1_{AC}$ is called a strong solution of Problem B, if there exists a sequence $(u_n)_{n=1}^\infty$, $u_n \in C^\infty_{AC}(\overline{G})$ such that

$$\|u_n - u\|_1 + \|Lu_n - f(x, y)\|_0 \to 0 \quad \text{as } n \to \infty.$$

Our approach is based on functional-analytic methods. We obtain by an energy-integral method the necessary a priori estimates in Section 2.

In Section 3 the existence of generalized solution of Problem B is proved (in an appropriate weighted Sobolev space) assuming only that $a(x, y)$ is strictly positive on the line of degeneration AB. This result improves the existence theorem in [6], where a similar statement was proved under the following additional assumption: $\ell'(0) > 0$.

In Section 4 we show that each generalized solution of Problem B is also a strong solution and use this fact to prove uniqueness of generalized solution.

Our final Section 5 contains examples and comments.

2. A priori estimates

Lemma 1. *If $k, \ell \in C[0, Y]$ and $v \in C^\infty_{BC}(\overline{G})$, then the boundary problem*

$$h(u) := e^{-\lambda x} u_x = v \ in\, G, \qquad u|_{AC} = 0,$$

has a unique solution $u \in C^1_{AC}(\overline{G})$.

Indeed, consider

$$u(x,y) = \int_{F(y)}^{x} e^{\lambda t} v(t,y)dt. \tag{4}$$

Remark. Although $v \in C^\infty(\overline{G})$, the function u given by (4) may not have second partial derivative u_{yy} because $F'(y) = \sqrt{k/\ell}$ may not be differentiable. Observe, however, that $u_x = e^{\lambda x} v \in C^\infty(\overline{G})$, which allows us to use Green's formula in the next lemma.

Lemma 2. *Suppose $k(y) \in C[0,Y]$, $\ell(y) \in C^1[0,Y]$, $a(x,y), r(x,y) \in C(\overline{G})$ and*

$$m_0 := \inf_{[0,1]} a(x,0) > 0. \tag{5}$$

Then for every $m \in (0, m_0)$ there is $\lambda > 0$ such that:

(a) for $u \in C^\infty_{AC}(\overline{G})$

$$(Lu, e^{-\lambda x} u_x)_0 \geq m \int_G e^{-\lambda x}[u_x^2 + \ell u_y^2 + u^2]dx\,dy; \tag{6}$$

(b) for $v \in C^\infty_{BC}(\overline{G})$ and u related to v as in Lemma 1, it holds

$$B[u,v] = B[u,h(u)] \geq m \int_G e^{-\lambda x}[u_x^2 + \ell u_y^2 + u^2]dx\,dy. \tag{7}$$

Proof.

(a) By Green's formula,

$$(Lu, e^{-\lambda x} u_x)_0 = \int_G (Lu) e^{-\lambda x} u_x dx dy$$

$$= \int_G e^{-\lambda x}\left[\frac{1}{2}\lambda k u_x^2 + \frac{1}{2}\lambda \ell u_y^2 + a u_x^2 + r u u_x\right] dx\,dy$$

$$+\frac{1}{2}\int_{\partial G} e^{-\lambda x}\left[2\ell u_x u_y dx + (k u_x^2 + \ell u_y^2)dy\right].$$

The line integral $\int_{\partial G} = \int_{AB} + \int_{BC} + \int_{CA}$ is nonnegative. Indeed, $\int_{AB} = 0$ because $k(0) = \ell(0) = 0$. On $BC : x = 1 - F(y)$ we have $dx = -\sqrt{k/\ell}dy$, therefore

$$\int_{BC} = \frac{1}{2}\int_0^Y e^{-\lambda(1-F(y))}(\sqrt{k}u_x - \sqrt{\ell}u_y)^2 dy \geq 0.$$

On $AC : x = F(y)$ we have $dx = \sqrt{k/\ell}dy$, and, in addition, $u \equiv 0$ on AC implies $\sqrt{k}u_x + \sqrt{\ell}u_y \equiv 0$ on AC, therefore

$$\int_{CA} = \frac{1}{2}\int_Y^0 e^{-\lambda F(y)}(\sqrt{k}u_x + \sqrt{\ell}u_y)^2 dy = 0.$$

Hence,

(8) $$(Lu, e^{-\lambda x}u_x)_0 \geq I(\lambda),$$

where

(9) $$I(\lambda) = \frac{1}{2}\int_G e^{-\lambda x}\left[\lambda k u_x^2 + \lambda \ell u_y^2 + 2au_x^2 + 2ruu_x\right] dx\, dy.$$

Taking into account that

$$0 \leq \int_{\partial G} e^{-\lambda x}u^2 dy = \int_G \left(-\lambda e^{-\lambda x}u^2 + e^{-\lambda x}2uu_x\right) dx\, dy,$$

we obtain

(10) $$I(\lambda) \geq \frac{1}{2}\int_G e^{-\lambda x}\left[(\lambda k + 2a)u_x^2 + \lambda \ell u_y^2 + 2(r-1)uu_x + \lambda u^2\right] dx\, dy.$$

Fix $m < m_0$ and put $\varepsilon = (m_0 - m)/2$. Since $a(x,y)$ is continuous in $\overline{G}$, it is uniformly continuous in $\overline{G}$, so there is a $\delta > 0$ such that

(11) $$a(x,y) > m + \varepsilon \qquad \forall (x,y) \in G_\delta^1,$$

where

$$G_\delta^1 = \{(x,y) \in \overline{G} : \ 0 \leq y \leq \delta\}.$$

Using the inequality

$$2(r-1)uu_x \geq -\varepsilon^{-1}(r-1)^2u^2 - \varepsilon u_x^2,$$

we obtain from (10)

(12) $$I(\lambda) \geq \frac{1}{2}\int_G e^{-\lambda x}\left[A(x,y)u_x^2 + \lambda \ell u_y^2 + C(x,y)u^2\right] dx\, dy,$$

where

$$A(x,y) = \lambda k + 2a - \varepsilon, \quad C(x,y) = \lambda - \varepsilon^{-1}(r-1)^2.$$

Set

$$G_\delta^2 = \{(x,y) \in \overline{G} : \ \delta \leq y\}$$

and choose $\lambda > 0$ so that

$$\lambda \geq \sup_{G_\delta^2} \frac{2m + \varepsilon + 2|a(x,y)|}{k(y)}, \quad \lambda \geq 2m + \sup_G \varepsilon^{-1}(r(x,y) - 1)^2.$$

Then, by (11) and the choice of λ we obtain for any $(x,y) \in \overline{G}$,

$$A(x,y) \geq 2m, \qquad C(x,y) \geq 2m,$$

which completes the proof of (a).

(b) By Green's formula we obtain with $h(u) = \exp(-\lambda x)u_x$,

$$B[u,h(u)] = \int_G [-ku_x\partial_x h(u) + \ell u_y \partial_y h(u) + au_x h(u) + ruh(u)]dx\,dy$$

$$= \int_G e^{-\lambda x}[\lambda k u_x^2 - ku_x u_{xx} + \ell u_y u_{xy} + au_x^2 + ruu_x]dx\,dy$$

$$= \int_G e^{-\lambda x}\left[\frac{\lambda}{2}ku_x^2 + \frac{\lambda}{2}\ell u_y^2 + au_x^2 + ruu_x\right]dxdy + \frac{1}{2}\int_{\partial G} e^{-\lambda x}\left(-ku_x^2 + \ell u_y^2\right)dy.$$

It is easy to see that the line integral

$$\int_{\partial G} e^{-\lambda x}\left(-ku_x^2 + \ell u_y^2\right)dy = \int_{AB} + \int_{BC} + \int_{CA}$$

is nonnegative. Indeed, $\int_{AB} = 0$ because $k(0) = \ell(0) = 0$; on BC we have $h(u) = e^{-\lambda x}u_x = v(x,y) = 0$, thus $u_x = 0$, *so* the line integral on BC equals $\int_0^Y \exp(-\lambda(1 - F(y))\ell u_y^2 dy \geq 0$. Finally, on CA we have $u = 0$, which implies $ku_x^2 - \ell u_y^2 = 0$ on CA, so $\int_{CA} = 0$.

Hence

$$B[u,h(u)] \geq I(\lambda). \tag{13}$$

Since the right-hand side in this inequality is the same as in (8), the argument used to prove (a) proves as well (b).

3. Existence of generalized solution

Theorem 1. *If $k(y) \in C[0,Y]$, $\ell(y) \in C^1[0,Y]$, $a(x,y), r(x,y) \in C(\overline{G})$ and*

$$m_0 = \inf_{[0,1]} a(x,0) > 0,$$

then for every $f(x,y) \in L^2(G)$ there exists a generalized solution $w \in H^1_{AC}$ of Problem B.

Proof. Fix $m \in (0, m_0)$ and choose $\lambda > 0$ as in Lemma 2. Then,

$$\tilde{m}\|v\|_0 \leq \sup_{u \in H^1_{AC}} \frac{|B[u,v]|}{\|u\|_1} \leq C\|v\|_1 \qquad \forall v \in H^1_{BC}, \tag{14}$$

where $\tilde{m} = m\exp(-\lambda)$, $C = const > 0$. Indeed, the right-hand side of (14) follows immediately from Cauchy inequality. Since the set $C^\infty_{BC}(\overline{G})$ is dense in the space H^1_{BC}, it is enough to prove the left-hand side of (14) for $v \in C^\infty_{BC}(\overline{G})$. Fix $v \in C^\infty_{BC}(\overline{G})$. Then by Lemma 1, there exists $u \in C^1_{AC}(\overline{G})$ such that $h(u) = \exp(-\lambda x)u_x = v$, and from part (b) of Lemma 2 it follows

$$B[u, h(u)] \geq \tilde{m}\|u\|_1^2.$$

Thus,

$$\tilde{m}\|v\|_0 = \tilde{m}\|e^{-\lambda x}u_x\|_0 \leq \tilde{m}\|u\|_1 \leq \frac{B[u,v]}{\|u\|_1},$$

which proves (14).

Due to (14) for every fixed $v \in H^1_{BC}$ the linear functional

$$S_v(u) = B[u,v], \quad u \in H^1_{AC}$$

is continuous and its norm satisfies

$$\tilde{m}\|v\|_0 \leq \|S_v\| \leq C\|v\|_1.$$

Thus, by Riesz Representation Theorem there exists a linear continuous operator $T : H^1_{BC} \to H^1_{AC}$ such that

$$B[u,v] = S_v(u) = (u, Tv)_1,$$

and

$$\tilde{m}\|v\|_0 \leq \|Tv\|_1 \leq C\|v\|_1.$$

Fix $f \in L^2(G)$ and consider on the range $R(T)$ of T a linear functional

$$K(Tv) = \int_G f v\,dx\,dy.$$

Since

$$\left|\int_G f v\,dx\,dy\right| \leq \|f\|_0\|v\|_0 \leq \frac{1}{\tilde{m}}\|f\|_0\,\|Tv\|_1,$$

the linear functional K is continuous and its norm does not exceed $(1/\tilde{m})\|f\|_0$. Obviously, there is a unique linear continuous extension of K on the Hilbert space $\overline{R(T)}$. Therefore by Riesz Representation Theorem there exists a unique $w \in \overline{R(T)} \subset H^1_{AC}$ such that

$$K(u) = (w, u)_1 \qquad \forall u \in \overline{R(T)}.$$

So, for $u = Tv$, $v \in H^1_{BC}$ we have

$$B[w, u] = (w, Tv)_1 = K(Tv) = \int_G fv\,dx\,dy,$$

hence w is a generalized solution of Problem B.

Remark. We prove that for each $f \in L^2(G)$ there is a unique generalized solution $w \in \overline{R(T)}$ of Problem B. But, if $\overline{R(T)} \neq H^1_{AC}$, then there are other generalized solutions of the same problem, namely for every $z \in H^1_{AC}$ with $z \perp \overline{R(T)}$ the sum $w + z$ is also a solution.

4. Strong solutions

Choose $\eta \in C_0^\infty(-\infty, \infty)$ such that $\eta(x) = 0$ if $|x| \geq 1$ and $\int_{-1}^{1} \eta(x)dx = 1$. Consider for $\varepsilon > 0$

$$\varphi_\varepsilon(x, y) = \frac{1}{\varepsilon^2}\eta\left(\frac{x}{\varepsilon} - 3E - 2\right)\eta\left(\frac{y}{\varepsilon} + 2\right), \tag{15}$$

where $E = \sup_{[0,Y]} \sqrt{k(y)/\ell(y)}$, so

$$|F(y) - F(y_1)| \leq E|y - y_1| \quad \text{for} \quad y, y_1 \in [0, Y].$$

The following lemma makes an important observation: if (x, y) is fixed in an appropriate neighborhood of AC, then for each $(x_1, y_1) \in \overline{G}$ the function $\varphi_\varepsilon(x - x_1, y - y_1)$ vanishes. Symmetrically, if (x_1, y_1) is fixed in a suitable neighborhood of BC, then the same function vanishes for all $(x, y) \in \overline{G}$.

Lemma 3. *Suppose* $(x, y), (x_1, y_1) \in \overline{G}$. *Then*

$$\varphi_\varepsilon(x - x_1, y - y_1) \equiv 0 \quad \textit{if } x < F(y) + \varepsilon, \tag{16}$$

and

$$\varphi_\varepsilon(x - x_1, y - y_1) \equiv 0 \quad \textit{if } 1 - F(y_1) - \varepsilon < x_1. \tag{17}$$

In addition,

$$\varphi_\varepsilon(x - x_1, y - y_1) \equiv 0 \quad \textit{if } y_1 < \varepsilon. \tag{18}$$

Proof. By (15), $\varphi_\varepsilon(x - x_1, y - y_1) \neq 0$ only if

$$(3E+1)\varepsilon < x - x_1 < (3E+3)\varepsilon, \quad -3\varepsilon < y - y_1 < -\varepsilon.$$

Suppose $-3\varepsilon < y - y_1 < -\varepsilon$. If $x \leq F(y) + \varepsilon$ then, since for each $(x_1, y_1) \in \overline{G}$ we have $F(y_1) \leq x_1$, it follows

$$x - x_1 \leq F(y) + \varepsilon - F(y_1) \leq E|y - y_1| + \varepsilon < (3E+1)\varepsilon,$$

therefore (16) holds. An an analogous way, if $1 - F(y_1) - \varepsilon \leq x_1$ then, since for $(x, y) \in \overline{G}$ it holds $x \leq 1 - F(y)$ we obtain

$$x - x_1 \leq 1 - F(y) - [1 - F(y_1) - \varepsilon] \leq E|y - y_1| + \varepsilon < (3E+1)\varepsilon,$$

which proves (17). Finally, if $y_1 < \varepsilon$ then (since $y > 0$) the difference $y - y_1$ is greather than $-\varepsilon$, thus (18) holds.

Consider for $g \in L^2(G)$ the averaging operators

$$(J_\varepsilon g)(x, y) = \int_G \varphi_\varepsilon(x - x_1, y - y_1) g(x_1, y_1) dx_1 dy_1$$

and

$$(J_\varepsilon^* g)(x_1, y_1) = \int_G \varphi_\varepsilon(x - x_1, y - y_1) g(x, y) dx dy.$$

The next lemma summarize some of the properties of operators J_ε and J_ε^*. We only sketch the proof because the arguments used are standard.

Lemma 4.
(i) $J_\varepsilon g \in C^\infty_{AC}(\overline{G})$; *moreover,* $J_\varepsilon g$ *vanishes in a neighborhood of* AC.
(ii) $J_\varepsilon^* g \in C^\infty_{BC}(\overline{G})$; *moreover,* $J_\varepsilon^* g$ *vanishes in a neighborhood of* BC.
(iii) $(J_\varepsilon g, \psi)_0 = (g, J_\varepsilon^* \psi)_0 \quad \forall g, \psi \in L^2(G)$.
(iv) If $g \in C_0(\overline{G})$ *then* $(J_\varepsilon g)(x, y) \to g(x, y)$ *as* $\varepsilon \to 0$ *uniformly in* $\overline{G}$.
(v) $\|J_\varepsilon g - g\|_0 \to 0$ *as* $\varepsilon \to 0$.
(vi) Suppose $\psi \in C^1(\overline{G})$; *if* $D = \partial_{x_1}$, $D = \partial_{y_1}$, *or* $D = id$ *and*

$$(I_\varepsilon g)(x, y) := \int_G D\left([\psi(x, y) - \psi(x_1, y_1)]\varphi_\varepsilon(x - x_1, y - y_1)\right) g(x_1, y_1) dx_1 dy_1,$$

then $\|I_\varepsilon g\|_0 \to 0$ *as* $\varepsilon \to 0$.

Proof. (Sketch) Obviously (i) and (ii) follows, respectively, from (16) and (17). A change of order of integration gives (iii). If g is a continuous function with compact support in G, then it is uniformly continuous, which implies (iv). From (iv) follows (v), because $\|J_\varepsilon g\|_0 \leq \|g\|_0$ and $C_0(\overline{G})$ is dense in $L^2(G)$.

Finally, in case $D = id$ (vi) follows from (iv). If $D = \partial_{x_1}$ and $g \in C_0^\infty(\overline{G})$ then by Green's formula

$$(I_\varepsilon g)(x,y) = -\int_G [\psi(x,y) - \psi(x_1,y_1)]\varphi_\varepsilon(x - x_1, y - y_1)\partial_{x_1} g(x_1, y_1)dx_1dy_1,$$

so the claim follows from the case where $D = id$. On the other hand, one can easily see that

$$\|I_\varepsilon g\|_0 \le \tilde{C}\|g\|_0,$$

where the constant $\tilde{C}$ depends only on ψ. From here, since $C_0^\infty(\overline{G})$ is dense in $L^2(G)$, the claim follows for every $g \in L^2(G)$. The same argument works if $D = \partial_{y_1}$.

Theorem 2. *If $k(y) \in C[0,Y], \ell(y) \in C^1[0,Y], a(x,y) \in C^1(\overline{G})$, $r(x,y) \in C(\overline{G})$ and*

$$m_0 = \inf_{[0,1]} a(x,0) > 0,$$

then:

(a) each generalized solution of Problem B is also a strong solution;

(b) for each $f(x,y) \in L^2(G)$ there exists a unique generalized (strong) solution $w \in H^1_{AC}$ of Problem B.

P r o o f.

(a) Consider

$$u_\varepsilon := J_\varepsilon u, \quad \varepsilon > 0.$$

Since by Lemma 4, $u_\varepsilon \in C^\infty_{AC}$ and $\|u_\varepsilon - u\|_0 \to 0$ as $\varepsilon \to 0$, to prove the claim it is enough to show that

$$\|Lu_\varepsilon - f\|_0 \to 0 \quad \text{as } \varepsilon \to 0.$$

It is easy to see that the generalized solution u is also a weak solution of Problem B, that is we have

$$(u, L^+v)_0 = (f,v)_0 \qquad \forall v \in C^2_{BC}(\overline{G}), \tag{19}$$

where

$$L^+v = kv_{xx} - \partial_y(\ell v_y) - \partial_x(av) + rv$$

is the formally adjoint to L differential operator.

By substituting in (19) $J^*_\varepsilon v$ instead of v we obtain

$$\left((L^+J^*_\varepsilon)^* u, v\right)_0 = \left(u, L^+J^*_\varepsilon v\right)_0 = (f, J^*_\varepsilon v)_0 = (J_\varepsilon f, v)_0$$

for every $v \in L^2(G)$. Therefore,

$$J_\varepsilon f = \left(L^+ J_\varepsilon^*\right)^* u.$$

Since by part (v) of Lemma 4,

$$\|J_\varepsilon f - f\|_0 \to 0 \quad \text{as } \varepsilon \to 0,$$

it is enough to prove

$$\|LJ_\varepsilon u - \left(L^+ J_\varepsilon^*\right)^* u\|_0 \to 0 \quad \text{as } \varepsilon \to 0.$$

A computation involving Green's formula and using (16), (17) and (19) shows that

$$\left[LJ_\varepsilon u - \left(L^+ J_\varepsilon^*\right)^* u\right](x,y)$$

$$= -\int_G \partial_{x_1}\left([k(y) - k(y_1)]\varphi_\varepsilon(x - x_1, y - y_1)\right) \partial_{x_1} u(x_1, y_1) dx_1 dy_1$$

$$+ \int_G \partial_{y_1}\left([\ell(y) - \ell(y_1)]\varphi_\varepsilon(x - x_1, y - y_1)\right) \partial_{y_1} u(x_1, y_1) dx_1 dy_1$$

$$- \int_G [\ell'(y) - \ell'(y_1)]\, \varphi_\varepsilon(x - x_1, y - y_1) \partial_{y_1} u(x_1, y_1) dx_1 dy_1$$

$$+ \int_G [a(x,y) - a(x_1, y_1)]\, \varphi_\varepsilon(x - x_1, y - y_1) \partial_{x_1} u(x_1, y_1) dx_1 dy_1$$

$$+ \int_G [r(x,y) - r(x_1, y_1)]\, \varphi_\varepsilon(x - x_1, y - y_1) u(x_1, y_1) dx_1 dy_1.$$

In view of (vi) in Lemma 4, each integral term in the above expression tends to 0 in $L^2(G)$ as $\varepsilon \to 0$, which proves the claim.

(b) If $u^{(1)}, u^{(2)} \in H^1_{AC}$ are two generalized solutions of Problem B, then their difference $\tilde{u} = u^{(1)} - u^{(2)}$ is a generalized solution of homogeneous Problem B. By (a) $\tilde{u}$ is also a strong solution, so there exists a sequence $(u_n)_{n=1}^\infty$ in $C^\infty_{AC}(\overline{G})$ such that

$$\|u_n - \tilde{u}\|_1 + \|Lu_n\|_0 \to 0 \quad \text{as } n \to \infty.$$

On the other hand, by part (a) of Lemma 2 there exist $m \in (0, m_0)$ and $\lambda > 0$ such that for any $u \in C^\infty_{AC}(\overline{G})$

$$\tilde{m}\|u\|_1^2 \le (Lu, e^{-\lambda x} u_x)_0 \le \|Lu\|_0\, \|e^{-\lambda x} u_x\|_0 \le \|Lu\|_0\, \|u\|_1$$

with $\tilde{m} = m\exp(-\lambda)$. From here it follows

$$\tilde{m}\|u\|_1 \leq \|Lu\|_0 \qquad \forall u \in C^{\infty}_{AC}(\overline{G}),$$

therefore

$$\tilde{m}\|u_n\|_1 \leq \|Lu_n\|_0 \to 0 \quad \text{as} \quad n \to \infty,$$

hence $\tilde{u} = \lim u_n = 0$.

5. Comments

1. The following example shows the crucial role of the sign of the coefficient $a(x,y)$ for existence and uniqueness of solutions of Problem B.

Example. Consider

$$Lu = yu_{xx} - \partial_y(y\partial_y) - u_x,$$

that is, $k(y) = \ell(y) = y$, $a(x,y) \equiv -1$ and $r(x,y) \equiv 0$. The formally adjoint operator is

$$L^+u = yu_{xx} - \partial_y(y\partial_y) + u_x.$$

One can easily see that

$$L\phi(x-y) = 0 \qquad \forall \phi \in C^2(-\infty, \infty)$$

and

$$L^+\phi(1-x-y) = 0 \qquad \forall \phi \in C^2(-\infty, \infty).$$

Hence for a given $f(x,y) \in L^2(G)$, either Problem B has no solution (e.g. if $f(x,y) = \phi(1-x-y)$) or it has infinitely many solutions.

2. The use of the weighted Sobolev norm

$$\|u\|_1 = \left(\int_G [u_x^2 + \ell u_y^2 + u^2]dxdy\right)^{1/2}$$

and the corresponding Sobolev space H^1 is essential in order to have existence theorem.

Indeed, consider the equation (1) with $k(y) = \ell(y) = y^2$ and $a(x,y) \equiv 1$, that is,

$$Lu = y^2u_{xx} - \partial_y(y^2u_y) + u_x = f(x,y).$$

Then the characteristic AC has equation $x - y = 0$. It is easy to see that for each $\alpha \in (0, 1/2)$,

$$u(x,y) = (x-y)y^{\alpha} \in H^1_{AC}, \quad Lu \in L^2(G),$$

but $u(x,y)$ does not belong to the usual Sobolev space.

References

[1] A. V. B i t s a d z e, *Some Classes of Partial Differential Equations.* Gordon and Breach Science Publishers, New York (1988).

[2] A. V. B i t s a d z e, Theory of equations of mixed type, whose order degenerates along the line of change of type. In: *Continuous Medium Mechanics and Related Problems of Analysis*, Moscow, 47-52 (1972) (In Russian).

[3] R. W. C a r r o l l, R. E. S h o w a l t e r, *Singular and Degenerate Cauchy Problems.* Academic Press, New York (1976).

[4] N. Yu. K a p u s t i n, Solvability in the class L^2 of a characteristic problem for a hyperbolic equation with order degeneration. *Dokl. Acad. Nauk USSR* **311** (1990), 784-788 (In Russian).

[5] N. Yu. K a p u s t i n, Solvability in the class L^2 of a problem with a boundary condition on the characteristic for a hyperbolic equation with order degeneration. *Differents. Uravn.* **26** (1990), 66-75 (In Russian).

[6] R. I. S e m e r d j i e v a, A hyperbolic equation with degeneration of order. *Serdica (Bulg. Math. Publ.)* **8** (1982), 57-63 (In Russian).

[7] R. I. S e m e r d j i e v a, A difference method for solution of a class of hyperbolic equations with degeneration of order. *Differens. Uravn.* **25** (1989), 1755-1766 (In Russian).

[8] M. M. S m i r n o v, *Degenerated Hyperbolic Equations.* Vyshaya Skola, Minsk (1977) (In Russian).

[9] S. A. T e r s e n o v, *Introduction to the Theory of Equations, Degenerating on the Boundary.* Novosibirsk State University, Novosibirsk (1973) (In Russian).

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
1090 Sofia, BULGARIA

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