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Some Extensions of a Problem of Biehler and Hermite

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A starting point for this article is the famous theorem of Biehler and Hermite: Let $f(z) = \sum_{k=0}^{n} a_k z^k$, where $a_k = \alpha_k + i\beta_k$, $\alpha_k, \beta_k \in \mathbf{R}$ and $u(z) = \sum_{k=0}^{n} \alpha_k z^k$, $v(z) = \sum_{k=0}^{n} \beta_k z^k$. If the zeros z_k of f satisfy $Im\ z_k > 0$ $(k = 1, 2, ..., n)$, then all the zeros of $u(z)$ and $v(z)$ are real.

All the considered functions are of finite genre. We localize the zeros of $g_1 = [u^2(z) + v^2(z)]' = 2[u(z)u'(z) + v(z)v'(z)]$ and $g_2 = \left\{\mathrm{Log}\left[\frac{u(z)}{v(z)}\right]\right\}' = \frac{u'(z)}{u(z)} - \frac{v'(z)}{v(z)}$.

Under the conditions of Theorem 2 and Theorem 3, we localize the nonreal zeros of g_1 and g_2 in the circles which are similar to the Jensen circles for polynomials. A similar result is obtained for the real entire functions in [4].

Further: all the sequences $\{z_k\}_{k=1}^{\infty}$ will satisfy $\lim_{k\to\infty} |z_k| = \infty$ and the Maclaurin series of f is $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where $a_k = \alpha_k + i\beta_k, \alpha_k, \beta_k \in \mathbf{R}$ and $u(z) = \sum_{k=0}^{\infty} \alpha_k z^k$, $v(z) = \sum_{k=0}^{\infty} \beta_k z^k$.

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Theorem 1. *Let*

$$f(z) = \exp(d + cz^m + bz^{m+1} + az^{m+2}) \prod_{k=1}^{\infty} E(\frac{z}{z_k})$$

be an entire function, where m is a positive integer, $Re\ a \geq 0$, $Re\ b \leq 0$, $Re\ c \geq 0$, and the Weierstrass factors are:

$$E(\zeta) = (1 - \zeta)\exp(\zeta + \frac{\zeta^2}{2} + ... + \frac{\zeta^m}{m}).$$

Let all the zeros z_k of $f(z)$ satisfy $\varphi_k = \arg(z_k) \in (-\frac{\pi}{2m+2}, \frac{\pi}{2m+2})$. Then all the roots z of $g_1(z) = \left[u^2(z) + v^2(z)\right]' = 2\left[u(z)u'(z) + v(z)v'(z)\right]$ satisfy $Re(z) \geq 0$.

Proof. We have

$$f(z) = u(z) + iv(z) = \exp(d + cz^m + bz^{m+1} + az^{m+2}) \prod_{k=1}^{\infty} E(\frac{z}{z_k}).$$

Let z_0 be such that $g_1(z_0) = 0$. Then $u(z_0)u'(z_0) + v(z_0)v'(z_0) = 0$, and therefore

$$\frac{u'(z_0) + iv'(z_0)}{u(z_0) + iv(z_0)} = -\frac{u'(z_0) - iv'(z_0)}{u(z_0) - iv(z_0)}, \quad \text{i.e.} \quad \frac{f'(z_0)}{f(z_0)} = -\frac{\overline{f'(z_0)}}{\overline{f(z_0)}}.$$

We denote $z_0 = x + iy$, where $x, y \in \mathbf{R}$, supposing that $\mathrm{Re}(z_0) = x < 0$.

Then

$$A = \frac{f'(z_0)}{f(z_0)} + \frac{\overline{f'(z_0)}}{\overline{f(z_0)}} = 0, \qquad \text{i.e.}$$

$$A = m(c+\overline{c})z_0 + (m+1)(b+\overline{b})z_0 + (m+2)(a+\overline{a})z_0$$

$$+\sum_{k=1}^{\infty}\left(\frac{1}{z_0 - z_k} + \frac{1}{z_k} + \dots + \frac{z_0^{m-1}}{z_k^m}\right) + \sum_{k=1}^{\infty}\left(\frac{1}{z_0 - \overline{z}_k} + \frac{1}{\overline{z}_k} + \dots + \frac{z_0^{m-1}}{\overline{z}_k^m}\right)$$

$$= \frac{2m\mathrm{Re}(c)z_0^m\overline{z_0}}{|z_0|^2} + 2(m+1)\mathrm{Re}(b)z_0^m + 2(m+2)\mathrm{Re}(a)z_0^m z_0 + \sum_{k=1}^{\infty}\frac{\left(\frac{z_0}{z_k}\right)^m}{z_0 - z_k} + \sum_{k=1}^{\infty}\frac{\left(\frac{z_0}{\overline{z}_k}\right)^m}{z_0 - \overline{z}_k}$$

$$= z_0^m\Big[\frac{2m\mathrm{Re}(c)\overline{z_0}}{|z_0|^2} + 2(m+1)\mathrm{Re}(b) + 2(m+2)\mathrm{Re}(a)z_0$$

$$+\sum_{k=1}^{\infty}\frac{\overline{z_0 - z_k}}{z_k^m\,|z_0 - z_k|^2} + \sum_{k=1}^{\infty}\frac{\overline{z_0} - z_k}{\overline{z}_k^m\,|z_0 - \overline{z}_k|^2}\Big] = 0.$$

Let

$$B = \frac{2m\mathrm{Re}(c)\overline{z_0}}{|z_0|^2} + 2(m+1)\mathrm{Re}(b) + 2(m+2)\mathrm{Re}(a)z_0$$

$$+\sum_{k=1}^{\infty}\frac{\overline{z_0 - z_k}}{z_k^m\,|z_0 - z_k|^2} + \sum_{k=1}^{\infty}\frac{\overline{z_0} - z_k}{\overline{z}_k^m\,|z_0 - \overline{z}_k|^2},$$

$z_k = x_k + iy_k$, $\overline{z_k} = x_k - iy_k$, where $x_k, y_k \in \mathbf{R}$.

Then,

$$C = B - \left[\frac{2m\mathrm{Re}\,(c)\overline{z_0}}{|\,z_0\,|^2} + 2(m+1)\mathrm{Re}\,(b) + 2(m+2)\mathrm{Re}\,(a)z_0\right]$$

$$= \sum_{k=1}^{\infty} \frac{\overline{z_0 - z_k}}{z_k^m \,|\, z_0 - z_k \,|^2} + \sum_{k=1}^{\infty} \frac{\overline{z_0} - z_k}{\overline{z}_k^m \,|\, z_0 - \overline{z}_k \,|^2}$$

$$= \sum_{k=1}^{\infty} \frac{\overline{z}_k^m \,|\, z_0 - \overline{z}_k \,|^2 \,(\overline{z_0 - z_k}) + z_k^m \,|\, z_0 - z_k \,|^2 \,(\overline{z_0} - z_k)}{D_k},$$

where $D_k = |\, z_0 - \overline{z}_k \,|^2 |\, z_k^m \,|^2 |\, z_0 - z_k \,|^2$

Let $r = x - x_k$, $q = y = y_k$, $s = y + y_k$, $z_k = p_k(\cos\varphi_k + \sin\varphi_k)$,

$$\Delta_k = \frac{\overline{z}_k^m \,|\, z_0 - \overline{z}_k \,|^2 \,(\overline{z_0 - z_k}) + z_k^m \,|\, z_0 - z_k \,|^2 \,(\overline{z_0} - z_k)}{p_k^m}$$

$$= [\cos(m\varphi_k) - i\sin(m\varphi_k)]\,(r - iq)\left(r^2 + s^2\right)$$

$$+ [\cos(m\varphi_k) + i\sin(m\varphi_k)]\,(r - is)\left(r^2 + q^2\right).$$

Then,

$$\mathrm{Re}\,\Delta_k = [r\cos(m\varphi_k) - q\sin(m\varphi_k)]\left(r^2 + s^2\right) + [r\cos(m\varphi_k) + s\sin(m\varphi_k)]\left(r^2 + q^2\right)$$

$$= r\cos(m\varphi_k)\left(2r^2 + s^2 + q^2\right) + \sin(m\varphi_k)\left(r^2 s + q^2 s - r^2 q - s^2 q\right)$$

$$= 2r\cos(m\varphi_k)\left(r^2 + y^2 + y_k^2\right) + 2y_k\sin(m\varphi_k)\left(r^2 - y^2 + y_k^2\right)$$

$$= 2\left(r^2 + y_k^2\right)[r\cos(m\varphi_k) + y_k\sin(m\varphi_k)] + 2y^2\,[r\cos(m\varphi_k) - y_k\sin(m\varphi_k)].$$

Since, by assumption, $\varphi_k \in (-\frac{\pi}{2m+2}, \frac{\pi}{2m+2})$, we have $\cos(m\varphi_k) > 0$, and if we assume that $r\cos(m\varphi_k) - y_k\sin(m\varphi_k) \geq 0$, then

$$(x_k - x)\cos(m\varphi_k) + y_k\sin(m\varphi_k) \leq 0.$$

Since $x < 0$, $x_k = p_k\cos\varphi_k$ and $y_k = p_k\sin\varphi_k$, we obtain:

$$x_k\cos(m\varphi_k) + y_k\sin(m\varphi_k) \leq 0, \quad \text{or} \quad \cos\varphi_k\cos(m\varphi_k) + \sin\varphi_k\sin(m\varphi_k) \leq 0,$$

i.e. $\cos[(m-1)\varphi_k] \le 0$, which is impossible since $\varphi_k \in (-\frac{\pi}{2m+2}, \frac{\pi}{2m+2})$. Therefore, $r\cos(m\varphi_k) - y_k \sin(m\varphi_k) < 0$. If we assume that $r\cos(m\varphi_k) + y_k\sin(m\varphi_k) \ge 0$, then by the same way we obtain $\cos[(m+1)\varphi_k] \le 0$ which is impossible since $\varphi_k \in (-\frac{\pi}{2m+2}, \frac{\pi}{2m+2})$. Therefore, $r\cos(m\varphi_k) + y_k\sin(m\varphi_k) < 0$ and $\mathrm{Re}\,\Delta_k < 0$, i.e. $\mathrm{Re}\,B < 0$, because $\mathrm{Re}\,a \ge 0$, $\mathrm{Re}\,b \le 0$, $\mathrm{Re}\,c \ge 0$. But $A = z_0^m B = 0$, i.e. $B = 0$. This contradiction completes the proof. ■

Theorem 2. *Let* $f(z) = \exp(d + cz^m + bz^{m+1} + az^{m+2})\prod_{k=1}^{\infty} E(\frac{z}{z_k})$ *be an entire function, where* m *is a positive integer,* $Im\,a \ge 0$, $Im\,c \le 0$ *and all the zeros* z_k *of* $f(z)$ *satisfy* $\varphi_k = \arg(z_k) \in (0, \frac{\pi}{m})$. *Let* V_k *be the disk:*

$$V_k = \left\{ |\, z - Re\, z_k \,| \le |\, Im\, z_k \,| \, \frac{1 + |\cos(m\varphi_k)|}{\sin(m\varphi_k)} \right\},$$

and let $M = \cup_{k=1}^{\infty} V_k$. *Then all nonreal roots* z *of*

$$g_2(z) = \left[\log\frac{u(z)}{v(z)}\right]' = \frac{u'(z)}{u(z)} - \frac{v'(z)}{v(z)}$$

are in M.

Proof. We have

$$f(z) = u(z) + iv(z) = \exp(d + cz^m + bz^{m+1} + az^{m+2}) \prod_{k=1}^{\infty} E(\frac{z}{z_k}).$$

Let z_0 be such that $g_2(z_0) = 0$. Then $\frac{u'(z_0)}{u(z_0)} - \frac{v'(z_0)}{v(z_0)} = 0$, and therefore

$$\frac{u'(z_0) + iv'(z_0)}{u(z_0) + iv(z_0)} = \frac{u'(z_0) - iv'(z_0)}{u(z_0) - iv(z_0)}, \quad \text{i.e.} \quad \frac{f'(z_0)}{f(z_0)} = \frac{\overline{f'(z_0)}}{\overline{f(z_0)}}.$$

We denote $z_0 = x + iy$, where $x, y \in \mathbf{R}$, supposing that $z_0 \notin M$. Then

$$A = \frac{f'(z_0)}{f(z_0)} - \frac{\overline{f'(z_0)}}{\overline{f(z_0)}} = 0,$$

i.e. $z_k = x_k + iy_k$, $\overline{z_k} = x_k - iy_k$, where $x_k, y_k \in \mathbf{R}$. Then

$$A = m(c - \overline{c})z_0^{m-1} + (m+1)(b - \overline{b})z_0^m + (m+2)(a - \overline{a})z_0^{m+1}$$

$$+ \sum_{k=1}^{\infty}\left(\frac{1}{z_0 - z_k} + \frac{1}{z_k} + \dots + \frac{z_0^{m-1}}{z_k^m}\right) - \sum_{k=1}^{\infty}\left(\frac{1}{z_0 - \overline{z}_k} + \frac{1}{\overline{z}_k} + \dots + \frac{z_0^{m-1}}{\overline{z}_k^m}\right)$$

$$= \frac{2m\mathrm{Im}(c)z_0^m\overline{z_0}}{|z_0|^2} + 2(m+1)\mathrm{Im}(b)z_0^m + 2(m+2)\mathrm{Im}(a)z_0^m z_0$$

$$+\sum_{k=1}^{\infty}\frac{\left(\frac{z_0}{z_k}\right)^m}{z_0 - z_k} - \sum_{k=1}^{\infty}\frac{\left(\frac{z_0}{\overline{z}_k}\right)^m}{z_0 - \overline{z}_k}$$

$$= z_0^m\left[\frac{2m\mathrm{Im}(c)\overline{z_0}}{|z_0|^2} + 2(m+1)\mathrm{Im}(b) + 2(m+2)\mathrm{Im}(a)z_0\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{\overline{z_0 - z_k}}{z_k^m\,|z_0 - z_k|^2} - \sum_{k=1}^{\infty}\frac{\overline{z_0} - z_k}{\overline{z}_k^m\,|z_0 - \overline{z}_k|^2}\right] = 0.$$

Let

$$B = \frac{2m\mathrm{Im}(c)\overline{z_0}}{|z_0|^2} + 2(m+1)\mathrm{Im}(b) + 2(m+2)\mathrm{Im}(a)z_0$$

$$+\sum_{k=1}^{\infty}\frac{\overline{z_0 - z_k}}{z_k^m\,|z_0 - z_k|^2} - \sum_{k=1}^{\infty}\frac{\overline{z_0} - z_k}{\overline{z}_k^m\,|z_0 - \overline{z}_k|^2},$$

$z_k = x_k + iy_k, \overline{z_k} = x_k - iy_k$, where $x_k, y_k \in \mathbf{R}$. Then

$$C = B - \left[\frac{2m\mathrm{Im}(c)\overline{z_0}}{|z_0|^2} + 2(m+1)\mathrm{Im}(b) + 2(m+2)\mathrm{Im}(a)z_0\right]$$

$$= \sum_{k=1}^{\infty}\frac{\overline{z_0 - z_k}}{z_k^m\,|z_0 - z_k|^2} - \sum_{k=1}^{\infty}\frac{\overline{z_0} - z_k}{\overline{z}_k^m\,|z_0 - \overline{z}_k|^2}$$

$$= \sum_{k=1}^{\infty}\frac{\overline{z}_k^m\,|z_0 - \overline{z}_k|^2\,(\overline{z_0 - z_k}) - z_k^m\,|z_0 - z_k|^2\,(\overline{z_0} - z_k)}{D_k},$$

where $D_k = |z_0 - \overline{z}_k|^2 |z_k^m|^2 |z_0 - z_k|^2$. If we put $r = x - x_k, q = y = y_k, s = y + y_k, z_k = p_k(\cos\varphi_k + \sin\varphi_k)$, then as in Theorem 1, we obtain

$$\Delta_k = \frac{\overline{z}_k^m\,|z_0 - \overline{z}_k|^2\,(\overline{z_0 - z_k}) - z_k^m\,|z_0 - z_k|^2\,(\overline{z_0} - z_k)}{p_k^m}$$

$$= [\cos(m\varphi_k) - i\sin(m\varphi_k)](r - iq)\left(r^2 + s^2\right)$$

$$-[\cos(m\varphi_k) + i\sin(m\varphi_k)](r - is)\left(r^2 + q^2\right).$$

Thus,

$$\mathrm{Re}\Delta_k = [r\cos(m\varphi_k) - q\sin(m\varphi_k)]\left(r^2 + s^2\right) - [r\cos(m\varphi_k) + s\sin(m\varphi_k)]$$

$$\times\left(r^2+q^2\right)=4yy_kr\cos(m\varphi_k)-2y(r^2+y^2-y_k^2)\sin(m\varphi_k)$$

$$=2y\left[2y_kr\cos(m\varphi_k)-(r^2+y^2-y_k^2)\sin(m\varphi_k)\right].$$

Let $C_k=(r^2+y^2-y_k^2)\sin(m\varphi_k)-2y_kr\cos(m\varphi_k)$ and $R=\mid z_0-\mathrm{Re}z_k\mid$, so that $r^2+y^2=R^2$, and therefore $r=\varepsilon\sqrt{R^2-y^2}$, where $\varepsilon=\pm1$. We have

$$C_k\geq(R^2-y_k^2)\sin(m\varphi_k)-2\mid y_k\mid R\mid\cos(m\varphi_k)\mid$$

$$=\sin(m\varphi_k)R^2-2\mid y_k\mid\mid\cos(m\varphi_k)\mid R-\sin(m\varphi_k)y_k^2$$

$$=\sin(m\varphi_k)(R-R_1)(R-R_2),$$

where $R_1=\mid y_k\mid\frac{|\cos(m\varphi_k)|-1}{\sin(m\varphi_k)}$, $R_2=\mid y_k\mid\frac{|\cos(m\varphi_k)|+1}{\sin(m\varphi_k)}$. Since $\varphi_k=\arg(z_k)\in(0,\frac{\pi}{m})$, we have that $\cos(m\varphi_k)>0$. If $R>\mid\mathrm{Im}z_k\mid\frac{1+|\cos(m\varphi_k)|}{\sin(m\varphi_k)}$, then we obtain that $C_k>0\,(k=1,2,\ldots)$. Hence $\mathrm{Re}\Delta_k=-2yC_k<0$, i.e. $\mathrm{Re}B<0$, since $\mathrm{Im}a\geq0$ and $\mathrm{Im}c\leq0$. But $A=z_0^mB=0$, i.e. $B=0$. This contradiction proves the theorem. ∎

Theorem 3. *Let* $f(z)=\exp(d+cz^m+bz^{m+1}+az^{m+2})\prod_{k=1}^{\infty}E(\frac{z}{z_k})$ *be an entire function, where* m *is a positive integer,* $\mathrm{Re}a\leq0$, $\mathrm{Re}c\geq0$, *and all the zeros* z *of* $f(z)$ *satisfy* $\varphi_k=\arg(z_k)\in(-\frac{\pi}{2m},\frac{\pi}{2m})$. *Let* V_k *be the disk:*

$$V_k=\left\{\mid z-\mathrm{Re}z_k\mid\leq\mid\mathrm{Im}z_k\mid\frac{1+\mid\sin(m\varphi_k)\mid}{\cos(m\varphi_k)}\right\},\qquad \textit{and let}\quad M=\cup_{k=1}^{\infty}V_k.$$

Then all nonreal roots z *of* $g_1(z)=\left[u^2(z)+v^2(z)\right]'=2\left[u(z)u'(z)+v(z)v'(z)\right]$ *are in* M.

P r o o f. We have

$$f(z)=u(z)+iv(z)=\exp(d+cz^m+bz^{m+1}+az^{m+2})\prod_{k=1}^{\infty}E(\frac{z}{z_k}).$$

Let z_0 be such that $g_1(z_0)=0$ and $z_0\notin M$. Then $u(z_0)u'(z_0)+v(z_0)v'(z_0)=0$, and therefore

$$\frac{u'(z_0)+iv'(z_0)}{u(z_0)+iv(z_0)}=-\frac{u'(z_0)-iv'(z_0)}{u(z_0)-iv(z_0)},\qquad\text{i.e.}\quad\frac{f'(z_0)}{f(z_0)}=-\frac{\overline{f'(z_0)}}{\overline{f(z_0)}}.$$

The same expression was obtained in Theorem 1. With the above denotations, after some computations we consider

$$\Delta_k = \frac{\overline{z}_k^m \mid z_0 - \overline{z}_k \mid^2 (\overline{z_0 - z_k}) + z_k^m \mid z_0 - z_k \mid^2 (\overline{z_0} - z_k)}{p_k^m}$$

$$= [\cos(m\varphi_k) - i\sin(m\varphi_k)]\,(r - iq)\,(r^2 + s^2)$$

$$+ [\cos(m\varphi_k) + i\sin(m\varphi_k)]\,(r - is)\,(r^2 + q^2)\,.$$

Then,

$$-\mathrm{Im}\Delta_k = [q\cos(m\varphi_k) + r\sin(m\varphi_k)]\,(r^2 + s^2)$$

$$+ [s\cos(m\varphi_k) - r\sin(m\varphi_k)]\,(r^2 + q^2)$$

$$= 2y(r^2 + y^2 - y_k^2)\cos(m\varphi_k) + 4yy_k r\sin(m\varphi_k).$$

Let $C_k = (r^2 + y^2 - y_k^2)\cos(m\varphi_k) + 2y_k r\sin(m\varphi_k)$ and $R = \mid z_0 - \mathrm{Re} z_k \mid$, so that $r^2 + y^2 = R^2$, and therefore, $r = \varepsilon\sqrt{R^2 - y^2}$, where $\varepsilon = \pm 1$. We have

$$C_k \geq (R^2 - y_k^2)\cos(m\varphi_k) - 2 \mid y_k \mid R \mid \sin(m\varphi_k) \mid$$

$$= \cos(m\varphi_k)R^2 - 2 \mid y_k \mid\mid \sin(m\varphi_k) \mid R - \cos(m\varphi_k)y_k^2$$

$$= \cos(m\varphi_k)(R - R_1)(R - R_2),$$

where $R_1 = \mid y_k \mid \frac{|\sin(m\varphi_k)| - 1}{\cos(m\varphi_k)}$, $R_2 = \mid y_k \mid \frac{|\sin(m\varphi_k)| + 1}{\cos(m\varphi_k)}$. Since $\varphi_k = \arg(z_k) \in (-\frac{\pi}{2m}, \frac{\pi}{2m})$, $z_0 \notin M$, we have that $\cos(m\varphi_k) > 0$ and $R > \mid \mathrm{Im} z_k \mid \frac{1 + |\sin(m\varphi_k)|}{\cos(m\varphi_k)}$. Then we obtain that $C_k > 0\,(k = 1, 2, \ldots)$. Hence, $\mathrm{Im}\Delta_k = -2yC_k < 0$, i.e. $\mathrm{Im}B < 0$, since $\mathrm{Re}a \leq 0$ and $\mathrm{Re}c \geq 0$. But $A = z_0^m B = 0$, i.e. $B = 0$. This contradiction proves the theorem. ■

Theorem 4. *Let* $f(z) = \exp(d + cz^m + bz^{m+1} + az^{m+2})\prod_{k=1}^{\infty} E(\frac{z}{z_k})$ *be an entire function, where* m *is a positive integer,* $Ima \leq 0, Imb \geq 0$, $Imc \leq 0$. *Let all the zeros* z *of* $f(z)$ *satisfy* $\varphi_k = \arg(z_k) \in (0, \frac{\pi}{m+1})$. *Then all the roots* z_k *of* $g_2(z) = \left[\log \frac{u(z)}{v(z)}\right]' = \frac{u'(z)}{u(z)} - \frac{v'(z)}{v(z)}$ *satisfy* $Re(z) \geq 0$.

Proof. We have

$$f(z) = u(z) + iv(z) = \exp(d + cz^m + bz^{m+1} + az^{m+2})\prod_{k=1}^{\infty} E(\frac{z}{z_k}).$$

Let z_0 be such that $g_2(z_0) = 0$. Then, $\frac{u'(z_0)}{u(z_0)} - \frac{v'(z_0)}{v(z_0)} = 0$, and therefore

$$\frac{u'(z_0) + iv'(z_0)}{u(z_0) + iv(z_0)} = \frac{u'(z_0) - iv'(z_0)}{u(z_0) - iv(z_0)}, \qquad \text{i.e.} \quad \frac{f'(z_0)}{f(z_0)} = \frac{\overline{f'(z_0)}}{\overline{f(z_0)}}.$$

We denote $z_0 = x + iy$, where $x, y \in \mathbf{R}$ supposing that $\mathrm{Re} z_0 = x < 0$. Then

$$A = \frac{f'(z_0)}{f(z_0)} - \frac{\overline{f'(z_0)}}{\overline{f(z_0)}} = 0.$$

The same expression was obtained in Theorem 2. We will consider

$$\Delta_k = \frac{\overline{z}_k^m \mid z_0 - \overline{z}_k \mid^2 (\overline{z_0 - z_k}) - z_k^m \mid z_0 - z_k \mid^2 (\overline{z_0} - z_k)}{p_k^m}$$
$$= [\cos(m\varphi_k) - i \sin(m\varphi_k)] (r - iq) \left(r^2 + s^2\right)$$
$$- [\cos(m\varphi_k) + i \sin(m\varphi_k)] (r - is) \left(r^2 + q^2\right).$$

Thus,

$$-\mathrm{Im}\Delta_k = [r \sin(m\varphi_k) + q \cos(m\varphi_k)] \left(r^2 + s^2\right)$$
$$- [s \cos(m\varphi_k) - r \sin(m\varphi_k)] \left(r^2 + q^2\right)$$

$$= 2r(r^2 + y^2 + y_k^2) \sin(m\varphi_k) - 2y_k(r^2 - y^2 + y_k^2) \cos(m\varphi_k)$$

$$= 2(r^2 + y_k^2) [r \sin(m\varphi_k) - y_k \cos(m\varphi_k)] + 2y^2 [r \sin(m\varphi_k) + y_k \cos(m\varphi_k)].$$

Since by assumption, $\varphi_k \in (0, \frac{\pi}{m+1})$ we have $\sin(m\varphi_k) > 0$, and if we assume that $r \sin(m\varphi_k) - y_k \cos(m\varphi_k) \geq 0$, then $(x_k - x) \sin(m\varphi_k) + y_k \cos(m\varphi_k) \leq 0$.

Since $x < 0$, $x_k = p_k \cos \varphi_k$ and $y_k = p_k \sin \varphi_k$, we obtain

$$x_k \sin(m\varphi_k) + y_k \cos(m\varphi_k) \leq 0, \quad \text{or} \quad \cos \varphi_k \sin(m\varphi_k) + \sin \varphi_k \cos(m\varphi_k) \leq 0,$$

i.e. $\sin [(m + 1) \varphi_k] \leq 0$, which is impossible since $\varphi_k \in (0, \frac{\pi}{m+1})$. Therefore

$$r \cos(m\varphi_k) - y_k \sin(m\varphi_k) < 0.$$

If we assume that $r \sin(m\varphi_k) + y_k \cos(m\varphi_k) \geq 0$, then in the same way we obtain $\sin [(m - 1) \varphi_k] \leq 0$ which is impossible since $\varphi_k \in (0, \frac{\pi}{m+1})$. Therefore

$$r \sin(m\varphi_k) + y_k \cos(m\varphi_k) < 0 \qquad \text{and} \quad \mathrm{Im}\Delta_k > 0,$$

i.e. $\text{Im}B > 0$, because $\text{Im}a \leq 0$, $\text{Im}b \geq 0$, $\text{Im}\,c \leq 0$. But $A = z_0^m B = 0$, i.e. $B = 0$. This contradiction completes the proof. ■

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