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Smooth an Classical Solution of Non-local Boundary Value Problem for a Class of High Order Partial Differential Equations

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In this paper we consider the problem

$$\sum_{i=1}^{2s} k_i(t,x)D_t^i u-(-1)^m \sum_{|\alpha|=|\beta|=m} D_x^\alpha\left[a^{\alpha\beta}(x)D_x^\beta u\right]+[c(t,x)-C]u=f(t,x),$$

$$D_x^\alpha u\,|_\Gamma=0 \quad \text{for} \quad |\alpha|\le m-1,$$

$$D_t^i u\,(T,x)=\lambda D_t^i u\,(0,x)\,, \quad i=\overline{0,2s-1},$$

where $m\ge 1, s\ge 1, \lambda = const \ne 0, |\lambda|<1, x=(x_1,x_2,...,x_n), \Gamma=\partial D\times(0,T), T>0$, D is a bounded domain in $R^n, n\ge 1$ and $k_{2s}(t,x)\le 0 \quad \forall (t,x)\in\overline{G}$, where $G=D\times(0,T)$. If $l\ge 1$, a smoothness of generalized solution $u\in W_{t,x}^{2s-1+l,2m+(l-1)[m/s]}(G)$ of the above problem is obtained. Sufficient conditions this solution to be classical are also found.

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1. Introduction

Let D be a bounded domain in $R^n, n\ge 1$, with a boundary ∂D. Denote:

$$x=(x_1,x_2,...,x_n),\ G=D\times(0,T),\ \Gamma=\partial D\times(0,T),\ T>0.$$

Suppose, that Γ is smooth and consider in G the equation

$$(1.1)\qquad Lu\equiv P_{2s}(t,x)u-(-1)^m M_{2m}(x)u+[c(t,x)-C]u=f(t,x),$$

where C is a sufficiently large, positive constant and

$$P_{2s}(t,x)u \equiv \sum_{i=1}^{2s} k_i(t,x)\,D_t^i u(t,x)\,,\quad M_{2m}(x)u = \sum_{|\alpha|=|\beta|=m} D_x^\alpha[a^{\alpha\beta}(x)D_x^\beta u(t,x)],$$

$$D_t^i u(t,x) = \frac{\partial^i}{\partial t^i}u(t,x),\quad D_x^\alpha u(t,x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}...\partial x_n^{\alpha_n}}u(t,x),$$

$\alpha_i \geq 0, m \geq 1, s \geq 1$ are integer numbers and the coefficients $k_i(t,x)$, $c(t,x)$, $a^{\alpha\beta}(x) \equiv a^{\beta\alpha}(x)$ are infinitely smooth functions in $\overline{G}$. We suppose that the conditions

$$k_{2s}(t,x) \leq 0\,\forall x \in \overline{G},\quad k_{2s}(T,x) = k_{2s}(0,x) < 0\quad \forall x \in \overline{D}$$

are satisfied and $M_{2m}(x)$ is a strong elliptic operator in G, i.e.

$$\sum_{|\alpha|=|\beta|=m} \xi^\alpha a^{\alpha\beta}(x)\xi^\beta \geq C_0\,|\xi|^{2m}\,,\forall\xi \in R^n,\forall x \in \overline{D},$$

where $C_0 = const > 0$, α, β are multi indices. In the case $s = m = 2q-1$ the equation (1.1) is an equation of hyperbolic-parabolic type in G∪Γ. In the case $s = m = 2q$ the equation (1.1) is an equation of elliptic-parabolic type in G∪Γ. If $s < m$ the equation is parabolic.

2. Boundary conditions and function spaces

Consider the following boundary value problem. To find a solution of equation (1.1) in G, satisfying the boundary conditions:

(2.1) $$D_x^\alpha u\,|_\Gamma = 0 \quad \text{for} \quad |\alpha| \leq m-1,$$

(2.2) $$D_t^i u(T,x) = \lambda D_t^i u(0,x)\,,\quad i = \overline{0,2s-1};\quad \forall x \in \overline{D},$$

where $\lambda = const \neq 0$, $|\lambda| < 1$.

For $l = \overline{0,2s-1}$; $j = \overline{0,2s-1}$; we introduce the functions

$$\sigma_{j,l}(t,x) = \sum_{i=j+l+1}^{2s} (-1)^{i-1-j}\binom{i-1-j}{l} D_t^{i-1-j-l}u(t,x)\,k_i(t,x)$$

and suppose that for such indices j, l

(2.3) $$\sigma_{j,l}(T,x) = \sigma_{j,l}(0,x)\quad \forall x \in \overline{D}.$$

Let $\widetilde{C}^\infty(\overline{G})$ be the space of infinitely smooth in $\overline{G}$ functions, satisfying the boundary conditions (2.1) and (2.2) and let $\widetilde{C}^\infty_*(\overline{G})$ be the corresponding space of infinitely smooth in $\overline{G}$ functions, satisfying the adjoint to (2.1) and (2.2) boundary conditions:

$$D_x^\alpha v\mid_\Gamma = 0 \quad \text{for} \quad |\alpha| \le m-1,$$
$$\lambda D_t^i v(T,x) = D_t^i v(0,x), \quad i = \overline{0, 2s-1}.$$

Let $p \ge 1$ and $q \ge 1$ are integer numbers. Define the space $H^{p,q}_{t,x}(G)$ as the closure of the function space $\widetilde{C}^\infty(\overline{G})$ with respect to the norm

$$\|u\|^2_{p,q} = \int_G \sum_{qi+p|\alpha|\le pq} (D_t^i D_x^\alpha u)^2 dx dt \tag{2.4}$$

and the space $H^{p,q}_{t,x,*}(G)$ as the closure of the function space $\widetilde{C}^\infty_*(\overline{G})$ with respect to the same norm.

Define the space $H^{p,0}_{t,x}(G)$ as the closure of the function space $\widetilde{C}^\infty(\overline{G})$ with respect to the norm

$$\|u\|^2_{p,0} = \int_G \sum_{i\le p} (D_t^i u)^2 dx dt.$$

If $p \ge 1$ and $q \ge 1$ are integer numbers we define the space $W^{p,q}_{t,x}(G)$ as the set of functions u $\in L_2(G)$, which have generalized derivatives $D_t^i D_x^\alpha u \in L_2(G)$ for each index i and multi index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, such that $\frac{i}{p} + \frac{|\alpha|}{q} \le 1$. $W^{p,q}_{t,x}(G)$ is a normed space with a norm (2.4).

If $q \ge 1$ is an integer number we define the space $W^{0,q}_{t,x}(G)$ as the set of functions u $\in L_2(G)$, which have generalized derivatives $D_x^\alpha u \in L_2(G)$ for each multi index α such that $|\alpha| \le q$.

The scalar product of the space $L_2(G) \equiv H^{0,0}_{t,x}(G)$ we shall denote by $(.,.)_{0,G}$ or only by $(.,.)_0$.

Definition 1. A function $u \in H^{2s-1,m}_{t,x}(G)$ is called a generalized solution for the problem (1.1) - (2.2), if

$$(u, L^* v)_0 = (f, v)_0 \quad \forall v \in \widetilde{C}^\infty_*(\overline{G}). \tag{2.5}$$

3. Main results

Theorem 1. *Let the following conditions be satisfied:*

(i) $2k_{2s-1}(t,x) - D_t k_{2s}(t,x) > 0, \quad \forall (t,x) \in \overline{G};$
(ii) *If* $s > 1$, *then* $k_i(T,x) = k_i(0,x), \quad \forall x \in \overline{D}, \quad i = \overline{1,s},$
If $s \geq 1$, *then* (2.3) *is satisfied;*
(iii) *If* $s > 1$, *then* $D_t^i c(t,x) \mid_{t=T} = D_t^i c(t,x) \mid_{t=0}, \quad i = \overline{0, 2s-2}, \quad \forall x \in \overline{D}.$

Then for any function $f \in L_2(G)$ *there exists a generalized solution for the problem (1.1)-(2.2).*

Let the functions $k_i^*(t,x)$, $c^*(t,x)$ be the corresponding coefficients of the operator L^*, formally adjoint to the differential operator L.

Theorem 2. *Let the following conditions be satisfied:*

(i) $2k_{2s-1}(t,x) - D_t k_{2s}(t,x) > 0 \quad \forall (t,x) \in \overline{G},$
(ii) *If* $s > 1$, *then* $k_i^*(T,x) = k_i^*(0,x), \quad \forall x \in \overline{D}, \quad i = \overline{1,s};$
If $s \geq 1$, *then* (2.3) *is satisfied.*
(iii) *If* $s > 1$, *then* $D_t^i c^*(t,x) \mid_{t=T} = D_t^i c^*(t,x) \mid_{t=0}, \quad i = \overline{0, 2s-2}, \quad \forall x \in \overline{D}.$

Then the problem (1.1)-(2.2) can have no more then one generalized solution.

Let us define the positive constant κ by the equality $e^{\kappa T} = \lambda^{-2}$.

Theorem 3. *Let the following conditions be satisfied:*

(i) $f \in W_{t,x}^{l,l[m/s]}(G);$
(ii) $D_t^i f(T,x) = \lambda D_t^i f(0,x)$ *almost everywhere in* D *for* $i = \overline{0, l-1};$
Suppose that the following conditions are satisfied in $\overline{G}$:
(iii) $2k_{2s-1}(t,x) - rD_t k_{2s}(t,x) > 0 \quad \forall (t,x) \in \overline{G},$
where $r = 2p-1, \; p = \overline{0,l}; \; r = 2p - 4s + 1, \; p = \overline{0,l};$
Suppose that the following conditions hold in D :
(iv) $k_{2s-1}(0,x) - (p+1)D_t k_{2s}(0,x) \neq 0, \; p = \overline{-l, 2s-2};$
(v) $D_t^i c(t,x) \mid_{t=T} = D_t^i c(t,x) \mid_{t=0}$ *for* $i = \overline{0, 2s-3+l};$
(vi) $D_t^i k_j(T,x) = D_t^i k_j(0,x), \quad j = \overline{1,s}; \; i = \overline{0, \max(l-1, 2s-2+j)}.$

Then the generalized solution of the problem (1.1)-(2.2) belongs to the space

$$W_{t,x}^{2s-1+l, 2m+(l-1)[m/s]}(G)$$

and $D_t^i u(T,x) = \lambda D_t^i u(0,x)$ *almost everywhere* D *for* $i = \overline{0, 2s-2+l}.$

4. Proofs

Proof of Theorem 1. Let u be an arbitrary function, belonging to the space $\widetilde{C}^\infty(\overline{G})$. If we denote

$$R(t)u = \sum_{l=1}^{2s-1} \binom{2s-1}{l} D_t^{(2s-1)-l}[e^{\kappa t}] D_t^l u + e^{\kappa t} D_t^{2s-1} u,$$

then we have

$$2\int_G Lu R(t)u\, dt dx = J_0 + \sum_{i=1}^{2s} J_i + 2 \sum_{|\alpha|=|\beta|=m} J_{\alpha\beta},$$

where

$$J_0 = 2\int_G c(t,x) u R(t)u\, dt dx, \quad J_i = 2\int_G k_i(t,x) D_t^i u R(t) u\, dt dx, \quad i = \overline{1,2s};$$

$$J_{\alpha\beta} = -(-1)^m \int_G D_x^\alpha [a^{\alpha\beta}(x) D_x^\beta u] R(t) u\, dt dx, \quad |\alpha| = |\beta| = m.$$

By integration by parts and by Lemma 1 from [8], for each function $u \in \widetilde{C}^\infty(\overline{G})$ we obtain the inequality

$$J_0 \geq -\varepsilon.C \|u\|^2_{2s-1,0} + (C - C_2.\varepsilon^{2-2s}) \|u\|_0^2, \tag{4.1}$$

where C_1, C_2 are positive constants, non depending on $\varepsilon \in (0,1)$.

Integrating by parts $J_{\alpha\beta}$ and summing by indices α, β, such that $|\alpha| = |\beta| = m$, we obtain

$$2 \sum_{|\alpha|=|\beta|=m} J_{\alpha\beta} = \int_G \sum_{|\alpha|=|\beta|=m} e^{\kappa t} \kappa^{2s-1} a^{\alpha\beta}(x) D_x^\beta u D_x^\alpha u\, dt dx.$$

If we fix $t \in [0,T]$, for the restriction $u(t,x)$ of this function $u \in \widetilde{C}^\infty(\overline{G})$ from the Garding inequality [4], we obtain

$$\int_D \sum_{|\alpha|=|\beta|=m} a^{\alpha\beta}(x) D_x^\beta u D_x^\alpha u\, dx \geq C_3 \|u(t,x)\|^2_{0,m,D} + C_4 \|u(t,x)\|^2_{0,D},$$

for appropriate constants C_3, C_4 and because the restriction $u(t,x) \in C^\infty(D_t)$ satisfies the boundary conditions

$$D_x^\alpha u \mid_{\partial D_t} = 0 \quad \text{for} \quad |\alpha| \leq m-1, \quad \text{where} \quad D_t = D \times \{t\},$$

and the Garding inequality, proved for $u \in C_0^\infty(D_t)$.

Multiplying the last inequality by $e^{\kappa t}\kappa^{2s-1}$ and integrating with respect to t from 0 to T, we obtain

$$(4.2)\quad \int_G \sum_{|\alpha|=|\beta|=m} e^{\kappa t}\kappa^{2s-1} a^{\alpha\beta}(x) D_x^\beta u D_x^\alpha u\, dt dx \geq C_3 \|u\|_{0,m}^2 + C_4 \|u\|_0^2$$

for each function $u \in \widetilde{C}^\infty(\overline{G})$, where C_3, C_4 are appropriate positive constants.

For the integrals J_i, $i = \overline{1, 2s}$ by integration by parts we obtain

$$\sum_{i=1}^{2s} J_i \geq -C_5 \|u\|_{2s-2,0}^2 + 2\delta \left\|D_t^{2s-1} u\right\|_0^2, \quad \forall u \in \widetilde{C}^\infty(\overline{G}),$$

where $C_5 = const > 0$ and δ is the constant from the condition (i) of the theorem.

From Lemma 1 from [8] we obtain that for each $\varepsilon \in (0,1)$ there exists a constant $C_6 > 0$, such that for each function $u \in \widetilde{C}^\infty(\overline{G})$, we have

$$(4.3)\sum_{i=1}^{2s} J_i \geq -C_5 \|u\|_{2s-1,0}^2 - C_6 \varepsilon^{2-2s} \|u\|_0^2 + 2\delta \left\|D_t^{2s-1} u\right\|_0^2, \ \forall u \in \widetilde{C}^\infty(\overline{G}).$$

Now from the inequalities (4.1)-(4.3), we obtain

$$\begin{gathered} 2\int_G Lu.R(t)u\, dt dx \geq -(C_5 + C_1) \|u\|_{2s-1,0}^2 + 2\delta \left\|D_t^{2s-1} u\right\|_0^2 \\ +(C - C_4 - C_2\varepsilon^{2-2s} - C_6\varepsilon^{2-2s}) \|u\|_0^2 + C_3 \|u\|_{0,m}^2, \ \forall u \in \widetilde{C}^\infty(\overline{G}), \end{gathered}$$

where $\varepsilon \in (0,1)$ is an arbitrary constant, $C_i = const > 0$, $i = \overline{1,6}$.

By Theorem 10.2, [2] it follows that there exists a constant $C_7 > 0$, such that

$$2\delta \left\|D_t^{2s-1} u\right\|_0^2 + C_3 \|u\|_{0,m}^2 \geq C_7 \|u\|_{2s-1,m}^2 \quad \forall u \in \widetilde{C}^\infty(\overline{G}),$$

from where it follows that

$$\begin{gathered} (Lu, R(t)u)_0 \geq \{C_7 - \varepsilon(C_5 + C_1)\} \|u\|_{2s-1,m}^2 \\ +\{C - C_4 - C_2\varepsilon^{2-2s} - C_6\varepsilon^{2-2s}\} \|u\|_0^2 \ \forall u \in \widetilde{C}^\infty(\overline{G}). \end{gathered}$$

The constants C_i, $i = \overline{1,7}$ do not depend on ε. If ε is a sufficiently small parameter and C is a sufficiently large positive constant, then there exists another positive constant C, such that

$$(4.4)\qquad (Lu, R(t)u)_0 \geq C \|u\|_{2s-1,m}^2 \quad \forall u \in \widetilde{C}^\infty(\overline{G}).$$

Let $v(t,x) \in \widetilde{C}_*^\infty(\overline{G})$ be an arbitrary fixed function. Consider the non-local problem

(4.5) $$R(t)u = v \quad \text{in} \quad G,$$

(4.6) $$D_x^\alpha u \mid_\Gamma = 0 \quad \text{for} \quad |\alpha| \leq m-1,$$

(4.7) $$D_t^i u(T,x) = \lambda D_t^i u(0,x), \quad i = \overline{0, 2s-2}; \quad \forall x \in \overline{D}.$$

Without loss of generality, we can consider (4.5) as an ordinary differential equation with respect to the variable t with constant coefficients (again with respect to t) in the form

(4.8) $$\sum_{l=1}^{2s-1} \binom{2s-1}{l} (\kappa)^{(2s-1)-l} D_t^l u + D_t^{2s-1} u = e^{-\kappa t} v.$$

Dividing the corresponding characteristic equation

(4.9) $$\sum_{l=1}^{2s-1} \binom{2s-1}{l} (\kappa)^{(2s-1)-l} \chi^l + \chi^{2s-1} = 0,$$

by κ^{2s-1} and set $\tau = \chi/\kappa + 1/2$, we obtain

$$(\tau + \frac{1}{2})^{2s-1} + (\tau - \frac{1}{2})^{2s-1} - 1 = 0,$$

from where by Lemma 2, [8] it follows that the last equation has only simple zeros. Hence the equation (4.9) also has only simple zeros.

The general solution of the equation (4.8) has the form

$$u(t,x) = \sum_{l=1}^{2s-1} b_i(x)\eta_i(t) + \eta_0(t,x),$$

where $\{\eta_i(t)\}_{i=1}^{2s-1}$ is a fundamental system of solutions of the corresponding homogeneous solution and $\eta_0(t,x)$ is a partial solution of the equation (4.8).

To build $\eta_0(t,x)$ we use a method of Cauchy [7, p.459]. We build a solution of the homogeneous equation, corresponding to the equation (4.8), satisfying the Cauchy conditions

$$\eta \mid_{t=\xi} = 0, \; D_t\eta \mid_{t=\xi} = 0, \; D_t^2\eta \mid_{t=\xi} = 0, \ldots, \; D_t^{2s-2}\eta \mid_{t=\xi} = 1,$$

where ξ is an arbitrary point in (0,T). Let we denote this solution by $\Phi(t,\xi)$. It exists and is unique, because $\det \|\eta(t)\| \neq 0 \; \forall \xi \in [0,T]$, where $\|\eta(t)\|$ is the fundamental matrix of considered equation. Then the function

(4.10)
$$\eta_0(t,x) = \int_{t_0}^{t} \Phi(t,\xi)e^{-\kappa\tau}v(\xi,x)d\xi,$$

where $t_0 \in (0,T)$ is fixed, is a partial solution of (4.8) [7]. Denote

$$\overrightarrow{B}(x) = \begin{bmatrix} b_1(x) \\ b_2(x) \\ b_3(x) \\ \dots \\ b_{2s-1}(x) \end{bmatrix}, \qquad \overrightarrow{D_t} = \begin{bmatrix} 1 \\ D_t \\ D_t^2 \\ \dots \\ D_t^{2s-2} \end{bmatrix}.$$

Now the boundary conditions (4.7) have the form

(4.11)
$$[\|\eta(T)\| - \lambda\,\|\eta(0)\|]\,\overrightarrow{B}(x) = \lambda \overrightarrow{D}_t\eta_0(0,x) - \overrightarrow{D}_t\eta_0(T,x).$$

The last system has a unique solution $\overrightarrow{B}(x)$ if and only if

(4.12)
$$\det\,[\|\eta(T)\| - \lambda\,\|\eta(0)\|] \neq 0,$$

where $\|\eta(t)\| = \left[D_t^i\eta_j(t)\right]_{j=1,2s-1}^{i=0,2s-2}$.

Using the results of Lemmas 2,3 from [8], it is easy to obtain that the inequality (4.12) is true.

If we solve the system (4.11) using the Cramer formula, then from (4.12) it follows, that the obtained solution fulfills the boundary conditions (4.6), because the function $\exp(-\kappa t)v(t,x)$ also fulfills the conditions (4.6) and the differentiation under the sign of the integral (4.10) is possible. In addition from the construction of this solution and from (4.5) and (4.7) it follows, that

$$D_t^{2s-1}u\,(T,x) = \lambda D_t^{2s-1}u\,(0,x) \quad \forall x \in \overline{D}.$$

Let us denote by $H_{t,x}^{-(2s-1,m)}(G)$ the space with a negative norm, adjoint to $H_{t,x}^{2s-1,m}(G)$. Then if $u(t,x)$ is a solution of the problem (4.5)-(4.7), where $v(t,x) \in \widetilde{C}_*^{\infty}(G)$ is an arbitrary fixed function, from the explicit construction of the solution it follows, that $u(t,x) \in \widetilde{C}^{\infty}(G)$ and from (4.4) we have

$$\|L^*v\|_{-(2s-1,m)} \cdot \|u\|_{2s-1,m} \geq (L^*v,u)_0 = (v,Lu)_0$$
$$= (R(t)u,Lu)_0 \geq C.\,\|u\|_{2s-1,m}^2,$$

from where

(4.13) $$\|L^*v\|_{-(2s-1,m)} \geq C.\|v\|_0 \quad \forall v \in \widetilde{C}^\infty_*(\overline{G}),$$

because from the equality $R(t)u = v$ it follows that $\|v\|_0 \leq \|u\|_{2s-1,m}$.

From (4.13) it follows that there exists a function $u \in H^{2s-1,m}_{t,x}(G)$, for which (2.5) is true [1]. Thus the theorem is proved. ■

P r o o f o f T h e o r e m 2. Let v be an arbitrary function, belonging to $\widetilde{C}^\infty_*(\overline{G})$. Denote

$$R_1(t)u = -\sum_{l=1}^{2s-1} \binom{2s-1}{l} D_t^{(2s-1)-l}[e^{-\kappa t}]D_t^l u - e^{-\kappa t} D_t^{2s-1} u.$$

Repeating the scheme of the proof of Theorem 1, we obtain the estimate

$$(L^*v, R_1(t)v)_0 \geq C' \|v\|^2_{2s-1,m} \quad \forall v \in \widetilde{C}^\infty_*(\overline{G}),$$

where $C' = const > 0$.

For any function $u(t,x) \in \widetilde{C}^\infty(\overline{G})$ consider the non-local problem

(4.14) $$R_1(t)v = u \quad \text{in} \quad G,$$

(4.15) $$D_x^\alpha v\,|_\Gamma = 0 \quad \text{for} \quad |\alpha| \leq m-1,$$

(4.16) $$\lambda D_t^i v\,(T,x) = D_t^i v\,(0,x)\,, \quad i = \overline{0, 2s-2}.$$

By Lemmas 1-3 from [8], analogously to the proof of Theorem 1, it is easy to obtain that the problem (4.14)-(4.16) has a unique solution $v \in H^{2s-1,m}_{t,x,*}(G)$. In addition, from the construction of this solution it follows that $v \in \widetilde{C}^\infty_*(\overline{G})$.

Denote by $H^{-(2s-1,m)}_{t,x,*}(G)$ the space with a negative norm, adjoint to the space $H^{2s-1,m}_{t,x,*}(G)$. Then, if $v(t,x)$ is a solution of the problem (4.14)-(4.16) for any fixed function $u \in \widetilde{C}^\infty(G)$, we have

$$\|Lu\|_{-(2s-1,m),*}\,\|v\|_{2s-1,m} \geq (Lu, v)_0 = (u, L^*v)_0$$

$$= (R_1(t)v, L^*v)_0 \geq C' \|v\|^2_{2s-1,m}\,,$$

from where

(4.17) $$\|Lu\|_{-(2s-1,m),*} \geq C' \|u\|_0 \quad \forall u \in \widetilde{C}^\infty(\overline{G}),$$

because the equality $R_1(t)v = u$ implies that $\|u\|_0 \leq \|v\|_{2s-1,m}$.

By the estimate (4.17) we obtain an uniqueness of the generalized solution of the considered problem [6]. The theorem is proved. ■

P r o o f o f T h e o r e m 3. Let first $l = 1$. From the conditions of the theorem, it follows that are fulfilled the conditions of Theorems 1,2. Hence the problem (1.1)-(2.2) has a unique solution $u \in H^{2s-1,m}_{t,x}(G)$. Let us set

$$L_1 w \equiv k_{2s}(t,x)D_t^{2s} w + \{k_{2s-1}(t,x) + D_t k_{2s}(t,x)\}\, D_t^{2s-1} w$$

$$+D_t k_{2s-1}(t,x)D_t^{2s-2} w - Nw - (-1)^m \sum |\alpha| = |\beta| = m D_x^\beta \left(a^{\alpha\beta} D_x^\beta w\right),$$

$$f_1 = D_t \left\{ f - c(t,x)u - Nu - [1 - \delta_{s1}] \sum_{i=1}^{2s-2} k_i(t,x) D_t^i u \right\},$$

where N = const ¿ 0 and δ_{s1} is the Kronecker symbol. Consider the problem

(4.18) $$L_1 w = f_1 \text{ in } G,$$

(4.19) $$D_x^\alpha w \,|_\Gamma = 0 \qquad \text{for } |\alpha| \le m - 1,$$

(4.20) $$D_t^i w\,(T,x) = \lambda D_t^i w\,(0,x), \quad i = \overline{0, 2s-1}.$$

Let us set:

$$\begin{aligned} \text{for } s > 1: \quad & k_i^{(1)}(t,x) = 0, \text{ for } i = \overline{1, 2s-3}, \\ \text{for } s \ge 1: \quad & k_{2s-2}^{(1)}(t,x) = [1 - \delta_{s1}]\, D_t k_{2s-1}(t,x), \\ & k_{2s-1}^{(1)}(t,x) = k_{2s-1}(t,x) + D_t k_{2s}(t,x), \\ & k_{2s}^{(1)}(t,x) = k_{2s}(t,x). \end{aligned}$$

We can write the equation (4.18) in the form

$$L_1 w \equiv k_{2s}^{(1)}(t,x)D_t^{2s} w + k_{2s-2}^{(1)}(t,x)D_t^{2s-1} w + k_{2s-2}^{(1)}(t,x)D_t^{2s-2} w$$

$$-(-1)^m \sum_{|\alpha|=|\beta|=m} D_x^\beta \left(a^{\alpha\beta} D_x^\beta w\right) + C_1(t,x)w = f_1(t,x).$$

If $f \in W^{1,[m/s]}_{t,x}(G)$, $u \in H^{2s-1,m}_{t,x}(G)$, then from the definition of the function f, we have that $f_1 \in L_2(G)$. From the conditions of the theorem it follows that if $N > 0$ is a sufficiently large positive constant, then for L_1 and f_1 are fulfilled all the conditions of Theorems 1, 2. Hence the problem (4.18)-(4.20) has a unique solution $w(t,x) \in H^{2s-1,m}_{t,x}(G)$.

For an arbitrary element $\zeta \in \widetilde{C}_*^\infty(\overline{G})$ it is easy to see that the function

$$v(t,x) = Z(t,x) + (\lambda - 1)^{-1} Z(0,x), \text{ where } Z(t,x) = \int_T^t \zeta(\tau,x)\,d\tau$$

also belongs to $\widetilde{C}_*^\infty(\overline{G})$ and $D_t v(t,x) = \zeta(t,x) \ \forall \ (t,x) \in \overline{G}$.

Hence the equality

$$(w, L_1^* v)_0 = (f_1, v)_0 \tag{4.21}$$

is true.

From the condition (ii) of the theorem, we have $f(T,x) = \lambda f(0,x)$ almost everywhere in D, $k_i(T,x) = k_i(0,x), i = \overline{0, 2s-2}$ for $s > 1$ $\forall x \in \overline{D}$ and $c(T,x) = c(0,x)$ $\forall x \in \overline{D}$.

Since $u \in H_{t,x}^{2s-1,m}(G)$, then $D_t^i u(T,x) = \lambda D_t^i u(0,x)$, $i = \overline{0, 2s-1}$ almost everywhere in D. Finally, from the definition of function ζ we have $D_t v = \zeta$.

Integrating by parts, we obtain

$$(f_1, v)_0 = -(f - cu - Nu - [1 - \delta_{s1}] \sum_{i=1}^{2s-2} k_i(t,x) D_t^i u(t,x), \zeta)_0. \tag{4.22}$$

From Lemma 1 from [9] for the function w, there exists a unique function $\Phi \in H_{t,x}^{2s-1,m}(G)$ such that $D_t \Phi = w$, from where

$$(D_t \Phi, L_1^* v)_0 = (f_1, v)_0 \ \ \forall \ v \in \widetilde{C}_*^\infty(\overline{G}).$$

Integrating by parts using the fact that the corresponding boundary integrals vanish, and using the definition of the function v, we obtain

$$(D_t \Phi, L_1^* v)_0 = -(\Phi, D_t \{ D_t^{2s}[k_{2s}(t,x) v] - D_t^{2s-1}[(D_t k_{2s}(t,x) + k_{2s-1}(t,x)) v]$$

$$+ D_t^{2s-2}[(D_t k_{2s-1}(t,x) v] - (-1)^m \sum_{|\alpha|=|\beta|=m} D_x^\beta \left(a^{\alpha\beta} D_x^\beta v \right) - Nv \})_0$$

$$= -(\Phi, D_t^{2s-1} \{ D_t k_{2s}(t,x) D_t v + k_{2s}(t,x) D_t^2 v - k_{2s-1}(t,x) D_t v \}$$

$$-(-1)^m \sum_{|\alpha|=|\beta|=m} D_x^\beta \left(a^{\alpha\beta}(x) D_x^\alpha D_t v \right) - N D_t v)_0 = -(\Phi, D_t^{2s-1} \{ D_t [k_{2s}(t,x) \zeta]$$

$$-k_{2s-1}(t,x) \zeta \} - (-1)^m \sum_{|\alpha|=|\beta|=m} D_x^\beta \left(a^{\alpha\beta}(x) D_x^\alpha \zeta \right) - N\zeta)_0 . \tag{4.23}$$

Consider the operator

$$L_2 \psi \equiv k_{2s}(t,x) D_t^{2s} \psi + k_{2s-1}(t,x) D_t^{2s-1} \psi$$

$$- (-1)^m \sum_{|\alpha|=|\beta|=m} D_x^\alpha \left(a^{\alpha\beta}(x) D_x^\beta \psi\right) - N\psi ,$$

and his formally adjoint operator

$$L_2^* \zeta \equiv D_t^{2s}[k_{2s}(t,x)\zeta] - D_t^{2s-1}[k_{2s-1}(t,x)\zeta]$$

$$- (-1)^m \sum_{|\alpha|=|\beta|=m} D_x^\beta \left(a^{\alpha\beta}(x) D_x^\alpha \zeta\right) - N\zeta . \qquad (4.24)$$

From the equalities (4.21)-(4.24) it follows that

$$(\Phi, L_2^*\zeta)_0 = (f - Cu - Nu - [1-\delta_{s1}] \sum_{i=1}^{2s-2} k_i(t,x) D_t^i u(t,x), \zeta)_0 \ \forall\, \zeta \in \tilde{C}_*^\infty(\overline{G}).$$

Hence Φ is a generalized solution of the problem

$$L_2\psi \equiv f - Cu - Nu - [1-\delta_{s1}] \sum_{i=1}^{2s-2} k_i(t,x) D_t^i u(t,x) , \qquad (4.25)$$

$$D_x^\alpha \psi \,|_\Gamma = 0 \quad \text{for} \quad |\alpha| \le m-1, \qquad (4.26)$$

$$D_t^i \psi(T,x) = \lambda D_t^i \psi(0,x), \quad i = \overline{0, 2s-1}. \qquad (4.27)$$

But u is a generalized solution of the problem (1.1)-(2.2) and the equality

$$(u, L^*\zeta)_0 = (f, \zeta)_0 \quad \forall\, \zeta \in \tilde{C}_*^\infty(\overline{G}) \qquad (4.28)$$

is true. Hence we have

$$(u, L_2^*\zeta)_0 = (f - Cu - Nu - [1-\delta_{s1}] \sum_{i=1}^{2s-2} k_i(t,x) D_t^i u(t,x), \zeta)_0 \ \forall\, \zeta \in \tilde{C}_*^\infty(\overline{G}).$$

If we choose a constant $N > 0$ sufficiently large, then from the condition of the theorem it follows that the problem (4.25)-(4.27) has a unique solution belonging to the class $H_{t,x}^{2s-1,m}(G)$. Hence from the uniqueness $u = \Phi$ almost everywhere in G, follows $D_t u = w$ almost everywhere in G. Then $D_t u \in H_{t,x}^{2s-1,m}(G)$ and for $D_t u$ are fulfilled the boundary conditions (4.27) almost everywhere in D.

From the equality (4.28) we obtain

$$\int_G \sum_{|\alpha|=|\beta|=m} a^{\alpha\beta}(x) D_x^\alpha u D_x^\beta \zeta \, dt dx = (f - cu - \sum_{i=1}^{2s} k_i(t,x) D_t^i u(t,x), \zeta)_0$$

for each $\zeta \in \widetilde{C}_*^\infty(\overline{G})$, hence for each $\zeta \in C_0^\infty(G)$. Then from Theorem 3, [4], it follows that $u(t,x) \in W_{t,x}^{0,2m}(G)$.

From the estimates of the mixed derivatives (point 10.2 from [2]), we have that $u(t,x) \in W_{t,x}^{2s,2m}(G)$. Now from the equalities $D_t^i w(T,x) = \lambda D_t^i w(0,x)$, $i = \overline{0,2s-2}$ almost everywhere in D, changing the index we obtain $D_t^i u(T,x) = \lambda D_t^i u(0,x)$, $i = \overline{0,2s-2}$ almost everywhere in D if we add the equality $u(T,x) = \lambda u(0,x)$. Thus, the theorem is proved in the case when $l = 1$.

Let we suppose that the theorem is true for $l = l_0$, where $l_0 \geq 1$ is a fixed number, and that the conditions of the theorem are fulfilled for $l = l_0 + 1$. Then the problem (1.1)-(2.2) has a unique solution $u \in W_{t,x}^{2s-1+l_0,2m+(l_0-1)[m/s]}(G)$, such that almost everywhere in D, $D_t^i u(T,x) = \lambda D_t^i u(0,x)$, $i = \overline{0,2s-2+l_0}$.

Let us set in the conditions of Lemma 2, [9],

$$\mu(l,s,m) = l,\ \nu(l,s,m) = l\,[m/s],\ \chi(l,s,m) = 2m + (l-1)\,[m/s],$$

where [.] is the usual function "entire part of the argument". For this choice it is easy to check that are fulfilled the conditions (5)-(10) from [9]. Since we suppose that $f \in W_{t,x}^{l_0+1,(l_0+1)[m/s]}(G)$, $u \in W_{t,x}^{2s-1+l_0,2m+(l_0-1)[m/s]}(G)$, then from point (i) of Lemma 2, [9], it follows that $f_1 \in W_{t,x}^{l_0,l_0[m/s]}(G)$. Now for the operator L_1 and for the right hand f_1, all the cnditions of the theorem are fulfilled. By the suggestion in induction, we obtain that the problem (4.18)-(4.20) has a unique solution $w \in W_{t,x}^{2s-1+l_0,2m+(l_0-1)[m/s]}(G)$, such that $D_t^i w(T,x) = \lambda . D_t^i w(0,x)$, $i = \overline{0,2s-2+l_0}$ almost everywhere in D. Repeating the discourses conducted in the case $l = 1$, we obtain that $D_t u = w$ almost everywhere in G. Then $D_t u \in W_{t,x}^{2s-1+l_0,2m+(l_0-1)[m/s]}(G)$, from where $u \in W_{t,x}^{2s+l_0,0}(G)$. Almost everywhere in D we have $D_t^i u(t,x)\,|_{t=T} = \lambda D_t^i u(t,x)\,|_{t=0}$, $i = \overline{0,2s-1+l_0}$.

By integration by parts in the equality (2.5) and moving some summands to the right hand, we have

$$\int_G \sum_{|\alpha|=|\beta|=m} a^{\alpha\beta}(x) D_x^\alpha u D_x^\beta \zeta \, dtdx = (f - cu - \sum_{i=1}^{2s} k_i(t,x)\, D_t^i u(t,x), \zeta)_0$$

for each $\zeta \in \widetilde{C}_*^\infty(\overline{G})$, hence also for each $C_0^\infty(G)$. From point (ii) of Lemma 2, [9], it follows that the function $f_2 = f - cu - \sum_{i=1}^{2s} k_i(t,x)\, D_t^i u(t,x)$ belongs to the class $W_{t,x}^{0,l_0.[m/s]}(G)$, from where using again Theorem 3 from [3], we obtain that $u \in W_{t,x}^{0,2m+l_0[m/s]}(G)$. Now the estimates from point 10.2 of [2] give us that $u \in W_{t,x}^{2s+l_0,2m+l_0[m/s]}(G)$. The theorem is proved. ■

A cylindrical domain of the considered type fulfills a b-horn condition for a vector $b = (b_0, b_1, b_2, b_3, ..., b_n)$ such that $b_i > 0$, $i = \overline{0,n}$, $b_1 = b_2 = ... = b_n$. If $l \geq 1$ is an integer number, such that

$$(4.29) \qquad \frac{2s}{2s-1+l} + \frac{1}{2}\left\{\frac{1}{2s-1+l} + \frac{n}{2m+(l-1)[m/s]}\right\} < 1,$$

then from Theorem 10.4, [2], it follows that the derivatives $D_t^i u$, $i \leq 2s$ of the generalized solution $u \in W_{t,x}^{2s-1+l,2m+(l-1)[m/s]}(G)$, of the problem (1.1)-(2.2) are classical. Now again by Theorem 10.4 from [2], we have that if $l \geq 1$ fulfills the inequality

$$(4.30) \qquad \frac{2m}{2m+(l-1)[m/s]} + \frac{1}{2}\left\{\frac{1}{2s-1+l} + \frac{n}{2m+(l-1)[m/s]}\right\} < 1,$$

then the derivatives $D_x^\alpha u$, $|\alpha| \leq 2m$ of this solution are classical.

If both the inequalities (4.29) and (4.30) are true, from Definition 1, using integration by parts, we obtain that the generalized solution of the problem (1.1)-(2.2) fulfills the equation (1.1) in classical sense.

5. Example

Let $f \in L_2(G), n = 2, T = 1, A = const > 0, C = const > 0, X = const > 0$. Set $D = \{(x_1, x_2)/x_1^2 + x_1^2 < X\}$, $G = D \times (0,1), \Gamma = \partial D \times (0,1)$. Consider the problem

$$(5.1) \qquad \begin{gathered}[\sin(2\pi t) - 1]D_t^6 u + AD_t^5 u + D_{x_1}^6 u + D_{x_2}^6 u \\ -[\cos(\pi t) - C]u = f(t,x) \quad \text{in } G,\end{gathered}$$

$$(5.2) \qquad D_x^\alpha u \mid_\Gamma = 0 \quad \text{for} \quad |\alpha| \leq 2,$$

$$(5.3) \qquad D_t^i u(T,x) = (1/2)D_t^i u(0,x), \quad i = \overline{0,5}.$$

In this example we have $s = m = 3$, $k_6(t,x) \equiv \sin(2\pi t) - 1$, $k_5(t,x) \equiv A$, $k_i(t,x) \equiv 0$, $\overline{i = 1,4}$, $a^{\alpha\beta}(x) \equiv 1$, if $\alpha = \beta = (3,0)$, $a^{\alpha\beta}(x) \equiv 0$ for other multi indices, $c(t,x) \equiv \cos(\pi t)$.

The equation (5.1) is a sixth order hyperbolic-parabolic type equation. It is easy to see that if the constants A, C are sufficiently large, $f \in W_{t,x}^{l,l}(G)$ and $D_t^i f(T,x) = (1/2)D_t^i f(0,x)$, almost everywhere in D for $i = \overline{0,l-1}$, where $l \geq 1$ is a parameter, then all the conditions of the theorem are fulfilled. Hence the problem (5.1)-(5.3) has a unique generalized solution belonging to the class $W_{t,x}^{3+l,3+l}(G)$. If we put $l = 3$ in the conditions (4.29),(4.30), then the generalized solution of the problem (5.1)-(5.3) is a classical solution of this problem.

In the present paper we generalize the result of [5], where the case of second order equation is considered.

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