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On the Extensions, Refinements and Modifications of Relators [1]

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By establishing some intimate connections between unary operations and set-valued functions on relators (families of relations), we greatly extend and supplement some of the former results of Á. Száz and J. Mala on the various refinements and modifications of relators.

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0. Introduction

A nonvoid family $\mathcal{R}$ of binary relations on a nonvoid set X is called a relator on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is called a relator space. Relator spaces are straightforward generalizations of ordered sets and uniform spaces [13]. They deserve to be widely investigated because of the following two facts.

If $\mathcal{D}$ is a family of certain distance functions on X, then the relator $\mathcal{R}_{\mathcal{D}}$, consisting of all surroundings $B_r^d = \{ (x, y) \in X^2 : \ d(x, y) < r \}$, where $d \in \mathcal{D}$ and $r > 0$, is a more convenient mean of defining the basic notions of analysis in the space $X(\mathcal{D})$ than the family of all open subsets of $X(\mathcal{D})$, or even the family $\mathcal{D}$, itself.

Moreover, all the reasonable generalizations of the usual topological structures (such as proximities, closures, topologies, filters and convergences, for instance) can be easily derived from relators (according to the results of [16] and [12]), and thus they need not be studied separately.

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If $\mathcal{R}$ is a relator on X, then according to [16],, for any $A, B \subset X$ and $x, y \in X$, we write:

(1) $B \in \operatorname{Int}_{\mathcal{R}}(A)$ $(B \in \operatorname{Cl}_{\mathcal{R}}(A))$ if $R[B] \subset A$ $(R[B] \cap A \neq \emptyset)$ for some (all) $R \in \mathcal{R}$;

(2) $x \in \operatorname{int}_{\mathcal{R}}(A)$ $(x \in \operatorname{cl}_{\mathcal{R}}(A))$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(A)$ $(\{x\} \in \operatorname{Cl}_{\mathcal{R}}(A))$;

(3) $y \in \sigma_{\mathcal{R}}(x)$ $(y \in \rho_{\mathcal{R}}(x))$ if $y \in \operatorname{int}_{\mathcal{R}}(\{x\})$ $(y \in \operatorname{cl}_{\mathcal{R}}(\{x\}))$;

and moreover:

(4) $A \in \tau_{\mathcal{R}}$ $(A \in \tau\!\!\!-_{\mathcal{R}})$ if $A \in \operatorname{Int}_{\mathcal{R}}(A)$ $(X \setminus A \notin \operatorname{Cl}_{\mathcal{R}}(A))$;

(5) $A \in \mathcal{T}_{\mathcal{R}}$ $(A \in \mathcal{F}_{\mathcal{R}})$ if $A \subset \operatorname{int}_{\mathcal{R}}(A)$ $(\operatorname{cl}_{\mathcal{R}}(A) \subset A)$;

(6) $A \in \mathcal{E}_{\mathcal{R}}$ $(A \in \mathcal{D}_{\mathcal{R}})$ if $\operatorname{int}_{\mathcal{R}}(A) \neq \emptyset$ $(\operatorname{cl}_{\mathcal{R}}(A) = X)$.

The relations $\operatorname{Int}_{\mathcal{R}}$, $\operatorname{int}_{\mathcal{R}}$ and $\sigma_{\mathcal{R}}$ are called the proximal, the topological, and the infinitesimal interiors induced by $\mathcal{R}$ on X, respectively. And the members of the families $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$, and $\mathcal{E}_{\mathcal{R}}$ are called the proximally open, the topologically open, and the fat subsets of $X(\mathcal{R})$, respectively.

The proximally open sets and the fat sets are usually more important tools, then the topologically open sets. For instance, if $\prec$ is a preorder on X, then $\mathcal{T}_{\{\prec\}}$ and $\mathcal{E}_{\{\prec\}}$ are precisely the families of all ascending and residual subsets of the preordered set $X(\prec)$, respectively.

Therefore, it is not surprising that sometimes we also need the sets

$$E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}} \qquad \text{and} \qquad \mathcal{D}_{\mathcal{R}} = \bigcup \left(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}} \right).$$

By specializing some unpublished ideas of Á. Száz, a function x of a preordered set Γ into a set X is called a Γ–net in X. And the Γ–net x is said to be fatly (densely) in a subset A of X if $x^{-1}(A)$ is a fat (dense) subset of Γ. Note that these definitions would actually allow Γ to be an arbitrary relator space.

Moreover, if $\mathcal{R}$ is a relator on X, then for any Γ-nets x and y in X and $a \in X$ we write:

(7) $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $(y \in \operatorname{Adh}_{\mathcal{R}}(x))$ if the net (y, x) is fatly (densely) in each $R \in \mathcal{R}$;

(8) $a \in \lim_{\mathcal{R}}(x)$ $(a \in \operatorname{adh}_{\mathcal{R}}(x))$ if $a_\Gamma \in \operatorname{Lim}_{\mathcal{R}}(x)$ $(a_\Gamma \in \operatorname{Adh}_{\mathcal{R}}(x))$, where $a_\Gamma = \Gamma \times \{a\}$.

To provide a general framework for the investigation of the above basic tools, we introduce here several new definitions and establish their most important consequences.

A function $\mathfrak{F}$ of the family of all relators on X into a family of sets is called a set-valued function for relators on X. And we write $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

In particular, a function $\Box$ of the family of all relators on X into itself is called a unary operation for relators on X. And we write $\mathcal{R}^{\Box} = \Box(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

The set-valued function $\mathfrak{F}$ is called $\Box$–increasing ($\Box$–decreasing) if for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X we have $\mathcal{S} \subset \mathcal{R}^{\Box} \iff \mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ ($\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$). Namely, in this case, $\mathfrak{F}$ is increasing (decreasing) in the usual sense.

Moreover, an increasing unary operation $\Box$ for relators on X is called a refinement operation if it is expansive and idempotent in the sense that $\mathcal{R} \subset \mathcal{R}^{\Box}$ and $\mathcal{R}^{\Box} = \mathcal{R}^{\Box\Box}$ for every relator $\mathcal{R}$ on X.

Theorem 1. *If $\Box$ is a unary operation for relators on X, then the following assertions are equivalent:*

(1) $\Box$ is a refinement; *(2) $\Box$ is self–increasing;*

(3) there exists a $\Box$–monotonic set-valued function $\mathfrak{F}$ for relators on X.

Theorem 2. *If $\Box$ is a unary operation and $\mathfrak{F}$ is a $\Box$–monotonic set-valued function for relators on X, then, for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\Box}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box}}$.*

If $\mathfrak{F}$ is an increasing (decreasing) set-valued function for relators on X, then the operation $\Box_{\mathfrak{F}}$, defined by

$$\mathcal{R}^{\Box_{\mathfrak{F}}} = \{ \mathcal{S} \subset X^{\in} : \quad \mathfrak{F}_{\{\mathcal{S}\}} \subset \mathfrak{F}_{\mathcal{R}} \} \qquad \left(\mathcal{R}^{\Box_{\mathfrak{F}}} = \{ \mathcal{S} \subset X^{\in} : \quad \mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\{\mathcal{S}\}} \} \right)$$

for every relator $\mathcal{R}$ on X, is called the operation induced by $\mathfrak{F}$.

Moreover, the increasing (decreasing) set-valued function $\mathfrak{F}$ is called regular if it is $\Box_{\mathfrak{F}}$–increasing ($\Box_{\mathfrak{F}}$–decreasing).

Theorem 3. *If $\Box$ is a unary operation and $\mathfrak{F}$ is a $\Box$–monotonic set-valued function for relators on X, then $\Box = \Box_{\mathfrak{F}}$.*

Theorem 4. *If $\mathfrak{F}$ is a monotonic set-valued function for relators on X, then the following assertions are equivalent:*

(1) $\mathfrak{F}$ *is regular;* *(2)* $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}}$ *for every relator* $\mathcal{R}$ *on* X;

(3) $\mathfrak{F}$ *is* $\square$*-monotonic for some unary operation* $\square$ *for relators on* X.

An increasing (decreasing) set-valued function $\mathfrak{F}$ for relators on X is called normal if, for every relator $\mathcal{R}$ on X, we have

$$\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}} \qquad \left(\mathfrak{F}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}} \right).$$

Theorem 5. *If $\mathfrak{F}$ is a normal set-valued function for relators on X, then $\mathfrak{F}$ is, in particular, regular.*

Theorem 6.

(1) Int, int, σ, τ, $\tau\!\!\!-$, $\mathcal{E}$, *and* D *are normal increasing set-valued functions for relators on* X;

(2) Lim, Adh, lim, adh, Cl, cl, ρ, $\mathcal{D}$, *and* E *are normal decreasing set-valued functions for relators on* X.

Unfortunately, the increasing set-valued functions $\mathcal{T}$ and $\mathcal{F}$, on which topology was based on, are not even regular in general.

Therefore, if $\mathcal{R}$ is a relator on X, then in general there does not exist a largest relator $\mathcal{R}^{\square}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}}$ $(\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\square}})$.

1. Unary operations and set-valued functions for relators

Definition 1.1. A function $\square$ of the family of all relators on X into itself is called a unary operation for relators on X. And we write $\mathcal{R}^{\square} = \square(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

Moreover, a function $\mathfrak{F}$ of the family of all relators on X into a family of sets is called a set-valued function for relators on X. And we write $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

Remark 1.2. Note that thus a unary operation for relators is, in particular, a set-valued function for relators.

Important examples for set-valued functions and unary operations for relators have been given by Száz and Mala in [16], [17], [4], and [7].

Definition 1.3. If $\square$ is a unary operation and $\mathfrak{F}$ is a set-valued function for relators on X, then we say that:

(1) $\square$ is stable if $\mathcal{R} = \mathcal{R}^{\square}$ whenever $\mathcal{R} = \{ X^{\in} \}$;

(2) $\square$ is expansive if $\mathcal{R} \subset \mathcal{R}^{\square}$ for every relator $\mathcal{R}$ on X;

(3) $\square$ is idempotent if $\mathcal{R}^{\square} = \mathcal{R}^{\square\square}$ for every relator $\mathcal{R}$ on X;

(4) $\mathfrak{F}$ is increasing if $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X with $\mathcal{S} \subset \mathcal{R}$.

Remark 1.4. Analogously, the set-valued function $\mathfrak{F}$ is called decreasing if $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X with $\mathcal{S} \subset \mathcal{R}$.

Note that the expansivity property (1) implies that $\mathcal{R}^{\square} \subset \mathcal{R}^{\square\square}$ for every relator $\mathcal{R}$ on X. Therefore, an expansive operation $\square$ for relators is idempotent if the converse inclusion holds.

The usefulness of the expansivity and the idempotency properties are already apparent from the following theorem.

Theorem 1.5. *If $\square$ is a unary operation for relators on X, then the following assertions are equivalent:*

(1) $\square$ is expansive and idempotent;

(2) for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\square}$ is the largest relator on X such that $\mathcal{R}^{\square} = (\mathcal{R}^{\square})^{\square}$;

(3) there exists a set-valued function $\mathfrak{F}$ for relators on X such that, for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\square}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square}}$.

Proof. Assume that the assertion (1) holds, and let $\mathcal{R}$ be a relator on X. Then by the idempotency property of the operation $\square$ we have $\mathcal{R}^{\square} = (\mathcal{R}^{\square})^{\square}$. On the other hand, if $\mathcal{S}$ is a relator on X such that $\mathcal{R}^{\square} = \mathcal{S}^{\square}$, then by the expansivity property of the operation $\square$, we also have $\mathcal{S} \subset \mathcal{R}^{\square}$. Therefore, the assertion (2) also holds.

While if the assertion (2) holds, then we can observe that the set-valued function $\mathfrak{F} = \square$ has the properties required in the assertion (3). Therefore, the implication (2) $\Longrightarrow$ (3) is obviously true.

Assume now that the assertion (3) holds, and let $\mathcal{R}$ be a relator on X. Then, from the obvious equality $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}}$, by using the corresponding maximality property of the relator $\mathcal{R}^{\square}$, we can infer that $\mathcal{R} \subset \mathcal{R}^{\square}$. Therefore, the operation $\square$ is expansive. Moreover, by writing $\mathcal{R}^{\square}$ in place of $\mathcal{R}$ in the assertion (3), we can at once see that $\mathfrak{F}_{\mathcal{R}^{\square}} = \mathfrak{F}_{\mathcal{R}^{\square\square}}$. Hence, since $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square}}$, we also have $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square\square}}$. Hence, by using the corresponding maximality property of the relator $\mathcal{R}^{\square}$, we can infer that $\mathcal{R}^{\square\square} \subset \mathcal{R}^{\square}$. Therefore, by Remark 1.4, the operation $\square$ is idempotent. And thus the assertion (1) also holds. ■

Remark 1.6. Despite the above theorem, the most important property of a unary operation for relators is certainly the monotonicity property.

Namely, all the important set-valued functions for relators are monotone. But, some useful operations for relators are not either expansive or idempotent.

Definition 1.7. If $\square$ is a unary operation and $\mathfrak{F}$ is a set-valued function for relators on X, then we say that:

(1) $\square$ is an extension if it is expansive and increasing;

(2) $\square$ is a modification if it is idempotent and increasing;

(3) $\square$ is a refinement if it is expansive, idempotent and increasing;

(4) $\mathfrak{F}$ is $\square$–increasing if for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X we have $\mathcal{S} \subset \mathcal{R}^{\square} \iff \mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$.

Remark 1.8. Analogously, the set-valued function $\mathfrak{F}$ is called $\square$–decreasing if for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X we have $\mathcal{S} \subset \mathcal{R}^{\square} \iff \mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$.

Moreover, in particular, the operation $\square$ will be called self-increasing if it is $\square$–increasing. That is, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have $\mathcal{S} \subset \mathcal{R}^{\square} \iff \mathcal{S}^{\square} \subset \mathcal{R}^{\square}$.

The appropriateness of the above definitions is already apparent from the following theorem.

Theorem 1.9. *If $\square$ is a unary operation for relators on X, then the following assertions are equivalent:*

(1) $\square$ is a refinement; *(2) $\square$ is self-increasing;*

(3) there exists a $\square$–increasing set-valued function $\mathfrak{F}$ for relators on X.

P r o o f. Assume that the assertion (1) holds, and let $\mathcal{R}$ and $\mathcal{S}$ be relators on X. If $\mathcal{S} \subset \mathcal{R}^{\square}$, then by the increasingness and the idempotency properties of the operation $\square$, it is clear that $\mathcal{S}^{\square} \subset \mathcal{R}^{\square\square} = \mathcal{R}^{\square}$. While, if $\mathcal{S}^{\square} \subset \mathcal{R}^{\square}$, then by the expansivity property of the operation $\square$ alone it is clear that $\mathcal{S} \subset \mathcal{R}^{\square}$. Therefore, the assertion (2) also holds.

While if the assertion (2) holds, then we can note that the set-valued function $\mathfrak{F} = \square$ is $\square$–increasing. Therefore, the implication (2) $\Longrightarrow$ (3) is obviously true.

Assume now that the assertion (3) holds, and let $\mathcal{R}$ and $\mathcal{S}$ be relators on X. Then, from the obvious inclusion $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{R}}$, by using the assumption (3) we can infer that $\mathcal{R} \subset \mathcal{R}^{\square}$. Therefore, the operation $\square$ is expansive. On the other hand, from the obvious inclusion $\mathcal{R}^{\square} \subset \mathcal{R}^{\square}$, by using the assumption (3), we can infer that $\mathfrak{F}_{\mathcal{R}^{\square}} \subset \mathfrak{F}_{\mathcal{R}}$. Hence, by writing $\mathcal{R}^{\square}$ in place of $\mathcal{R}$, we can see that $\mathfrak{F}_{\mathcal{R}^{\square\square}} \subset \mathfrak{F}_{\mathcal{R}}$. Hence, by using the assumption (3), we can infer that $\mathcal{R}^{\square\square} \subset \mathcal{R}^{\square}$. Thus, by Remark 1.4, the operation $\square$ is idempotent.

Finally, if $\mathcal{S} \subset \mathcal{R}$, then by the expansivity property of the operation $\square$ it is clear that $\mathcal{S} \subset \mathcal{R}^{\square}$. Hence, by using the assumption (3), we can infer that $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$. Hence, since $\mathfrak{F}_{\mathcal{S}^{\square}} \subset \mathfrak{F}_{\mathcal{S}}$, it is clear that we also have $\mathfrak{F}_{\mathcal{S}^{\square}} \subset \mathfrak{F}_{\mathcal{R}}$. Hence, by using the assumption (3), we can infer that $\mathcal{S}^{\square} \subset \mathcal{R}^{\square}$. Therefore, the operation $\square$ is increasing. And thus the assertion (1) also holds. ■

Concerning refinement operations, we can also easily prove the following theorem.

Theorem 1.10. *If $\square$ is a refinement operation for relators on X and $(\mathcal{R}_i)_{i\in I}$ is a nonvoid family of relators on X, then*

$$\Big(\bigcup_{i\in I} \mathcal{R}_i\Big)^{\square} = \Big(\bigcup_{i\in I} \mathcal{R}_i^{\square}\Big)^{\square}.$$

Hence, it is clear that, in particular, we have

Corollary 1.11. *If $\square$ is a refinement operation for relators on X and $\mathcal{R}$ is a relator on X, then*

$$\mathcal{R}^{\square} = \Big(\bigcup_{R\in\mathcal{R}} \{R\}^{\square}\Big)^{\square}.$$

However, it is now more important to note that, in addition to Theorem 1.9, we can also prove the following theorem.

Theorem 1.12. *If $\Box$ is a unary operation and $\mathfrak{F}$ is a set-valued function for relators on X, then the following assertions are equivalent:*

(1) $\mathfrak{F}$ is $\Box$–increasing;

(2) $\mathfrak{F}$ is increasing and, for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\Box}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}^{\Box}} \subset \mathfrak{F}_{\mathcal{R}}$.

Proof.

Assume that the assertion (1) holds, and let $\mathcal{R}$ and $\mathcal{S}$ be relators on X. Then, if $\mathcal{S} \subset \mathcal{R}$, then by the assertion (1) and Theorem 1.9, it is clear that $\mathcal{S} \subset \mathcal{R}^{\Box}$, and hence $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$. Therefore, the function $\mathfrak{F}$ is increasing. On the other hand, from the obvious inclusion $\mathcal{R}^{\Box} \subset \mathcal{R}^{\Box}$, by using the assertion (1), we can infer that $\mathfrak{F}_{\mathcal{R}^{\Box}} \subset \mathfrak{F}_{\mathcal{R}}$. Moreover, if $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$, then by using the assertion (1) we can see that $\mathcal{S} \subset \mathcal{R}^{\Box}$. Therefore, the assertion (2) also holds.

Conversely, assume now that the assertion (2) holds, and let $\mathcal{R}$ and $\mathcal{S}$ be relators on X. Then, if $\mathcal{S} \subset \mathcal{R}^{\Box}$, then by the assertion (2) it is clear that $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}^{\Box}} \subset \mathfrak{F}_{\mathcal{R}}$. While, if $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$, then by the corresponding maximality of the relator $\mathcal{R}^{\Box}$ it is clear that $\mathcal{S} \subset \mathcal{R}^{\Box}$. Therefore, the assertion (1) also holds. ∎

Now, as an immediate consequence of Theorems 1.9 and 1.12, we can also state the following corollary.

Corollary 1.13 *If $\Box$ is a unary operation and $\mathfrak{F}$ is a $\Box$–increasing set-valued function for relators on X, then for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\Box}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box}}$.*

Proof. If $\mathcal{R}$ is a relator on X, then by Theorem 1.9 we have $\mathcal{R} \subset \mathcal{R}^{\Box}$. Hence, by Theorem 1.12, it follows that $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{R}^{\Box}}$. Moreover, by Theorem 1.12, we also have $\mathfrak{F}_{\mathcal{R}^{\Box}} \subset \mathfrak{F}_{\mathcal{R}}$. Therefore, $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box}}$.

On the other hand, if $\mathcal{S}$ is a relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$, then in particular $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$. Hence, since $\mathfrak{F}$ is $\Box$–increasing, it follows that $\mathcal{S} \subset \mathcal{R}^{\Box}$. Therefore, $\mathcal{R}^{\Box}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box}}$. ∎

Remark 1.14. Note that if $\Box$ is a unary operation and $\mathfrak{F}$ is a $\Box$–decreasing set-valued function for relators on X, then the pointwise complement $\mathfrak{F}^c$ of $\mathfrak{F}$, defined by

$$\mathfrak{F}^c_{\mathcal{R}} = \left(\bigcup \operatorname{rng}(\mathfrak{F})\right) \setminus \mathfrak{F}_{\mathcal{R}}$$

for every relator $\mathcal{R}$ on X, is $\square$–increasing. Therefore, the corresponding duals of Theorems 1.9 and 1.12 and Corollary 1.13 are also true.

2. The induced unary operations and regular set-valued functions

Definition 2.1. If $\mathfrak{F}$ is an increasing set-valued function for relators on X, then the operation $\square_{\mathfrak{F}}$, defined by

$$\mathcal{R}^{\square_{\mathfrak{F}}} = \left\{ S \subset X^{\in} : \quad \mathfrak{F}_{\{S\}} \subset \mathfrak{F}_{\mathcal{R}} \right\}$$

for every relator $\mathcal{R}$ on X, is called the operation induced by $\mathfrak{F}$.

Moreover, the increasing set-valued function $\mathfrak{F}$ is called regular if it is $\square_{\mathfrak{F}}$–increasing.

Remark 2.2. Analogously, if $\mathfrak{F}$ is a decreasing set-valued function for relators on X, then the function $\square_{\mathfrak{F}}$, defined by $\mathcal{R}^{\square_{\mathfrak{F}}} = \left\{ S \subset X^{\in} : \quad \mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\{S\}} \right\}$ for every relator $\mathcal{R}$ on X, is called the operation induced by $\mathfrak{F}$. Moreover, the decreasing set-valued function $\mathfrak{F}$ is called regular if it is $\square_{\mathfrak{F}}$–decreasing.

Note that if $\mathfrak{F}$ is a set-valued function for relators on X such that $\mathfrak{F}$ is both increasing and decreasing, then $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{P}(X^2)}$ for every relator $\mathcal{R}$ on X. Therefore, the two possible definitions cannot lead to confusions.

The appropriateness of Definition 2.1 is already apparent from the following theorem.

Theorem 2.3. *If $\square$ is a unary operation and $\mathfrak{F}$ is a $\square$–increasing set-valued function for relators on X, then $\square = \square_{\mathfrak{F}}$. And thus, in particular, $\mathfrak{F}$ is regular.*

Proof. If $S \in \mathcal{R}^{\square}$, i.e., $\{S\} \subset \mathcal{R}^{\square}$ for some relator $\mathcal{R}$ on X, then since $\mathfrak{F}$ is $\square$–increasing we have $\mathfrak{F}_{\{S\}} \subset \mathfrak{F}_{\mathcal{R}}$. Hence, by the definition of $\mathcal{R}^{\square_{\mathfrak{F}}}$, it follows that $S \in \mathcal{R}^{\square_{\mathfrak{F}}}$. Therefore, $\mathcal{R}^{\square} \subset \mathcal{R}^{\square_{\mathfrak{F}}}$.

While, if $S \in \mathcal{R}^{\square_{\mathfrak{F}}}$ for some relator $\mathcal{R}$ on X, then by the definition of $\mathcal{R}^{\square_{\mathfrak{F}}}$ we have $\mathfrak{F}_{\{S\}} \subset \mathfrak{F}_{\mathcal{R}}$. Hence, since $\mathfrak{F}$ is $\square$–increasing, it follows that $\{S\} \subset \mathcal{R}^{\square}$, i.e., $S \in \mathcal{R}^{\square}$. Therefore, $\mathcal{R}^{\square_{\mathfrak{F}}} \subset \mathcal{R}^{\square}$ is also true. ∎

Now, as an immediate consequence of Theorem 2.3, we can also state the next corollary.

Corollary 2.4. *If $\mathfrak{F}$ is a set-valued function for relators on X, then there exists at most one unary operation $\square$ for relators on X such that $\mathfrak{F}$ is $\square$–increasing.*

Remark 2.5. Later we shall see that even for a refinement operation $\square$ for relators there may exist more than one $\square$–increasing set-valued function.

Moreover, there are important increasing set-valued functions for relators which are not regular.

However, despite this, we can still prove the following theorem.

Theorem 2.6. *If $\mathfrak{F}$ is an increasing set-valued function for relators on X, then*

(1) $\square_{\mathfrak{F}}$ is an extension operation for relators on X;

(2) $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ implies $\mathcal{S} \subset \mathcal{R}^{\square_{\mathfrak{F}}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X.

Proof. By using Definition 2.1, we can easily see that

$$\mathcal{R}^{\square_{\mathfrak{F}}} = \bigcup \{ \mathcal{S} : \quad \mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}} \}$$

for every relator $\mathcal{R}$ on X. And hence, the required assertions are quite obvious. ■

Remark 2.7. Later, we shall see that the operation $\square_{\mathfrak{F}}$, induced by an increasing set-valued function $\mathfrak{F}$ for relators, need not be idempotent.

However, despite this, we can still prove the following

Theorem 2.8. *If $\mathfrak{F}$ is an increasing set-valued function for relators on X, then the following assertions are equivalent:*

(1) $\mathfrak{F}$ is regular;

(2) $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}}$ for every relator $\mathcal{R}$ on X;

(3) $\mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}} \subset \mathfrak{F}_{\mathcal{R}}$ for every relator $\mathcal{R}$ on X;

(4) $\mathcal{S} \subset \mathcal{R}^{\square_{\mathfrak{F}}}$ implies $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X;

(5) $\mathfrak{F}$ is $\square$–increasing for some unary operation $\square$ for relators on X.

Proof. If the assertion (1) holds, i.e., $\mathfrak{F}$ is $\square_{\mathfrak{F}}$–increasing, then from Corollary 1.13 we can see that the assertion (2) also holds.

While, if the assertion (3) holds, then since $\mathfrak{F}$ is increasing it is clear that $\mathcal{S} \subset \mathcal{R}^{\square_{\mathfrak{F}}}$ implies $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}} \subset \mathfrak{F}_{\mathcal{R}}$ for any two relator $\mathcal{R}$ and $\mathcal{S}$ on X. That is, the assertion (4) also holds.

Moreover, if the assertion (4) holds, then from Theorem 2.6, it is clear that $\mathfrak{F}$ is $\Box_{\mathfrak{F}}$–increasing. Therefore, the assertion (5) also holds.

Finally, if the assertion (5) holds, then by Theorem 2.3 the assertion (1) also holds. ■

Now, to guarantee the regularity of increasing set-valued functions for relators, we may naturally have the next definition.

Definition 2.9. An increasing set-valued function $\mathfrak{F}$ for relators on X is called normal if, for every relator $\mathcal{R}$ on X, we have

$$\mathfrak{F}_{\mathcal{R}} = \bigcup_{R\in\mathcal{R}} \mathfrak{F}_{\{R\}}.$$

Remark 2.10. Analogously, a decreasing set-valued function $\mathfrak{F}$ for relators on X is called normal, if $\mathcal{R} = \bigcap_{R\in\mathcal{R}} \mathfrak{F}_{\{R\}}$ for every relator $\mathcal{R}$ on X.

Note that if $\mathfrak{F}$ is an increasing set-valued function for relators on X, then $\bigcup_{R\in\mathcal{R}} \mathfrak{F}_{\{R\}} \subset \mathfrak{F}_{\mathcal{R}}$ for every relator $\mathcal{R}$ on X. Therefore, an increasing set-valued function $\mathfrak{F}$ for relators on X is normal if the converse inclusion holds.

Now, as a useful consequence of Theorem 2.8, we can also state the next theorem.

Theorem 2.11. *If $\mathfrak{F}$ is a normal increasing set-valued function for relators on X, then $\mathfrak{F}$ is, in particular, regular.*

Proof. In this case, by Definitions 2.9 and 2.1, we evidently have

$$\mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}} = \bigcup_{S\in\mathcal{R}^{\Box_{\mathfrak{F}}}} \mathfrak{F}_{\{S\}} \subset \mathfrak{F}_{\mathcal{R}}$$

for every relator $\mathcal{R}$ on X. Therefore, by Theorem 2.8, $\mathfrak{F}$ is regular. ■

Moreover, concerning normal set-valued functions, we can also easily prove the following.

Theorem 2.12. *If $\mathfrak{F}$ is a normal increasing set-valued function for relators on X and $(\mathcal{R}_i)_{i\in I}$ is a nonvoid family of relators on X, then*

$$\mathfrak{F}_{\bigcup_{i\in I} \mathcal{R}_i} = \bigcup_{i\in I} \mathfrak{F}_{\mathcal{R}_i}.$$

The importance of the regular set-valued functions lies mainly in the following consequence of Theorems 1.9 and 1.12 and Corollary 1.13.

Theorem 2.13. *If $\mathfrak{F}$ is a regular increasing set-valued function for relators on X, then*

(1) $\Box_{\mathfrak{F}}$ is a refinement (self-increasing) operation for relators on X;

(2) for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\Box_{\mathfrak{F}}}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box}_{\mathfrak{F}}}$ $\left(\mathfrak{F}_{\mathcal{R}^{\Box}_{\mathfrak{F}}} \subset \mathfrak{F}_{\mathcal{R}} \right)$.

R e m a r k 2.14. Note that because of Remark 1.14 the corresponding duals of the above results are also true.

Moreover, as an immediate consequence of Theorems 1.9 and 2.3, we can also state the following.

Theorem 2.15. *If $\Diamond$ is a refinement operation for relators on X, then $\Diamond = \Box_{\Diamond}$. And thus, in particular, $\Diamond$ is regular.*

Somewhat more generally, we can also easily prove the following theorem.

Theorem 2.16. *If $\Diamond$ is an increasing operation for relators on X, then*

(1) $\mathcal{R}^{\Box_{\Diamond}} \subset \mathcal{R}^{\Diamond}$ for every relator $\mathcal{R}$ on X whenever $\Diamond$ is expansive;

(2) $\mathcal{R}^{\Diamond} \subset \mathcal{R}^{\Box_{\Diamond}}$ for every relator $\mathcal{R}$ on X whenever $\Diamond$ is idempotent.

P r o o f. If $S \in \mathcal{R}^{\Box_{\Diamond}}$ and $\Diamond$ is expansive, then by the corresponding definitions we have $S \in \{S\}^{\Diamond} \subset \mathcal{R}^{\Diamond}$. Therefore, the assertion (1) holds.

While, $\Diamond$ is idempotent, then in particular we have $\left(\mathcal{R}^{\Diamond}\right)^{\Diamond} \subset \mathcal{R}^{\Diamond}$ for every relator $\mathcal{R}$ on X. And hence, by Theorem 2.6, the assertion (2) follows immediately. ■

R e m a r k 2.17. Note that, by Theorem 2.6, the converse of the assertion (1) is also true.

Moreover, by Theorem 2.8, $\Diamond$ is regular if and only if $\mathcal{R}^{\Diamond} = \left(\mathcal{R}^{\Box_{\Diamond}}\right)^{\Diamond}$ $\left(\left(\mathcal{R}^{\Box_{\Diamond}}\right)^{\Diamond} \subset \mathcal{R}^{\Diamond} \right)$.

To briefly express some further important properties of the refinement operations for relators, we must also have the following definition.

Definition 2.18. If $\Diamond$ and $\Box$ are unary operations for relators on X, then we say that:

(1) $\Box$ is $\Diamond$–dominating if $\mathcal{R}^{\Diamond} \subset \mathcal{R}^{\Box}$ for every relator $\mathcal{R}$ on X;

(2) $\Box$ is $\Diamond$–invariant if $\mathcal{R}^{\Box} = \mathcal{R}^{\Box\Diamond}$ for every relator $\mathcal{R}$ on X;

(3) $\Box$ is $\Diamond$–absorbing if $\mathcal{R}^{\Box} = \mathcal{R}^{\Diamond\Box}$ for every relator $\mathcal{R}$ on X;

(4) $\Box$ is $\Diamond$–compatible if $\mathcal{R}^{\Box\Diamond} = \mathcal{R}^{\Diamond\Box}$ for every relator $\mathcal{R}$ on X.

Remark 2.19 In particular, the operation $\Box$ will be called inversion compatible if $\left(\mathcal{R}^{\Box}\right)^{-1} = \left(\mathcal{R}^{-1}\right)^{\Box}$.

Now, as some useful consequences of the corresponding definitions, we can also prove the following theorems.

Theorem 2.20. *If $\Diamond$ is an expansive and $\Box$ is a $\Diamond$–dominating idempotent operation for relators on X, then $\Box$ is $\Diamond$–invariant.*

Proof. For every relator $\mathcal{R}$ on X, we have $\mathcal{R}^{\Box} \subset \mathcal{R}^{\Box\Diamond} \subset \mathcal{R}^{\Box\Box} = \mathcal{R}^{\Box}$, and hence $\mathcal{R}^{\Box} = \mathcal{R}^{\Box\Diamond}$. ■

Theorem 2.21. *If $\Diamond$ is an expansive and $\Box$ is a $\Diamond$–dominating modification operation for relators on X, then $\Box$ is $\Diamond$–absorbing.*

Proof. For any relator $\mathcal{R}$ on X, we have $\mathcal{R}^{\Box} \subset \mathcal{R}^{\Diamond\Box} \subset \mathcal{R}^{\Box\Box} = \mathcal{R}^{\Box}$, and hence $\mathcal{R}^{\Box} = \mathcal{R}^{\Diamond\Box}$. ■

Remark 2.22. Note that if $\Diamond$ is an expansive and $\Box$ is a $\Diamond$–dominating operation for relators on X, then $\Box$ is also expansive.

Moreover, if $\Diamond$ is an arbitrary (increasing) and $\Box$ is an expansive operation for relators on X such that $\mathcal{R}^{\Diamond\Box} \subset \mathcal{R}^{\Box}$ $\left(\mathcal{R}^{\Box\Diamond} \subset \mathcal{R}^{\Box}\right)$ for every relator $\mathcal{R}$ on X, then $\Box$ is $\Diamond$–dominating.

3. The most important set-valued functions for relators

Definition 3.1. If $\mathcal{R}$ is a relator on X, then for any $A, B \subset X$ and $x, y \in X$ we write:

(1) $B \in \operatorname{Int}_{\mathcal{R}}(A)$ $\left(B \in \operatorname{Cl}_{\mathcal{R}}(A)\right)$ if $R[B] \subset A$ $\left(R[B] \cap A \neq \emptyset\right)$ for some (all) $R \in \mathcal{R}$;

(2) $x \in \operatorname{int}_{\mathcal{R}}(A)$ $\left(x \in \operatorname{cl}_{\mathcal{R}}(A)\right)$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(A)$ $\left(\{x\} \in \operatorname{Cl}_{\mathcal{R}}(A)\right)$;

(3) $y \in \sigma_{\mathcal{R}}(x)$ $\left(y \in \rho_{\mathcal{R}}(x)\right)$ if $y \in \operatorname{int}_{\mathcal{R}}(\{x\})$ $\left(y \in \operatorname{cl}_{\mathcal{R}}(\{x\})\right)$.

The relations $\operatorname{Int}_{\mathcal{R}}$ ($\operatorname{Cl}_{\mathcal{R}}$), $\operatorname{int}_{\mathcal{R}}$ ($\operatorname{cl}_{\mathcal{R}}$), and $\sigma_{\mathcal{R}}$ ($\rho_{\mathcal{R}}$) are called the proximal, the topological, and the infinitesimal interiors (closures) induced by $\mathcal{R}$ on X, respectively.

R e m a r k 3.2. If $\mathcal{R}$ is, in particular, a uniformity, then the relations $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$ are just the inverses of the induced proximity $\delta_{\mathcal{R}}$ and strong inclusion $\Subset_{\mathcal{R}}$, respectively. (See [1, p. 12]).

Concerning the above relations, we shall only quote here the following theorems from [16].

Theorem 3.3. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then*

$$\operatorname{Cl}_{\mathcal{R}}(A) = \mathcal{P}(X) \setminus \operatorname{Int}_{\mathcal{R}}(X \setminus A) \qquad \text{and} \qquad \operatorname{cl}_{\mathcal{R}}(A) = X \setminus \operatorname{int}_{\mathcal{R}}(X \setminus A).$$

Theorem 3.4. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then*

$$\operatorname{cl}_{\mathcal{R}}(A) = \bigcap_{R \in \mathcal{R}} R^{-1}[A] \qquad \text{and} \qquad \rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-\infty} = \left(\bigcap \mathcal{R}\right)^{-\infty}.$$

R e m a r k 3.5. The proximal and infinitesimal closures are usually more convenient tools than the topological closures since we have $\operatorname{Cl}_{\mathcal{R}}^{-1} = \operatorname{Cl}_{\mathcal{R}^{-\infty}}$ and $\rho_{\mathcal{R}}^{-1} = \rho_{\mathcal{R}^{-1}}$.

Definition 3.6. If $\mathcal{R}$ is a relator on X, then for any $A \subset X$ we write:

(1) $A \in \tau_{\mathcal{R}}$ ($A \in \tau\!\!\!-_{\mathcal{R}}$) if $A \in \operatorname{Int}_{\mathcal{R}}(A)$ ($X \setminus A \notin \operatorname{Cl}_{\mathcal{R}}(A)$);

(2) $A \in \mathcal{T}_{\mathcal{R}}$ ($A \in \mathcal{F}_{\mathcal{R}}$) if $A \subset \operatorname{int}_{\mathcal{R}}(A)$ ($\operatorname{cl}_{\mathcal{R}}(A) \subset A$);

(3) $A \in \mathcal{E}_{\mathcal{R}}$ ($A \in \mathcal{D}_{\mathcal{R}}$) if $\operatorname{int}_{\mathcal{R}}(A) \neq \emptyset$ ($\operatorname{cl}_{\mathcal{R}}(A) = X$).

The members of the families $\tau_{\mathcal{R}}$ ($\tau\!\!\!-_{\mathcal{R}}$), $\mathcal{T}_{\mathcal{R}}$ ($\mathcal{F}_{\mathcal{R}}$), and $\mathcal{E}_{\mathcal{R}}$ ($\mathcal{D}_{\mathcal{R}}$) are called the proximally open (closed), the topologically open (closed), and the fat (dense) subsets of $X(\mathcal{R})$, respectively.

R e m a r k 3.7. The fat sets are usually more important tools than the open sets. For instance, if $\prec$ is a preorder on X, then $\mathcal{T}_{\{\prec\}}$ and $\mathcal{E}_{\{\prec\}}$ are just the families of all ascending and residual subsets of the preordered set $X(\prec)$, respectively.

Moreover, if for instance $X = \mathbb{R}$ and R is a relation on X such that $R(x) = \,]-\infty, x] \cup \{x+1\}$ for all $x \in X$, then $\mathcal{T}_{\{R\}} = \{\emptyset, X\}$, but $\mathcal{E}_{\{R\}} \neq \{X\}$. Thus, the fat sets may be powerful tools even in a topologically well-chained relator space [2].

Concerning the above families, we shall only quote here the following theorems from [16].

Theorem 3.8. *If $\mathcal{R}$ is a relator on X, then*

$$\tau_{\mathcal{R}} = \{ A \subset X : \quad X \setminus A \in \tau_{\mathcal{R}} \} \qquad \textit{and} \qquad \mathcal{F}_{\mathcal{R}} = \{ A \subset X : \quad X \setminus A \in \mathcal{T}_{\mathcal{R}} \} .$$

Theorem 3.9. *If $\mathcal{R}$ is a relator on X, then*

$$\mathcal{D}_{\mathcal{R}} = \{ A \subset X \quad X \setminus A \notin \mathcal{E}_{\mathcal{R}} \} = \{ A \subset X : \quad \forall\ B \in \mathcal{E}_{\mathcal{R}} : \quad A \cap B \neq \emptyset \} .$$

Remark 3.10. The proximally open sets are usually more convenient tools than the topologically open sets and the fat sets since we also have $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{-\infty}}$.

Definition 3.11. If $\mathcal{R}$ is a relator on X, then we write

$$E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}} \qquad \text{and} \qquad D_{\mathcal{R}} = \bigcup \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}} \big).$$

Remark 3.12. Unfortunately, the above sets have not been introduced and investigated by Á. Száz.

Therefore, to reveal the relationship between the relation $\rho_{\mathcal{R}}$ and the set $E_{\mathcal{R}}$, we must prove the following remark.

Theorem 3.13. *If $\mathcal{R}$ is a relator on X, then*

$$E_{\mathcal{R}} = \bigcap_{x \in X} \rho_{\mathcal{R}}^{-1}(x) \qquad \textit{and} \qquad D_{\mathcal{R}} = X \setminus E_{\mathcal{R}} .$$

Proof. By the corresponding definitions and Theorem 3.4, we have

$$E_{\mathcal{R}} = \bigcap_{A \in \mathcal{E}_{\mathcal{R}}} A = \bigcap_{x \in X} \bigcap_{R \in \mathcal{R}} R(x) = \bigcap_{x \in X} \Big(\bigcap_{R \in \mathcal{R}} R \Big)(x) = \bigcap_{x \in X} \rho_{\mathcal{R}}^{-1}(x) .$$

Moreover, by the corresponding definitions and Theorem 3.9, we also have

$$D_{\mathcal{R}} = \bigcup_{A \notin \mathcal{D}_{\mathcal{R}}} A = \bigcup_{X \setminus A \in \mathcal{E}_{\mathcal{R}}} A = \bigcup_{B \in \mathcal{E}_{\mathcal{R}}} X \setminus B = X \setminus \bigcap_{B \in \mathcal{E}_{\mathcal{R}}} B = X \setminus E_{\mathcal{R}},$$

where it is tacitly assumed that $A \subset X$. ■

Remark 3.14. Note that if R is a relation and $\mathcal{R}$ is a relator on X, then by Theorems 3.13 and 3.4 we have

$$E_{\{R\}} = \bigcap_{x \in X} \rho_{\{R\}}^{-1}(x) = \bigcap_{x \in X} R(x) \qquad \text{and} \qquad E_{\mathcal{R}} = \bigcap_{x \in X} \rho_{\mathcal{R}}^{-1}(x) = E_{\{\rho_{\mathcal{R}}^{-1}\}}.$$

Definition 3.15. A function x of a preordered set Γ into a set X is called a Γ–net in X. The Γ–net x is said to be fatly (densely) in a subset A of X if $x^{-1}(A)$ is a fat (dense) subset of Γ.

If $\mathcal{R}$ is a relator on X, then for any Γ-nets x and y in X and $a \in X$ we write:

(1) $y \in \mathrm{Lim}_{\mathcal{R}}(x)$ $(y \in \mathrm{Adh}_{\mathcal{R}}(x))$ if the net (y, x) is fatly (densely) in each $R \in \mathcal{R}$;

(2) $a \in \lim_{\mathcal{R}}(x)$ $(a \in \mathrm{adh}_{\mathcal{R}}(x))$ if $a_{\Gamma} \in \mathrm{Lim}_{\mathcal{R}}(x)$ $(a_{\Gamma} \in \mathrm{Adh}_{\mathcal{R}}(x))$, where $a_{\Gamma} = \Gamma \times \{a\}$.

Remark 3.16 Note that the above definitions would actually allow Γ to be an arbitrary relator space.

However, the present generality is sufficient for several purposes since as a slight extension of [13, Theorem 3.1] we have the following theorem.

Theorem 3.17. *If $\mathcal{R}$ is a relator on X, then for any $A, B \subset X$, we have $B \in \mathrm{Cl}_{\mathcal{R}}(A)$ if and only if there exist nets x and y in A and B, respectively, such that $y \in \mathrm{Lim}_{\mathcal{R}}(x)$ $(y \in \mathrm{Adh}_{\mathcal{R}}(x))$.*

Hence, it is clear that in particular we also have the following.

Corollary 3.18. *If $\mathcal{R}$ is a relator on X, then for any $a \in X$ and $A \subset X$, we have $a \in \mathrm{cl}_{\mathcal{R}}(A)$ if and only if there exists a net x in A such that $a \in \lim_{\mathcal{R}}(x)$ $(a \in \mathrm{adh}_{\mathcal{R}}(x))$.*

In this respect, it is also worth mentioning that as a slight extension of [13, Theorem 3.10] we also have the next theorem.

Theorem 3.19. *If $\mathcal{R}$ is a relator on X, then for any Γ-net x in X we have*

$$\lim_{\mathcal{R}}(x) = \bigcap_{A \in \mathcal{D}_{\Gamma}} \mathrm{cl}_{\mathcal{R}}(x[A]) \qquad \text{and} \qquad \mathrm{adh}_{\mathcal{R}}(x) = \bigcap_{A \in \mathcal{E}_{\Gamma}} \mathrm{cl}_{\mathcal{R}}(x[A]).$$

Remark 3.20. In connection with Definition 3.15, it should also be noted that the family of all nets in the set X is not, even in the case of a simpler definition of nets, a well-defined collection.

Therefore, by defining the limit and adherence relations induced by a relator $\mathcal{R}$ on X, we should rather make some cardinality restrictions on domains of nets considered in X than to allow them to be arbitrary relator spaces.

Concerning the basic tools defined up till now, we shall mainly need here the following theorem.

Theorem 3.21. *(1)* Int, int, σ, τ, $\tau\!\!\!-$, $\mathcal{E}$, *and* D *are normal increasing set-valued functions for relators on* X *;*

(2) Lim, Adh, lim, adh, Cl, cl, ρ, $\mathcal{D}$, *and* E *are normal decreasing set-valued functions for relators on* X .

Hint of Proof. To prove the monotonicity and the normality of the set-valued functions E and D, note that for every relator $\mathcal{R}$ on X we have

$$E_{\mathcal{R}} = \bigcap_{A \in \mathcal{E}_{\mathcal{R}}} A = \bigcap_{A \in \bigcup_{R \in \mathcal{R}} \mathcal{E}_R} A = \bigcap_{R \in \mathcal{R}} \bigcap_{A \in \mathcal{E}_{\{R\}}} A = \bigcap_{R \in \mathcal{R}} E_{\{R\}}$$

and

$$D_{\mathcal{R}} = X \setminus E_{\mathcal{R}} = X \setminus \bigcap_{R \in \mathcal{R}} E_{\{R\}} = \bigcup_{R \in \mathcal{R}} (X \setminus E_{\{R\}}) = \bigcup_{R \in \mathcal{R}} D_{\{R\}}.$$

■

Remark 3.22. Later we shall see that the increasing set-valued functions $\mathcal{T}$ and $\mathcal{F}$ are not even regular in general.

Therefore, if $\mathcal{R}$ is a relator on X, then in general there does not exist a largest relator $\mathcal{R}^{\square}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}}$ $(\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\square}})$.

This reveals a further serious disadvantage of the topologically open sets in comparison to the proximally open sets and the fat sets.

4. The most important refinement operations for relators

Definition 4.1. If $\mathcal{R}$ is a relator on X, then the relators

$$\begin{aligned}
\mathcal{R}^{*} &= \{ S \subset X^2 : \quad \exists\ R \in \mathcal{R} : \quad \mathcal{R} \subset \mathcal{S} \}, \\
\mathcal{R}^{\#} &= \{ S \subset X^2 : \quad \forall\ A \subset X : \quad A \in \operatorname*{Int}_{\mathcal{R}}(S(A)) \}, \\
\mathcal{R}^{\wedge} &= \{ S \subset X^2 : \quad \forall\ x \in X : \quad x \in \operatorname*{int}_{\mathcal{R}}(S(x)) \}, \\
\mathcal{R}^{\Delta} &= \{ S \subset X^2 : \quad \forall\ x \in X : \quad S(x) \in \mathcal{E}_{\mathcal{R}} \}
\end{aligned}$$

are called the uniform, the proximal, the topological, and the paratopological refinements of $\mathcal{R}$, respectively. Moreover, the relators

$$\mathcal{R}^{\bullet} = \{ \rho_{\mathcal{R}}^{-1} \}^{*} \qquad \text{and} \qquad \mathcal{R}^{\blacktriangle} = \{ X \times E_{\mathcal{R}} \}^{*}$$

are called the infinitesimal and the parainfinitesimal refinements of $\mathcal{R}$, respectively.

Remark 4.2. Our notations of the paratopological and the infinitesimal refinements differ here from those of Á. Száz [17] in order that, analogously to $\mathcal{R}^{\vee} = (\mathcal{R}^{\wedge})^{-1}$, we could also write $\mathcal{R}^{\nabla} = (\mathcal{R}^{\Delta})^{-1}$.

The appropriateness of the above definitions is already apparent from the following theorem.

Theorem 4.3. *(1)* Lim *and* Adh *are* $*$*-decreasing set-valued functions for relators on* X.

(2) Int *is a* $\#$*-increasing and* Cl *is a* $\#$*-decreasing set-valued function for relators on* X.

(3) lim *and* adh *are* $\wedge$*-decreasing set-valued functions for relators on* X.

(4) int *is a* $\wedge$*-increasing and* cl *is a* $\wedge$*-decreasing set-valued function for relators on* X.

(5) $\mathcal{E}$ *is a* Δ*-increasing and* $\mathcal{D}$ *is a* Δ*-decreasing set-valued function for relators on* X.

(6) E *is a* $\blacktriangle$*-decreasing and* D *is a* $\blacktriangle$*-increasing set-valued function for relators on* X.

(7) ρ *is a* $\bullet$*-decreasing set-valued function for relators on* X.

Hint of Proof. Because of the corresponding definitions, we should show that, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have

(a) $\mathcal{S} \subset \mathcal{R}^{*} \iff \mathrm{Lim}_{\mathcal{R}} \subset \mathrm{Lim}_{\mathcal{S}} \iff \mathrm{Adh}_{\mathcal{R}} \subset \mathrm{Adh}_{\mathcal{S}}$;

(b) $\mathcal{S} \subset \mathcal{R}^{\#} \iff \mathrm{Int}_{\mathcal{S}} \subset \mathrm{Int}_{\mathcal{R}} \iff \mathrm{Cl}_{\mathcal{R}} \subset \mathrm{Cl}_{\mathcal{S}}$;

(c) $\mathcal{S} \subset \mathcal{R}^{\wedge} \iff \mathrm{lim}_{\mathcal{R}} \subset \mathrm{lim}_{\mathcal{S}} \iff \mathrm{adh}_{\mathcal{R}} \subset \mathrm{adh}_{\mathcal{S}}$;

(d) $\mathcal{S} \subset \mathcal{R}^{\wedge} \iff \mathrm{int}_{\mathcal{S}} \subset \mathrm{int}_{\mathcal{R}} \iff \mathrm{cl}_{\mathcal{R}} \subset \mathrm{cl}_{\mathcal{S}}$;

(e) $\mathcal{S} \subset \mathcal{R}^{\vartriangle} \iff \mathcal{E}_{\mathcal{S}} \subset \mathcal{E}_{\mathcal{R}} \iff \mathcal{D}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{S}}$;

(f) $\mathcal{S} \subset \mathcal{R}^{\blacktriangle} \iff E_{\mathcal{R}} \subset E_{\mathcal{S}} \iff D_{\mathcal{S}} \subset D_{\mathcal{R}}$;

(g) $\mathcal{S} \subset \mathcal{R}^{\bullet} \iff \rho_{\mathcal{R}} \subset \rho_{\mathcal{S}}$.

However, the assertions (a) through (e) have already been established in [13] and [17]. And the assertions (f) and (g) are quite obvious from the corresponding definitions. Therefore, the proofs may be omitted. ■

From the above theorem, by Theorem 1.9 and Corollary 1.13, and their duals, it is clear that we have the following theorem.

Theorem 4.4. *The operations given in Definition 4.1 are refinement operations for relators on* X *such that, for any relator* $\mathcal{R}$ *on* X,

(1) $\mathcal{R}^{*}$ *is the largest relator on* X *such that* $\mathrm{Lim}_{\mathcal{R}} = \mathrm{Lim}_{\mathcal{R}^{*}}$, *or equivalently* $\mathrm{Adh}_{\mathcal{R}} = \mathrm{Adh}_{\mathcal{R}^{*}}$;

(2) $\mathcal{R}^{\#}$ *is the largest relator on* X *such that* $\mathrm{Int}_{\mathcal{R}} = \mathrm{Int}_{\mathcal{R}^{\#}}$, *or equivalently* $\mathrm{Cl}_{\mathcal{R}} = \mathrm{Cl}_{\mathcal{R}^{\#}}$;

(3) $\mathcal{R}^{\wedge}$ *is the largest relator on* X *such that* $\mathrm{lim}_{\mathcal{R}} = \mathrm{lim}_{\mathcal{R}^{\wedge}}$, *or equivalently* $\mathrm{adh}_{\mathcal{R}} = \mathrm{adh}_{\mathcal{R}^{\wedge}}$;

(4) $\mathcal{R}^{\wedge}$ *is the largest relator on* X *such that* $\mathrm{int}_{\mathcal{R}} = \mathrm{int}_{\mathcal{R}^{\wedge}}$, *or equivalently* $\mathrm{cl}_{\mathcal{R}} = \mathrm{cl}_{\mathcal{R}^{\wedge}}$;

(5) $\mathcal{R}^{\vartriangle}$ *is the largest relator on* X *such that* $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^{\vartriangle}}$, *or equivalently* $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^{\vartriangle}}$;

(6) $\mathcal{R}^{\blacktriangle}$ *is the largest relator on* X *such that* $E_{\mathcal{R}} = E_{\mathcal{R}^{\blacktriangle}}$, *or equivalently* $D_{\mathcal{R}} = D_{\mathcal{R}^{\blacktriangle}}$.

(7) $\mathcal{R}^{\bullet}$ *is the largest relator on* X *such that* $\rho_{\mathcal{R}} = \rho_{\mathcal{R}^{\bullet}}$.

Simple applications of the corresponding definitions and Remark 3.14 give the following theorem.

Theorem 4.5. *If* R *is a relation on* X, *then*

(1) $\{R\}^* = \{R\}^\square$ *whenever* $\square \in \{\#, \wedge, \bullet\}$*;*

(2) $\{R\}^\Delta = \left(R \circ X^X\right)^*$*;* *(3)* $\{R\}^\blacktriangle = \left\{X \times \bigcap_{x \in X} R(x)\right\}^*$.

Hint of Proof. Note that if $S \in \{R\}^\Delta$, then for each $x \in X$ we have $S(x) \in \mathcal{E}_{\{R\}}$. Therefore, there exists $f_x \in X$ such that $R(f_x) \subset S(x)$. Hence, we can see that $R \circ f \subset S$ for some $f \in X^X$, and thus $S \in (R \circ X^X)^*$. Therefore, $\{R\}^\Delta \subset (R \circ X^X)^*$. The converse inclusion can be proved quite similarly. ■

From the above theorem, it is clear that in particular we have:

Corollary 4.6 *The operations given in Definition 4.1 are stable.*

Therefore, we may also naturally introduce the following

Definition 4.7. If $\mathcal{R}$ is a relator on X, then the relator $\mathcal{R}^\blacklozenge$, given by

$$\mathcal{R}^\blacklozenge = \{X^2\} \quad \text{if} \quad \mathcal{R} = \{\mathcal{X}^\in\} \qquad \text{and} \qquad \mathcal{R}^\blacklozenge = \mathcal{P}(\mathcal{X}^\in) \quad \text{if} \quad \mathcal{R} \neq \{\mathcal{X}^\in\},$$

is called the ultimate stable refinement of $\mathcal{R}$.

Namely, as a close analogue of Theorem 4.4, we have the following

Theorem 4.8. $\blacklozenge$ *is a stable refinement operation for relators on* X *such that, for every relator* $\mathcal{R}$ *on* X, $\mathcal{R}^\blacklozenge$ *is the largest relator on* X *such that* $\mathcal{R} = \mathcal{R}^\blacklozenge$ *whenever* $\mathcal{R} = \{\mathcal{X}^\in\}$.

Hint of Proof. To see the required maximality property of the relator $\mathcal{R}^\blacklozenge$, note that if $\mathcal{S}$ is a relator on X such that $\mathcal{S} \subset \mathcal{R}$ whenever $\mathcal{R} = \{\mathcal{X}^\in\}$, then by Definition 4.7, we have $\mathcal{S} \subset \mathcal{R}^\blacklozenge$. Namely, $\mathcal{R}^\blacklozenge = \mathcal{P}(X^2)$ whenever $\mathcal{R} \neq \{\mathcal{X}^\in\}$. ■

By using the corresponding definitions and Theorem 3.4, we can also easily establish the following

Theorem 4.9. *If* $\mathcal{R}$ *is a relator on* X, *then*

$$\mathcal{R} \subset \mathcal{R}^* \subset \mathcal{R}^\# \subset \mathcal{R}^\wedge \subset \mathcal{R}^\Delta \cap \mathcal{R}^\bullet \qquad \textit{and} \qquad \mathcal{R}^\Delta \cup \mathcal{R}^\bullet \subset \mathcal{R}^\blacktriangle \subset \mathcal{R}^\blacklozenge.$$

Remark 4.10. Later, we shall see that the relators $\mathcal{R}^\Delta$ and $\mathcal{R}^\bullet$ are, in general, incomparable.

Therefore, by Theorems 2.20 and 2.21, we can only state the following

Theorem 4.11. *If* $\Diamond, \Box \in \{*, \#, \wedge, \circ, \blacktriangle, \blacklozenge\}$, *where* $\circ = \Delta$ *or* $\bullet$, *such that* $\Diamond$ *precedes* $\Box$ *in the above list, then* $\Box$ *is both* $\Diamond$*–invariant and* $\Diamond$*–absorbing.*

Remark 4.12. From the equality $\mathcal{R}^{\bullet} = \{\rho_{\mathcal{R}}^{-1}\}^{*}$, by using the above theorem, we can infer that $\mathcal{R}^{\blacktriangle} = \{\rho_{\mathcal{R}}^{-1}\}^{\blacktriangle}$ and $\mathcal{R}^{\blacklozenge} = \{\rho_{\mathcal{R}}^{-1}\}^{\blacklozenge}$.

In addition to Theorem 4.11, it is also worth proving the following

Theorem 4.13. *The operations* $*$, $\#$ *and* $\bullet$ *are inversion compatible. While, the operation* $\blacklozenge$ *is both inversion invariant and inversion absorbing.*

Hint of Proof. Everything stated here is quite obvious, except that the operation $\#$ is inversion compatible. The latter fact was first established in [13] and [17]. ■

Remark 4.14. The above theorem shows in particular that the operation $\blacklozenge$ is also inversion compatible, and we have $\mathcal{R}^{-1} \subset \mathcal{R}^{\blacklozenge}$ for every relator $\mathcal{R}$ on X.

Finally, it is also worth mentioning that we also have the following

Theorem 4.15 *The operations* $*$ *and* $\blacklozenge$ *are normal.*

Remark 4.16. It can be shown that the operations $\wedge$, Δ, and $\blacktriangle$ are not, in general, inversion compatible. Moreover, the operations $\#$, $\wedge$, Δ, $\bullet$, and $\blacktriangle$ are not, in general, normal.

5. Some further properties of the basic refinement operations

As a useful consequence of [17, Theorem 3.10], we have the following

Theorem 5.1. *If* $\mathcal{R}$ *is a relator on* X, *then*

$$\tau_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}} \qquad \text{and} \qquad \tau_{\mathcal{R}^{\wedge}} = \mathcal{F}_{\mathcal{R}}.$$

Moreover, by using the operation Δ, we can easily prove the following addition to Theorem 3.13.

Theorem 5.2. *If $\mathcal{R}$ is a relator on X and $x \in X$, then*

$$\mathcal{E}_{\mathcal{R}} = \mathcal{R}^{\Delta}(x) \qquad \text{and} \qquad \rho_{\mathcal{R}^{\Delta}} = E_{\mathcal{R}} \times X .$$

Proof. If $S \in \mathcal{R}^{\Delta}$, then by the definition of $\mathcal{R}^{\Delta}$ we have $S(x) \in \mathcal{E}_{\mathcal{R}}$. Therefore, $\mathcal{R}^{\Delta}(x) \subset \mathcal{E}_{\mathcal{R}}$.

While, if $A \in \mathcal{E}_{\mathcal{R}}$, then by defining $S = X \times A$, we can at once see that $S \in \mathcal{R}^{\Delta}$ such that $A = S(x)$. Therefore, $\mathcal{E}_{\mathcal{R}} \subset \mathcal{R}^{\Delta}(x)$, and thus the corresponding equality is also true.

Now, by Theorem 3.4, it is clear that we also have

$$\rho_{\mathcal{R}^{\Delta}}^{-1}(x) = \left(\bigcap \mathcal{R}^{\Delta}\right)(x) = \bigcap \mathcal{R}^{\Delta}(x) = \bigcap \mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}} .$$

Therefore, $\rho_{\mathcal{R}^{\Delta}}^{-1} = X \times E_{\mathcal{R}}$, and thus, the required equality is also true. ∎

The following theorem shows that operations $\bullet$ and $\blacktriangle$ are of no primary importance for us since they can be expressed in terms of the operations $\wedge$, Δ, and -1.

Theorem 5.3. *If $\mathcal{R}$ is a relator on X, then*

$$\mathcal{R}^{\bullet} = \mathcal{R}^{\vee\vee} \qquad \text{and} \qquad \mathcal{R}^{\blacktriangle} = \mathcal{R}^{\Delta\bullet}$$

Proof. For each $x \in X$, let V_x be a relation on X such that

$$V_x(y) = X \quad \text{if} \quad y \in \rho_{\mathcal{R}}(x) \qquad \text{and} \qquad V_x(y) = X \setminus \{x\} \quad \text{if} \quad y \in X \setminus \rho_{\mathcal{R}}(x) .$$

Then, by the corresponding definitions, it is clear that $V_x \in \mathcal{R}^{\wedge}$. Namely, if $y \in \rho_{\mathcal{R}}(x)$, then $R(y) \subset X = V_x(y)$ for any $R \in \mathcal{R}$. While, if $y \in X$ such that $y \notin \rho_{\mathcal{R}}(x)$, then since $\rho_{\mathcal{R}}(x) = \mathrm{cl}_{\mathcal{R}}(\{x\})$ there exists an $R \in \mathcal{R}$ such that $R(y) \cap \{x\} = \emptyset$, and hence $R(y) \subset X \setminus \{x\} = V_x(y)$.

Now, since $V_x \in \mathcal{R}^{\wedge}$, it is clear that $V_x^{-1} \in \mathcal{R}^{\wedge -1} = \mathcal{R}^{\vee}$. Moreover, we can easily see that

$$V_x^{-1}(x) = \rho_{\mathcal{R}}(x) .$$

Namely, for any $y \in X$, we have $y \in V_x^{-1}(x)$, i.e., $x \in V_x(y)$ if and only if $y \in \rho_{\mathcal{R}}(x)$. Now, since $V_x^{-1} \in \mathcal{R}^{\vee}$ and $V_x^{-1}(x) \subset \rho_{\mathcal{R}}(x)$ for all $x \in X$, it is clear that

$$\rho_{\mathcal{R}} \in \mathcal{R}^{\vee\wedge},$$

and hence $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}^{\vee\vee}$. Hence, by using the corresponding properties of the operations $*$, -1 and $\wedge$, we can infer that

$$\mathcal{R}^{\bullet} = \{\rho_{\mathcal{R}}^{-1}\}^{*} \subset \mathcal{R}^{\vee\vee *} = \mathcal{R}^{\vee\vee}.$$

Moreover, by Theorem 3.4, it is clear that $\mathcal{R} \subset \{\rho_{\mathcal{R}}^{-\infty}\}^{*}$. Hence, by using the corresponding properties of the operations $\wedge$, -1 and $*$, we can infer that

$$\mathcal{R}^{\vee\vee} \subset \{\rho_{\mathcal{R}}^{-1}\}^{*\vee\vee} = \{\rho_{\mathcal{R}}^{-1}\}^{*} = \mathcal{R}^{\bullet}.$$

Therefore, the corresponding equality is also true.

Hence, by Theorem 5.2, it is clear that we also have

$$\mathcal{R}^{\blacktriangle} = \{X \times E_{\mathcal{R}}\}^{*} = \{\rho_{\mathcal{R}^{\Delta}}^{-1}\}^{*} = \mathcal{R}^{\Delta\bullet}.$$

■

Remark 5.4. The first statement of the above theorem was already proved by J. Mala [3, Theorem 1] and Á. Száz [17, Theorem 3.14]. However, our present proof is more direct than the ones given by the above mentioned authors.

Now, in view of Definition 4.1 and second statement of Theorem 5.3, it seems also convenient to introduce the following

Definition 5.5. If $\mathcal{R}$ is a relator on X, then the relator

$$\mathcal{R}^{\star} = \{\rho_{\mathcal{R}}^{-1}\}^{\Delta}$$

is called the parainfinitesimal extension of $\mathcal{R}$.

Remark 5.6. Hence, by Theorem 4.5, it is clear that $\mathcal{R}^{\star} = \left(\rho_{\mathcal{R}}^{-1} \circ X^{X}\right)^{*}$ for every relator $\mathcal{R}$ on X.

Moreover, by using the above definition, we can also easily prove the following addition to Theorems 5.3 and 4.9.

Theorem 5.7. *If $\mathcal{R}$ is a relator on X, then*

$$\mathcal{R}^{\star} = \mathcal{R}^{\bullet\Delta} \qquad \textit{and} \qquad \mathcal{R}^{\Delta} \cup \mathcal{R}^{\bullet} \subset \mathcal{R}^{\star} \subset \mathcal{R}^{\blacktriangle}.$$

Proof. By the corresponding definitions and Theorem 4.11, we evidently have

$$\mathcal{R}^{\star} = \{\rho_{\mathcal{R}}^{-1}\}^{\Delta} = \{\rho_{\mathcal{R}}^{-1}\}^{*\Delta} = \mathcal{R}^{\bullet\Delta}.$$

Hence, by the expansivity and the increasingness properties of the operations $\bullet$ and Δ, it is clear that $\mathcal{R}^{\Delta} \subset \mathcal{R}^{\bullet\Delta} = \mathcal{R}^{\star}$ and $\mathcal{R}^{\bullet} \subset \mathcal{R}^{\bullet\Delta} = \mathcal{R}^{\star}$. Moreover, by using Theorems 4.9 and 4.11, we can easily see that $\mathcal{R}^{\star} = \mathcal{R}^{\bullet\Delta} \subset \mathcal{R}^{\bullet\blacktriangle} = \mathcal{R}^{\blacktriangle}$. ■

From the first statement of Theorem 5.7, by the corresponding properties of the operation $\bullet$ and Δ, it is clear that we also have

Corollary 5.8. $\star$ *is a stable extension operation for relators on* X.

Remark 5.9. Later, we shall see that the operation $\star$ is not, in general, idempotent.

Therefore, the assertions (3) of Theorems 1.5 and 1.9 cannot, in general, hold for the operation $\square = \star$.

As a close analogue of the first statement of Theorem 5.3, we can also prove the following

Theorem 5.10. *If* $\mathcal{R}$ *is a relator on* X, *then*

$$\mathcal{R}^{\blacklozenge} = \mathcal{R}^{\nabla\Delta} = \mathcal{R}^{\nabla\nabla} \qquad \text{and} \qquad \mathcal{R}^{\blacklozenge} = \mathcal{R}^{\blacktriangledown\blacktriangle} = \mathcal{R}^{\blacktriangledown\blacktriangledown}.$$

Proof. If in particular $\mathcal{R} = \{\mathcal{X}^{\in}\}$, then by Corollary 4.6 we have $\mathcal{R}^{\Delta} = \mathcal{R}$. Hence, it is clear that $\mathcal{R}^{\nabla} = \mathcal{R}^{\Delta-1} = \mathcal{R}^{-1} = \mathcal{R}$ is also true. Therefore, we also have $\mathcal{R}^{\nabla\Delta} = \mathcal{R}^{\Delta} = \mathcal{R} = \mathcal{R}^{\blacklozenge}$.

While, if $\mathcal{R} \neq \{\mathcal{X}^{\in}\}$, then there exist $R \in \mathcal{R}$ and $x, y \in X$ such that $y \notin R(x)$. Hence, it follows that $R(x) \subset X \setminus \{y\}$, and thus $X \setminus \{y\} \in \mathcal{E}_{\mathcal{R}}$. Therefore, under the notation $S = (X \setminus \{y\}) \times X$, we have

$$S^{-1} = X \times (X \setminus \{y\}) \in \mathcal{R}^{\Delta},$$

and hence $S \in \mathcal{R}^{\Delta-1} = \mathcal{R}^{\nabla}$. Hence, since $S(y) = \emptyset$, it is clear that $\mathcal{E}_{\mathcal{R}^{\nabla}} = \mathcal{P}(X)$. Therefore, $\mathcal{R}^{\nabla\Delta} = \mathcal{P}(X^2) = \mathcal{R}^{\blacklozenge}$.

Thus, we have proved that $\mathcal{R}^{\blacklozenge} = \mathcal{R}^{\nabla\Delta}$. Hence, by Theorem 4.13, it is clear that we also have $\mathcal{R}^{\blacklozenge} = \mathcal{R}^{\blacklozenge-1} = \mathcal{R}^{\nabla\Delta-1} = \mathcal{R}^{\nabla\nabla}$. Moreover, by using Theorem 5.3 and some of the basic properties of corresponding refinement operations, we can easily see that

$$\mathcal{R}^{\blacklozenge} = \mathcal{R}^{\blacktriangle\blacklozenge\bullet} = \mathcal{R}^{\blacktriangle\nabla\Delta\bullet} = \mathcal{R}^{\blacktriangle\Delta-1\Delta\bullet} = \mathcal{R}^{\blacktriangle-1\blacktriangle} = \mathcal{R}^{\blacktriangledown\blacktriangle},$$

and thus $\mathcal{R}^{\blacklozenge} = \mathcal{R}^{\blacklozenge-1} = \mathcal{R}^{\blacktriangledown\blacktriangle-1} = \mathcal{R}^{\blacktriangledown\blacktriangledown}$ is also true. ■

Finally, we note that from Theorem 4.3, by using Theorem 2.3 and its dual, we can also at once get the following

Theorem 5.11. *If $\mathcal{R}$ is a relator on X, then:*

(1) $\mathcal{R}^{*} = \mathcal{R}^{\square_{\mathrm{Lim}}} = \mathcal{R}^{\square_{\mathrm{Adh}}}$;

(2) $\mathcal{R}^{\#} = \mathcal{R}^{\square_{\mathrm{Int}}} = \mathcal{R}^{\square_{\mathrm{Cl}}}$;

(3) $\mathcal{R}^{\wedge} = \mathcal{R}^{\square_{\mathrm{lim}}} = \mathcal{R}^{\square_{\mathrm{adh}}}$;

(4) $\mathcal{R}^{\wedge} = \mathcal{R}^{\square_{\mathrm{int}}} = \mathcal{R}^{\square_{\mathrm{cl}}}$;

(5) $\mathcal{R}^{\Delta} = \mathcal{R}^{\square_{\mathcal{E}}} = \mathcal{R}^{\square_{\mathcal{D}}}$;

(6) $\mathcal{R}^{\blacktriangle} = \mathcal{R}^{\square_{\mathcal{E}}} = \mathcal{R}^{\square_{\mathcal{D}}}$;

(7) $\mathcal{R}^{\bullet} = \mathcal{R}^{\square_{\rho}}$.

Remark 5.12. Note that, because of Definition 2.1 and Theorem 3.21, the operations $\square_{\tau}$, $\square_{\mathcal{T}}$, and $\square_{\sigma}$ must also be investigated.

Namely, for instance, by Theorem 3.21 and 2.13, $\square_{\sigma}$ is a refinement operation for relators on X such that, for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\square_{\sigma}}$ is the largest relator on X such that $\sigma_{\mathcal{R}} = \sigma_{\mathcal{R}^{\square_{\sigma}}}$.

6. Some further useful operations for relators

Definition 6.1. If R is a relation on X, then the relation

$$R^{\infty} = \bigcup_{n=0}^{\infty} R^{n} ,$$

where $R^{n} = R \circ R^{n-1}$ and $R^{0} = \Delta_{X}$, is called the preorder hull of R.

Namely, we have the following well-known theorem.

Theorem 6.2. *If R is a relation on X, then R^{∞} is the smallest preorder relation on X such that $R \subset R^{\infty}$.*

Hence, by taking some analogous definitions as in Sections 1 and 2, we can easily get

Corollary 6.3. *∞ is an inversion compatible refinement operation for relations on X.*

Definition 6.4. If $\mathcal{R}$ is a relator on X, then the relator

$$\mathcal{R}^{\infty} = \{ R^{\infty} : \quad R \in \mathcal{R} \}$$

is called the preorder modification of $\mathcal{R}$.

Simple applications of the corresponding definitions give the following

Theorem 6.5. *∞ is an inversion compatible normal modification operation for relators on X such that, for every relator $\mathcal{R}$ on X, we have*

$$\mathcal{R}^{\infty} \subset \mathcal{R}^{*\infty} \subset \mathcal{R}^{\infty *} \subset \mathcal{R}^{*}.$$

Hence, by using Theorems 2.20 and 2.21, we can easily get

Corollary 6.6. *If $\Box$ is a $*$–dominating refinement operation for relators on X, then $\Box\infty$ is a modification operation for relators on X such that $\mathcal{R}^{\Box\infty} \subset \mathcal{R}^{\Box}$ for every relator $\mathcal{R}$ on X.*

Remark 6.7. If $\mathcal{R}$ is a relator on X and $\Box$ is a $*$–dominating refinement operation for relators on X, then the relator $\mathcal{R}^{\Box\infty}$ is called the $\Box$–modification of $\mathcal{R}$.

The importance of the proximal and the topological modifications $\mathcal{R}^{\#\infty}$ and $\mathcal{R}^{\wedge\infty}$ is apparent from the following two theorems which were already proved in [4] and [7].

Theorem 6.8. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then the following assertions are equivalent:*

(1) $\mathcal{S}^{\infty} \subset \mathcal{R}^{\#}$; *(2)* $\tau_{\mathcal{S}} \subset \tau_{\mathcal{R}}$; *(3)* $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{R}}$;

(4) $\mathcal{S}^{\#\infty} \subset \mathcal{R}^{\#\infty}$; *(5)* $\mathcal{S}^{\#\infty} \subset \mathcal{R}^{\infty\#}$; *(6)* $\mathcal{S}^{\infty\#} \subset \mathcal{R}^{\infty\#}$.

Hence, by using Corollary 6.6, we can easily get

Corollary 6.9. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{\#\infty}$ is the largest preorder relator on X such that* $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\#\infty}}$ $\left(\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\#\infty}} \right)$.

Remark 6.10. From Theorem 6.8, it is also clear that, for any two relations R and S on X, we have $R \subset S^{\infty}$ if and only if $\tau_{\{S\}} \subset \tau_{\{R\}}$ $\left(\mathcal{T}_{\{S\}} \subset \mathcal{T}_{\{R\}} \right)$.

Therefore, by taking an analogous definition as in Remark 1.8, we can also state that τ and $\mathcal{T}$ are ∞–decreasing set-valued functions for relations on X.

Hence, by using a similar argument as in the proofs of Theorem 1.12 and Corollary 1.13, we can easily see that, for each relation R on X, R^{∞} is the largest relation on X such that $\tau_R = \tau_{R^{\infty}}$ $\left(\digamma_R = \digamma_{R^{\infty}} \right)$.

However, it is now more important to note that from Theorem 6.8, by using Theorems 5.1 and 4.11, we can also easily get the following

Theorem 6.11. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then the following assertions are equivalent:*

(1) $\mathcal{S}^{\wedge\infty} \subset \mathcal{R}^{\wedge}$; *(2)* $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{R}}$; *(3)* $\mathcal{F}_{\mathcal{S}} \subset \mathcal{F}_{\mathcal{R}}$;

(4) $\mathcal{S}^{\wedge\infty} \subset \mathcal{R}^{\wedge\infty}$; *(5)* $\mathcal{S}^{\wedge\infty\#} \subset \mathcal{R}^{\wedge\infty\#}$; *(6)* $\mathcal{S}^{\wedge\infty\wedge} \subset \mathcal{R}^{\wedge\infty\wedge}$.

Moreover, analogously to Corollary 6.9, we can also state

Corollary 6.12. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{\wedge\infty}$ is the largest preorder relator on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\wedge\infty}}$ $\left(\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\wedge\infty}} \right)$.*

By using the operations $\#$, $\wedge$, and ∞, we may also naturally introduce the following

Definition 6.13. If $\mathcal{R}$ is a relator on X, then the relators

$$\mathcal{R}^{\sharp} = \left\{ S \subset X^2 : \quad S^{\infty} \in \mathcal{R}^{\#} \right\} \qquad \text{and} \qquad \mathcal{R}^{\curlywedge} = \left\{ S \subset X^2 : \quad S^{\infty} \in \mathcal{R}^{\wedge} \right\}$$

are called the quasi-proximal refinement and the quasi-topological extension of $\mathcal{R}$, respectively.

The appropriateness of the above operations is apparent from the following

Theorem 6.14. *If $\mathcal{R}$ is a relator on X, then*

$$\mathcal{R}^{\sharp} = \mathcal{R}^{\square_{\tau}} = \mathcal{R}^{\square_{\tau}} \qquad \textit{and} \qquad \mathcal{R}^{\curlywedge} = \mathcal{R}^{\square_{\mathcal{T}}} = \mathcal{R}^{\square_{\mathcal{F}}}.$$

P r o o f. By the corresponding definitions and Theorems 6.8 and 6.11, it is clear that, for any $S \subset X^2$, we have

$$S^{\infty} \in \mathcal{R}^{\#} \iff \{ S \}^{\infty} \subset \mathcal{R}^{\#} \iff \tau_{\{S\}} \subset \tau_{\mathcal{R}} \iff \digamma_{\{S\}} \subset \digamma_{\mathcal{R}}$$

and

$$S^{\infty} \in \mathcal{R}^{\wedge} \iff \{ S \}^{\wedge\infty} \subset \mathcal{R}^{\wedge} \iff \mathcal{T}_{\{S\}} \subset \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\{S\}} \subset \mathcal{F}_{\mathcal{R}}.$$

Therefore, by Definitions 6.13 and 2.1, the required assertions are also true. ■

Remark 6.15. A similar application of Theorems 6.8 and 6.11 gives also that $\mathcal{R}^{\sharp} = \mathcal{R}^{\Box\#\infty}$ and $\mathcal{R}^{\curlywedge} = \mathcal{R}^{\Box\wedge\infty}$.

From the above theorem, by Theorems 3.21, 2.13 and 2.6, it is clear that we have the following

Theorem 6.16.

(1) $\curlywedge$ *is an extension operation for relators on* X *such that* $\mathcal{T}_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{R}}$ $(\mathcal{F}_{\mathcal{S}} \subset \mathcal{F}_{\mathcal{R}})$ *implies* $\mathcal{S} \subset \mathcal{R}^{\curlywedge}$ *for any two relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X.

(2) $\sharp$ *is a refinement operation for relators on* X *such that, for every relator* $\mathcal{R}$ *on* X, $\mathcal{R}^{\sharp}$ *is the largest relator on* X *such that* $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{\sharp}}$ $(\digamma_{\mathcal{R}} = \digamma_{\mathcal{R}^{\sharp}})$.

Remark 6.17. Later, we shall see that the operation $\curlywedge$ is not, in general idempotent.

Therefore, the assertions (3) of Theorems 1.5 and 1.9, cannot, in general, hold for the operation $\Box = \curlywedge$.

Theorem 6.18. *If* $\mathcal{R}$ *is a relator on* X, *then*

(1) $\mathcal{R}^{\#} \subset \mathcal{R}^{\sharp}$; *(2)* $\mathcal{R}^{\wedge} \subset \mathcal{R}^{\curlywedge}$;

(3) $\mathcal{R}^{\#\infty} = \mathcal{R}^{\sharp\infty}$; *(4)* $\mathcal{R}^{\wedge\infty} = \mathcal{R}^{\curlywedge\infty}$.

Hint of Proof. If $S \in \mathcal{R}^{\#}$, then by Corollary 6.6, we also have $S^{\infty} \in \mathcal{R}^{\#\infty} \subset \mathcal{R}^{\#}$. Hence, by Definition 6.13, it follows that $S \in \mathcal{R}^{\sharp}$. Therefore, the assertion (1) is true.

From the assertion (1), we can immediately infer that $\mathcal{R}^{\#\infty} \subset \mathcal{R}^{\sharp\infty}$. On the other hand, by Definitions 6.4 and 6.13, we evidently have that $\mathcal{R}^{\sharp\infty} \subset \mathcal{R}^{\#}$. Hence, it is clear that $\mathcal{R}^{\sharp\infty} = \mathcal{R}^{\sharp\infty\infty} \subset \mathcal{R}^{\#\infty}$. Therefore, the assertion (3) is also true. ■

By the corresponding definitions and Theorem 4.5, it is clear that in particular we also have

Theorem 6.19. *If* R *is a relation on* X, *then*

$$\{R\}^{\sharp} = \{R\}^{\curlywedge} = \{S \subset X^2 : \quad R \subset S^{\infty}\}.$$

R e m a r k 6.20. Hence, in particular, it is clear that

$$\{X^2\}^\sharp = \{X^2\}^\curlywedge = \{S \subset X^2 : \quad X^2 = S^\infty\}.$$

Therefore, the operations $\sharp$ and $\curlywedge$ is not, in general, stable. That is, these operations is not, in general, dominated by $\blacklozenge$.

7. Two illustrating examples

E x a m p l e 7.1. If $X = \{1, 2, 3\}$ and $R_1, R_2 \subset X^2$ such that:

$R_1(1) = \{1\}$, $R_1(2) = \{2, 3\}$, $R_1(3) = \{2, 3\}$;

$R_2(1) = \{1, 3\}$, $R_2(2) = \{2\}$, $R_2(3) = \{1, 3\}$;

then $\mathcal{R} = \{\mathcal{R}_\infty, \mathcal{R}_\in\}$ is an equivalence relator on X such that:

(1) $\mathcal{R}^\bullet \not\subset \mathcal{R}^\Delta$; (2) $\mathcal{R}^\Delta \not\subset \mathcal{R}^\bullet$; (3) $(\mathcal{R}^{\bullet\Delta})^{\bullet\Delta} \not\subset \mathcal{R}^{\bullet\Delta}$.

Note that $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = \Delta_X$, and hence $\Delta_X \in \{\rho_{\mathcal{R}}^{-1}\}^* = \mathcal{R}^\bullet$. But, $\{3\} \notin \mathcal{E}_{\mathcal{R}}$, and hence $\Delta_X \notin \mathcal{R}^\Delta$. Moreover, if $S = X \times \{2\}$, then $S \in \mathcal{R}^\Delta$, since $\{2\} \in \mathcal{E}_{\mathcal{R}}$. But, $S \notin \mathcal{R}^\bullet$, since $\mathcal{R}^\bullet = \{\Delta_X\}^*$ and $\Delta_X \not\subset S$. Therefore, the assertions (1) and (2) are true.

On the other hand, $E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}} = \emptyset$. Therefore,

$$(\mathcal{R}^{\bullet\Delta})^{\bullet\Delta} = \mathcal{R}^{\bullet\blacktriangle\Delta} = \mathcal{R}^\blacktriangle = \{X \times E_{\mathcal{R}}\}^* = \{\emptyset\}^* = \mathcal{P}(X^2).$$

But,

$$\mathcal{R}^{\bullet\Delta} = \{\Delta_X\}^{*\Delta} = \{\Delta_X\}^\Delta = \{\Delta_X \circ X^X\}^* = \{X^X\}^*.$$

Therefore, the assertion (3) is also true.

E x a m p l e 7.2. If $X = \{1, 2, 3\}$ and $\mathcal{R} = \{X^\in\}$, then

(1) $\mathcal{T}_{\mathcal{R}^\curlywedge} \not\subset \mathcal{T}_{\mathcal{R}}$; (2) $\mathcal{R}^{\curlywedge\curlywedge} \not\subset \mathcal{R}^\curlywedge$; (3) $\mathcal{R}^\curlywedge \not\subset \mathcal{R}^\blacklozenge$.

Moreover, there is no largest relator $\mathcal{S}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$. And thus, there is no unary operation $\Box$ for relators on X such that $\mathcal{T}$ is $\Box$–increasing. That is, $\mathcal{T}$ is not a regular set-valued function for relators on X.

To prove the above assertions, for each $i = 1, 2, 3$, define $R_i \subset X^2$ such that:

$$R_1(1) = \{1, 2\}, \qquad R_1(2) = X, \qquad R_1(3) = X;$$

$$R_2(1) = X, \qquad R_2(2) = \{1, 2\}, \qquad R_2(3) = X;$$

$$R_3(1) = \{1, 2\}, \qquad R_3(2) = \{1, 2\}, \qquad R_3(3) = X.$$

Then, by the definitions of open sets, we evidently have

$$\mathcal{T}_{\{R_i\}} = \{\emptyset, X\} = \mathcal{T}_{\mathcal{R}},$$

for each $i = 1, 2$. Hence, by Definition 2.1 and Theorem 6.14, it is clear that

$$R_1, R_2 \in \mathcal{R}^{\Box_{\mathcal{T}}} = \mathcal{R}^{\wedge}.$$

Thus, in particular, the assertion (3) is true.

Moreover, now we can also easily see that

$$\mathcal{T}_{\{R_3\}} = \{\emptyset, \{1, 2\}, X\} = \mathcal{T}_{\{R_1, R_2\}} \subset \mathcal{T}_{\mathcal{R}^{\wedge}}.$$

Therefore, the assertion (1) is true. Moreover, $R_3 \in \mathcal{R}^{\wedge \Box_{\mathcal{T}}} = \mathcal{R}^{\wedge\wedge}$. But, since $\mathcal{T}_{\{R_3\}} \not\subset \mathcal{T}_{\mathcal{R}}$, it is clear that $R_3 \notin \mathcal{R}^{\Box_{\mathcal{T}}} = \mathcal{R}^{\wedge}$. Therefore, the assertion (2) is also true.

Finally, to prove the remaining assertions, assume on the contrary that $\mathcal{S}$ is a largest relator on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$. Then, since $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\{R_i\}}$ for each $i = 1, 2$, we necessarily have $R_1, R_2 \in \mathcal{S}$. Hence, by the increasingness of $\mathcal{T}$, it follows that $\mathcal{T}_{\{R_1, R_2\}} \subset \mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}$, and this is a contradiction. Hence, by Theorem 1.12, it is clear that there is no unary operation $\Box$ for relators on X such that $\mathcal{T}$ is $\Box$–increasing. That is, by Theorem 2.8, $\mathcal{T}$ is not a regular set-valued function for relators on X.

R e m a r k 7.3. Note that if $\mathcal{R}$ is as in Example 7.2, then by Remark 6.20 we have

$$\mathcal{R}^{\sharp} = \mathcal{R}^{\wedge} = \{S \subset X^2 : \quad X^2 = S^{\infty}\}$$

Therefore, we can also state that $\mathcal{R}^{\sharp\wedge} \not\subset \mathcal{R}^{\sharp}$ and $\mathcal{R}^{\sharp} \not\subset \mathcal{R}^{\blacklozenge}$. Thus, $\sharp$ is also not a stable operation for relators on X.

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