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# Multipliers of the space $S_w(G)$

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In this paper, the space of multipliers from $L^1_w(G)$ to $S_w(G)$ is examined by using the space $S_w(G)$ defined by Cigler in [2]. Also, it is discussed the space of multipliers from $S_w(G)$ onto itself. At the end of this work, it is showed that, in the case $S_w(G)$ is reflexive, the multipliers space from $L^1_w(G)$ to $S_w(G)$ is homeomorphic to $S_w(G)$.

*AMS Subj. Classification*: multipliers, weighted spaces, homeomorphic spaces, locally compact Hausdorff spaces

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## 1. Introduction

Throughout this work, $G$ denotes a locally compact Abelian group with dual group $\hat{G}$ and $d\mu$ denotes a Haar measure on $G$. We denote by $C_c(G)$ the vector spaces of continuous functions on $G$ with compact support. Let $A$ be a Banach algebra. If for all $x \in A$, $x.A = \{0\}$ implies $x = 0$, then $A$ is called without order. Let $(B, \|.\|_B)$ be a Banach space and $(A, \|.\|_A)$ be a Banach algebra. If $B$ is an algebraic $A$-module, and $\|a.b\|_B \leq \|a\|_A \|b\|_B$ for all $a \in A$, $b \in B$, then $B$ is called a Banach $A$-module. If the Banach module $B$ is continuously embedded in $A$ and module operation is given by the multiplication in $A$, we call $B$ a Banach ideal of $A$. The left (right) translation operators $L_y(R_y)$ are given by $L_y f(x) = f(x-y), (R_y f(x) = f(x+y))$ for all $x, y \in G$. The Fourier transform for any $f \in L^1(G)$ is denoted by $\hat{f}$ or $\mathrm{F}f$. It is known that $\left\|\hat{f}\right\|_\infty \leq \|f\|_1$. We will denote the space of pseudo-measures by $A^*(G)$ (see p.97, [9]).

A real valued measurable function $w$ on $G$ is said to be a weighted function (Beurling's weight) if $w(x) \geq 1$ and $w(x+y) \leq w(x).w(y)$ for all $x, y \in G$. We set for $1 \leq p < \infty$,

$$(1) \qquad L_w^p(G) = \{f \mid f\, w \in L^p(G)\}.$$

It is a Banach space under the norm $\|f\|_{p,w} = \|f.w\|_p$.

Particularly, for $p = 1$, $L_w^1(G)$ is Banach algebra under convolution called a Beurling algebra. A weight function $w$ is said to satisfy the Beurling Domar condition (shortly BD), if one has

$$(2) \qquad \sum_{n\geq 1} \frac{\log w(nx)}{n^2} < \infty$$

for all $x \in G$, [13].

Let $X$ be a locally compact Hausdorff space and $A(X)$ be an algebra of complex-valued continuous functions on $X$ with the ordinary pointwise algebraic operations. $A(X)$ is a standard algebra if it has the following properties:

1) If $f \in A(X)$ and $f(a) \neq 0$ at a point $a \in X$, then there is a element $g \in A(X)$ such that $g(x) = \frac{1}{f(x)}$ for all $x$ in some neighbourhood of $a$.

2) For any closed set $E \subset X$ and any point $a \in X - E$ there is an element $f \in A(X)$ vanishing on $E$ and such that $f(E) = 0$.

A normed standard algebra $A(X)$ is said to be a topological standard algebra. Let $A(X)$ be a topological standard algebra. If the functions with compact supports in $A(X)$ are dense in $A(X)$, then $A(X)$ will be called a Wiener algebra [13].

Let $(A, \|.\|_A)$ be a Banach algebra. The proper subalgebra $B$ of $A$ is called an A-Segal algebra if:

1) $B$ is a dense ideal of $A$.

2) $(B, \|.\|_B)$ is a Banach algebra.

3) There exists $M > 0$ such that $\|f\|_A \leq M \|f\|_B$ , $(f \in B)$.

4) There exists $C > 0$ such that $\|f.g\|_B \leq C \|f\|_A \|g\|_B$ for all $f, g \in B$.

If $B_1$ and $B_2$ are Banach $A-$modules, then a multiplier (or module homomorphism) from $B_1$ to $B_2$ is a bounded linear operator $T$ from $B_1$ to $B_2$ which commutes with the module multiplication, i.e. $T(a.b) = aT(b)$ for all $a \in A$ and $b \in B_1$. The space of multipliers from $B_1$ to $B_2$ is denoted by $M(B_1, B_2)$(or $Hom(B_1, B_2)$).

We let

$$M(w) = \left\{ \mu \in M(G) \,\middle|\, \int_G w \, d|\mu| < \infty \right\}, \tag{3}$$

where $M(G)$ is the space of bounded regular Borel measures. It is known that the space of multipliers from $L^1_w(G)$ to $L^1_w(G)$ is $M(w)$,[7].

Let $G$ be a locally compact Abelian group and $F$ be a dense ideal in $L^1_w(G)$. If $(F, \|.\|_F)$ is a Banach space satisfying $\|f\|_{1,w} \leq \|f\|_F$ and $\|f * g\|_F \leq \|f\|_F\|g\|_{1,w}$ for all $f \in L^1_w(G)$ and $g \in F$, then we call $F$ normed ideal in $L^1_w(G)$. We denote by $F_{0,w}$ the set of all $f \in L^1_w(G)$ such that $\hat{f}$ has compact support. If the weight $w$ satisfies (BD), then $F_{0,w}$ is a dense ideal in $L^1_w(G)$, [3].

## 2. Multipliers from $L^1_w(G)$ to $S_w(G)$ and multipliers of the space $S_w(G)$

**Lemma 1.** *If $F$ is a normed ideal in the space $L^1_w(G)$ and $w$ satisfies (BD), then $F_{0,w} \subset F$.*

P r o o f. We denote by $\mathrm{F}(F)$, $\mathrm{F}(L^1_w(G))$ the image of $F$ and $L^1_w(G)$ under the Fourier transforms, respectively. It is easily seen that the functions $\left\|\hat{f}\right\|_{F(F)} = \|f\|_F$ and $\left\|\hat{f}\right\|_{F(L^1_w(G))} = \|f\|_{L^1_w(G)}$ are norms on $\mathrm{F}(F)$ and $\mathrm{F}(L^1_w(G))$, respectively. Then it is also easy to see that $F(F)$ is a dense Banach ideal in the space $F(L^1_w(G))$ by using the definition of the norm on the spaces $F(F)$ and $F(L^1_w(G))$ and the properties of the space $F$. Also it is known that the space $\mathrm{F}(L^1_w(G))$ is a standard algebra if $w$ satisfies the condition (B.D), (see [13]). Now, we will show that $cospF(F) = \emptyset$. For this, let us assume that the set $F$

$$cospF(F) = \left\{ \hat{x} \in \hat{G} \,\middle|\, \hat{f}(\hat{x}) = 0, \forall f \in F \right\}) \tag{4}$$

is nonempty. Then for at least one element $\hat{x} \in \hat{G}$, there exists an $f \in F$ such that $\hat{f}(\hat{x}) \neq 0$. Hence we can choose $\varepsilon > 0$ such that $\left| \hat{f}(\hat{x}) \right| > \varepsilon$. Since $F$ is dense in $L^1_w(G)$, then there exists a sequence $(f_n)_{n\in N} \subset F$ and $n_0 \in N$ such that $\|f_n - f\|_{1,w} < \varepsilon$ for all $n \geq n_0$. Thus, from the inequalities

$$\left\|\hat{f}_n - \hat{f}\right\|_\infty \leq \|f_n - f\|_1 \leq \|f_n - f\|_{1,w} \tag{5}$$

the sequence $\left(\hat{f}_n\right)_{n\in N}$ convergences uniformly to the element $\hat{f}$. Since uniform convergence implies pointwise convergence, then $\left|\hat{f}_n(\hat{x}) - \hat{f}(\hat{x})\right| < \varepsilon$ for all $n \geq$

$n_0$. Thus we obtain

$$\left|\hat{f}(\hat{x})\right| \leq \left|\hat{f}(\hat{x}) - \hat{f}_n(\hat{x})\right| + \left|\hat{f}_n(\hat{x})\right| < \varepsilon + \left|\hat{f}_n(\hat{x})\right| \tag{6}$$

for all $n \geq n_0$. Also, since $\hat{x} \in cospF(F)$ and $f_n \in F$ for all $n$, hence $\left|\hat{f}_n(\hat{x})\right| = 0$. By using (6) we have $\left|\hat{f}(\hat{x})\right| < \varepsilon$. This contradicts with $\left|\hat{f}(\hat{x})\right| > \varepsilon$. Then $cospF(F) = \emptyset$. Hence $F(F)$ contains all functions in $F(L^1_w(G))$ with compact support ([13], p.20). This implies $F_{0,w} \subset F$. ■

**Lemma 2.** *Let $F$ be an essential normed ideal in the space $L^1_w(G)$. If $w$ satisfies (BD), then $\mu * f \in F$ and also there exists a constant $C > 0$ such that*

$$\|\mu * f\|_F \leq C\,\|\mu\|_w\,\|f\|_F.$$

*for all $\mu \in M(w)$ and $f \in F$.*

Proof. For any $g \in F_{0,w}$ and $\mu \in M(w)$, we have $g * \mu \in L^1_w(G)$ and the inequality

$$\|g * \mu\|_{1,w} \leq \|g\|_{1,w}\,\|\mu\|_w \tag{7}$$

is satisfied [7]. Moreover, since $\hat{g}$ has compact support, $\widehat{\mu * g} = \hat{\mu}.\hat{g}$ has compact support and we get $g * \mu \in F_{0,w}$. Since $w$ satisfies (BD), $L^1_w(G)$ has a bounded approximate identity $(e_\alpha)_{\alpha\in J}$ with compactly supported Fourier transforms [5]. Then, there exists a constant $C > 0$ such that $\|e_\beta\|_{1,w} \leq C$ for all $\beta \in I$. Hence $e_\beta * \mu \in L_{1,w}(G)$ and we write

$$\|\mu * e_\beta\|_{1,w} \leq \|e_\beta\|_{1,w}\|\mu\|_w \leq C.\|\mu\|_w \tag{8}$$

for all $\beta \in I$, $\mu \in M(w)$. Thus by Lemma 1 and (8), we have $\mu * g * e_\beta \in F$ and

$$\|\mu * g * e_\beta\|_F \leq \|\mu * e_\beta\|_{1,w}\,\|g\|_F \leq c\,\|\mu\|_w\|g\|_F \tag{9}$$

for all $g \in F$ and for all $\beta \in I$. Here by using the facts that $F$ is an essential ideal and $(e_\beta)_{\beta\in I}$ is a bounded approximate identity of the space $L^1_w(G)$ from Corollary 15.3. in [4], we get

$$\|\mu * g\|_F = \lim_{\beta\in I}\|\mu * g * e_\beta\|_F. \tag{10}$$

Using the inequality

$$\|\mu * g\|_F \leq C.\|g\|_F\|\mu\|_w, \tag{11}$$

we obtain $g * \mu \in F$.

The next step is to show $\mu * f \in F$ for any $f \in F$ and $\mu \in M(w)$. Since $F$ is an essential normed ideal in the space $L^1_w(G)$, then for given any $\varepsilon > 0$ and $f \in F$, there exists $\alpha_1 \in I$ such that $\|f - f * e_\alpha\|_F < \varepsilon$ for all $\alpha \geq \alpha_1$ by Corollary 15.3 in [4]. Moreover, since $f * e_\alpha \in L^1_w(G)$ and $\hat{e}_\alpha$ has compact support, then $f \hat{*} e_\alpha = \hat{f}\,\hat{e}_\alpha$ has compact support and therefore $f * e_\alpha \in F_{0,w}$ for all $\alpha \in I$. We let $g_\alpha = f * e_\alpha$. Then

$$\|g_\alpha - f\|_F < \varepsilon \tag{12}$$

for all $\alpha \geq \alpha_1$. Since the net $(g_\alpha)_{\alpha\in J} \subset F_{0.w}$ converges to $f$ in the space $F$, then $(g_\alpha)_{\alpha\in J}$ is a Cauchy net. Thus for same $\varepsilon > 0$, there exists $\alpha_2 \in J$ such that

$$\|g_\alpha - g_\beta\|_F < {}^{\varepsilon}/_{C\,\|\mu\|_w} \tag{13}$$

for all $\alpha, \beta \geq \alpha_2$. Also since $g_\alpha, g_\beta \in F_{0,w}$ for all $\alpha, \beta \in J$, by using the inequality (11) and (13), we have $\mu * g_\alpha$, $\mu * g_\beta \in F$ and

$$\|\mu * g_\alpha - \mu * g_\beta\|_F = \|\mu * (g_\alpha - g_\beta)\|_F \leq C\,\|\mu\|_w \|g_\alpha - g_\beta\|_F < \varepsilon$$

for all $\alpha, \beta \geq \alpha_2$. Hence $(\mu * g_\alpha)_{\alpha\in J}$ is a Cauchy net in the space $F$ and converges to a function $h \in F$. Then there exists $\alpha_3 \in J$ such that

$$\|\mu * g_\alpha - h\|_F < \varepsilon$$

for all $\alpha \geq \alpha_3$. On the other hand, by the inequality $\|.\|_{1,w} \leq \|.\|_F$, we get

$$\|\mu * g_\alpha - h\|_{1,w} < \varepsilon \tag{14}$$

for all $\alpha \geq \alpha_3$. Also from (7) and (12), we have

$$\|\mu * g_\alpha - \mu * f\|_{1,w} = \|\mu * (g_\alpha - f)\|_{1,w} < \|\mu\|_w . \varepsilon \tag{15}$$

for all $\alpha \geq \alpha_1$. Now, we let $\alpha_0 = mak\{\alpha_1, \alpha_3\}$. For a fixed $\alpha \geq \alpha_0$, we write

$$\|\mu * f - h\|_{1,w} \leq \|\mu * f - \mu * g_\alpha\|_{1,w} + \|\mu * g_\alpha - h\|_{1,w} < \|\mu\|_w \varepsilon + \varepsilon \tag{16}$$

by using (14) and (15). Since the right side of this inequality approaches to 0, then we have $\mu * f = h \in F$. Finally for given $f \in F$, we write

$$\begin{aligned}\|\mu * f\|_F &\leq \|(\mu * f) - (\mu * f) * e_\alpha\|_F + \|(\mu * f) * e_\alpha\|_F \\ &\leq \|(\mu * f) - (\mu * f) * e_\alpha\|_F + C.\|\mu\|_w \cdot \|f\|_F.\end{aligned}$$

Since $F$ is an essential normed ideal, we conclude that

$$\|\mu * f\|_F \leq C\,\|\mu\|_w\,\|f\|_F \tag{17}$$

by Corollary 15.3. in [4]. ■

A kind of generalization of Segal Algebra has been given in [2], as follows:

Let $S_w(G)$ be the subalgebra of the space $L^1_w(G)$ satisfying the following conditions:

1) $S_w(G)$ is dense in the space $L^1_w(G)$.

2) $S_w(G)$ is a Banach algebra under some norm $\|.\|_{S_w}$ and invariant under translations.

3) For each $f \in S_w(G)$, $\|L_y f\|_{S_w} \leq w(y)\ \|f\|_{S_w}$ for all $y \in G$.

4) Given any $f \in S_w(G)$ and $\varepsilon > 0$ then there exists a neighbourhood $U$ of the unit element $e$ of $G$ such that $\|L_y f - f\|_{S_w} < \varepsilon$ for all $y \in U$.

5) For all $f \in S_w(G)$, $\|f\|_{1,w} \leq \|f\|_{S_w}$.

Suppose that $w$ satisfies (BD). Let $S_w(G)$ be a Banach ideal in the space $L^1_w(G)$ and $(e_\alpha)_{\alpha\in I}$ be a bounded approximate identity of the space $L^1_w(G)$ with compactly supported Fourier transforms. Define

$$M_{S_w} = \{\mu \in M(w) \mid \|\mu * e_\alpha\|_{S_w} \leq c_\mu\},$$

where $c_\mu$ is a constant depending on the measure $\mu$. It is easy to show that $M_{S_w}$ is a vector space on the field of complex numbers $C$ and the function

$$\|\mu\|_{M_{S_w}} = \sup_\alpha \left\{ \frac{\|\mu * e_\alpha\|_{S_w}}{\|e_\alpha\|_{1,w}} \right\}$$

is a norm on $M_{S_w}$.

**Proposition 3.** *Let $w$ satisfies (BD) and $S_w(G)$ be an essential ideal. Then the space $M_{S_w}$ is uniquely defined as independent of approximate identity.*

Proof. Let $(u_\alpha)_{\alpha\in I}$ and $(v_\beta)_{\beta\in J}$ be two bounded approximate identities with compactly supported Fourier transformations of the space $L^1_w(G)$. Then, there exist $M_1, M_2 > 0$ such that $\|u_\alpha\|_{1,w} \leq M_1$ and $\|v_\beta\|_{1,w} \leq M_2$ for every $\alpha \in I$ and $\beta \in J$. We define

$$M_{S_w}(G) = \{\mu \in M(w) \mid \|\mu * u_\alpha\|_{S_w} \leq c_\mu, \text{ for all } \alpha \in I\}, \tag{18}$$

and

(19) $$A = \left\{ \mu \in M(w) \,\middle|\, \|\mu * v_\beta\|_{S_w} \le C'_\mu, \text{ for, all } \beta \in J. \right\}.$$

We also define the following norms on these spaces:

(20) $$\|\mu\|_{M_{S_w}} = \sup_\alpha \left\{ \frac{\|\mu * u_\alpha\|_{S_w}}{\|u_\alpha\|_{1,w}} \right\}, \quad \mu \in M_{S_w}$$

and

(21) $$\|\mu\|_A = \sup_\beta \left\{ \frac{\|\mu * v_\beta\|_{S_w}}{\|v_\beta\|_{1,w}} \right\}, \quad \mu \in A.$$

Let any $\mu \in M_{S_w}$ be given. Then $\|\mu * u_\alpha\|_{S_w} \le C_\mu$ for all $\alpha \in I$. Hence, $\mu * u_\alpha \in S_w(G)$ and

(22) $$\|\mu\|_{M_{S_w}} = \sup_\alpha \left\{ \frac{\|\mu * u_\alpha\|_{S_w}}{\|u_\alpha\|_{1,w}} \right\} < \infty.$$

Take a fixed element $\beta_0 \in J$. Since $S_w(G)$ is a Banach ideal in $L^1_w(G)$, then $\mu * u_\alpha * v_{\beta_0} \in S_w(G)$ for all $\alpha \in I$ and we write

(23) $$\|\mu * u_\alpha * v_{\beta_0} - \mu * v_{\beta_0}\|_{S_w} \le \|\mu\|_w \|u_\alpha * v_{\beta_0} - v_{\beta_0}\|_{S_w}$$

by Lemma 2. Also, since $u_\alpha, v_\beta \in F_{0,w}$, we get $u_\alpha, v_\beta \in S_w(G)$ for all $\alpha \in I$ and $\beta \in J$ by Lemma 1. In the inequality (23) using the facts that $v_{\beta_0} \in S_w(G)$ and $S_w(G)$ is an essential ideal, then for given $\varepsilon > 0$ there exists $\alpha_0 \in I$ such that

(24) $$\|\mu * u_\alpha * v_{\beta_0} - \mu * v_{\beta_0}\|_{S_w} \le \|\mu\|_w \|u_\alpha * v_{\beta_0} - v_{\beta_0}\|_{S_w} < \varepsilon \|\mu\|_w$$

for all $\alpha \ge \alpha_0$.Since $S_w(G)$ is a Banach space, we have $\mu * v_{\beta_0} \in S_w(G)$ and moreover, we write

$$\|\mu * v_{\beta_0}\|_{S_w} \le \lim_\alpha \|\mu * u_\alpha * v_{\beta_0}\|_{S_w} \le \lim_\alpha \left( \|\mu * u_\alpha\|_{S_w} \|v_{\beta_0}\|_{1,w} \right) \le M_2 C_\mu = C'_\mu.$$
(25)

Since (25) is satisfied for all $\beta \in J$, hence $\mu * v_\beta \in S_w(G)$ and $\|\mu * v_\beta\|_{S_w} \le C'_\mu$. Thus $\mu \in A$ and we have $M_{S_w} \subset A$. Similarly, it can be shown that $A \subset M_{S_w}$. Hence $A = M_{S_w}$. Also if we use the inequality (23), write

$$\|\mu * v_{\beta_0}\|_{S_w} \le \|\mu * v_{\beta_0} - \mu * u_\alpha * v_{\beta_0}\|_{S_w}$$

$$+ \|\mu * u_\alpha * v_{\beta_0}\|_{S_w} < \varepsilon \|\mu\|_w + \|\mu * u_\alpha\|_{S_w} \|v_{\beta_0}\|_{1,w}$$

for all $\alpha \geq \alpha_0$. That means

$$\|\mu * v_{\beta_0}\|_{S_w} \leq \|\mu * u_\alpha\|_{S_w} \|v_{\beta_0}\|_{1,w} \tag{26}$$

for all $\alpha \geq \alpha_0$. Since this inequality holds for all $\beta \in J$, we have

$$\|\mu * v_\beta\|_{S_w} \leq \|\mu * u_\alpha\|_{S_w} \|v_\beta\|_{1,w} = \|v_\beta\|_{1,w} \frac{\|\mu * u_\alpha\|_{S_w}}{\|u_\alpha\|_{1,w}} \|u_\alpha\|_{1,w}$$

$$\leq M_1 \|v_\beta\|_{1,w} \sup_\alpha \left\{ \frac{\|\mu * u_\alpha\|_{S_w}}{\|u_\alpha\|_{1,w}} \right\}. \tag{27}$$

Hence

$$\frac{\|\mu * v_\beta\|_{S_w}}{\|v_\beta\|_{1,w}} \leq M_1 \|\mu\|_{M_{S_w}}$$

and thus

$$\|\mu\|_A = \sup_\beta \left\{ \frac{\|\mu * v_\beta\|_{S_w}}{\|v_\beta\|_{1,w}} \right\} \leq M_1 \|\mu\|_{M_{S_w}}. \tag{28}$$

Similarly, it is also easy to show the inequality $\|\mu\|_{M_{S_w}} \leq M_2 \|\mu\|_A$. From this inequality and (28) we write

$$\frac{1}{M_1} \|\mu\|_A \leq \|\mu\|_{M_{S_w}} \leq M_2 \|\mu\|_A. \tag{29}$$

Therefore the norms $\|.\|_A$ and $\|.\|_{M_{S_w}}$ are equivalent. This completes the proof. ∎

**Theorem 4.** *Assume that $w$ satisfies (BD) and $S_w(G)$ is an essential ideal in $L^1_w(G)$. The followings are equivalent:*
*1) $T \in M\left(L^1_w(G), S_w(G)\right)$,*
*2) There exists a unique $\mu \in M_{S_w}$ such that $Tf = \mu * f$ for all $f \in L^1_w(G)$.*
*Moreover, the spaces $M\left(L^1_w(G), S_w(G)\right)$ and $M_{S_w}$ are homeomorphic.*

P r o o f. Let us assume the existence of an element $\mu \in M_{S_w}$, such that $Tf = \mu * f$ for every $f \in L^1_w(G)$. Let$(e_\alpha)_{\alpha\in I}$ is a bounded approximate identity with compactly supported Fourier transformation of the space $L^1_w(G)$ and $C = \sup_{\alpha\in I} \|e_\alpha\|_{1,w}$. Since $\mu \in M_{S_w}$, we have $\mu * e_\alpha \in S_w(G)$ for all $\alpha \in I$.Define the net $(g_\alpha)_{\alpha\in I}$, where $g_\alpha = \mu * e_\alpha * e_\alpha * f$. Since $S_w(G)$ is a Banach ideal $L^1_w(G)$, then we have $g_\alpha \in S_w(G)$ for all $\alpha \in I$. Moreover, we write

$$\|g_\alpha - g_\beta\|_{S_w} = \|\mu * e_\alpha * e_\alpha * f - \mu * e_\beta * e_\beta * f\|_{S_w}$$

$$= \|(\mu * e_\alpha + \mu * e_\beta) * (e_\alpha * f - e_\beta * f)\|_{S_w}$$

$$\le \|\mu * e_\alpha + \mu * e_\beta\|_{S_w} \|e_\alpha * f - e_\beta * f\|_{1,w} \le 2\, c_\mu \|e_\alpha * f - e_\beta * f\|_{1,w}$$

$$\le 2\, c_\mu \|e_\alpha * f - f\|_{1,w} + 2\, c_\mu \|e_\beta * f - f\|_{1,w}. \tag{30}$$

Since $(e_\alpha)_{\alpha\in I}$ is a bounded approximate identity in the space $L^1_w(G)$, the right side of the inequality (30) goes to 0. Thus $(g_\alpha)_{\alpha\in I}$ is a Cauchy net in the space $S_w(G)$ and there exists $g \in S_w(G)$ such that

$$\lim_a \|g_a - g\|_{1,w} \le \lim_a \|g_a - g\|_{S_w} = 0. \tag{31}$$

Also using the inequality

$$\|g_\alpha - \mu * f\|_{1,w} \le \|g_\alpha - \mu * e_\alpha * f\|_{1,w} + \|\mu * e_\alpha * f - \mu * f\|_{1,w}$$

$$= \|\mu * e_\alpha * e_\alpha * f - \mu * e_\alpha * f\|_{1,w} + \|\mu * e_\alpha * f - \mu * f\|_{1,w}$$

we obtain

$$\lim_a \|g_a - \mu * f\|_{1,w} = 0. \tag{32}$$

From (31) and (32), it is also obtained that $g = \mu * f = Tf \in S_w(G)$. Moreover, from the inequality

$$\|Tf\|_{S_w} = \|\mu * f\|_{S_w} = \|g\|_{S_w} = \lim_\alpha \|g_\alpha\|_{S_w} = \lim_\alpha \|\mu * e_\alpha * e_\alpha * f\|_{S_w}$$

$$\le \lim_\alpha \left(\|\mu * e_\alpha\|_{S_w} \|e_\alpha * f\|_{1,w}\right) = \lim_\alpha \left(\frac{\|\mu * e_\alpha\|_{S_w}}{\|e_\alpha\|_{1,w}} \|e_\alpha\|_{1,w} \|e_\alpha * f\|_{1,w}\right)$$

$$\le c \sup_\alpha \frac{\|\mu * e_\alpha\|_{S_w}}{\|e_\alpha\|_{1,w}} \lim_\alpha \|e_\alpha * f\|_{1,w} \le C.\|\mu\|_{M_{S_w}} \|f\|_{1,w},$$

we get

$$\|T\| = \sup_{f\neq 0} \frac{\|Tf\|_{S_w}}{\|f\|_{1,w}} \le c\, \|\mu\|_{M_{S_w}}. \tag{33}$$

Thus the operator $T$ is continuous. Since $L_x(f * \mu) = L_x f * \mu$ for every $f \in L^1_w(G)$ and $x \in G$, we have

$$(L_x T)(f) = L_x(Tf) = T(L_x f) = (TL_x)(f). \tag{34}$$

That means $T \in M\left(L^1_w(G), S_w(G)\right)$.

Conversely, any $T \in M\left(L^1_w(G), S_w(G)\right)$. Since the operator $T$ is continuous, there exists $a > 0$ such that $\|Tf\|_{S_w} \le a\|f\|_{1,w}$ for all $f \in L^1_w(G)$. If we use the inequality $\|\cdot\|_{1,w} \le \|\cdot\|_{S_w}$, we write $\|Tf\|_{1,w} \le \|Tf\|_{S_w} \le a\|f\|_{1,w}$.

Thus we have $T \in M\left(L^1_w(G)\right)$. But for given $T \in M\left(L^1_w(G)\right)$ there exists $\mu \in M(w)$ such that $Tf = \mu * f$ for all $f \in L^1_w(G)$ , [7]. Since $e_\alpha \neq 0$ for every $\alpha \in I$, then we write

$$\|\mu\|_{M_{S_w}} = \sup_{\alpha}\left\{\frac{\|\mu * e_\alpha\|_{S_w}}{\|e_\alpha\|_{1,w}}\right\} \tag{35}$$

$$\leq \sup_{f\neq 0}\left\{\frac{\|\mu * f\|_{S_w}}{\|f\|_{1,w}}\right\} = \sup_{f\neq 0}\left\{\frac{\|Tf\|_{S_w}}{\|f\|_{1,w}}\right\} = \|T\|.$$

Also since the operator $T$ is bounded, we obtain $\mu \in M_{S_w}$. Finally from (33) and (35) the spaces $M\left(L^1_w(G), S_w(G)\right)$ and$M_{S_w}$ are homeomorphic. ∎

**Theorem 5.** *Assume that $w$ satisfies (BD). If $S_w(G)$ is an essential normed ideal and $T \in M(S_w(G))$, then there exists a unique pseudo measure $\sigma \in A^*(G)$ such that $Tf = \sigma * f$ for all $f \in S_w(G)$.*

Proof. It is easy to see the space $S_w(G)$ is an abstract Segal algebra on $L^1_w(G)$. Then the regular maximal ideal space of $S_w(G)$ and $L^1_w(G)$ are homeomorphic (see Theorem 2.1, [1]). Since $w$ satisfies (BD), the regular maximal ideal space of $L^1_w(G)$ is homeomorphic to the dual group $\hat{G}$ (see p.15 and Theorem 2.11, [3]). Then the regular maximal ideal space of $S_w(G)$ is dual group $\hat{G}$. Also since $L^1_w(G)$ is a without order algebra under convolution, and $S_w(G)$ is a subalgebra of $L^1_w(G)$, then $S_w(G)$ also is a without order algebra under convolution. Hence $S_w(G)$ is a without order commutative Banach algebra. Let $T \in M(S_w(G))$ be given. Then, for given any $T \in M(S_w(G))$, there exists a unique bounded continuous function $\varphi$ defined on $\hat{G}$ such that $(Tf)^\wedge = \varphi \hat{f}$ for all $f \in S_w(G)$ (see Theorem 1.2.2, [9]). Let $f \in F_{0,w}$ be given. Then we get $f \in S_w(G)$ by Lemma 1. Moreover, since $T \in M(S_w(G))$, we have $Tf \in S_w(G) \subset L^1_w(G)$. Also since $\hat{f}$has compact support and $\varphi$ is continuous function, then $(Tf)^\wedge = \varphi \hat{f}$ has compactly support. Therefore $Tf \in F_{0,w}$. Using the inclusion $C_C\left(\hat{G}\right) \subset L_1\left(\hat{G}\right)$, we obtain

$$F_{0,w} \subset \left\{f \in L_1(G) \,\middle|\, \hat{f} \in L_1\left(\hat{G}\right)\right\} = A_1(G). \tag{36}$$

Also from the Fourier Inversion Theorem, $A_1(G) \subset A(G)$ and hence, $F_{0,w} \subset A(G)$. It is known that $F_{0,w}$ is dense in the space $A(G)$, [8]. Let us define a function $L$ from the space $F_{0,w}$ to the field of complex numbers $C$ as $L(f) = Tf(0)$. It is easy to see that $T$ is linear. Let $f, g \in F_{0,w}$ and $\varepsilon > 0$ be given. If we choose $\delta = \varepsilon / \|\varphi\|_\infty$ and $\|f - g\|_A < \delta$, then we have

$$|L(f) - L(g)| = |Tf(0) - Tg(0)| \leq \|Tf - Tg\|_\infty \leq \left\|\widehat{Tf} - \widehat{Tg}\right\|_1$$

$$(37) = \left\|\varphi\hat{f} - \varphi\hat{g}\right\|_1 \leq \|\varphi\|_\infty \left\|\hat{f} - \hat{g}\right\|_1 = \|\varphi\|_\infty \|f - g\|_A < \|\varphi\|_\infty \frac{\varepsilon}{\|\varphi\|_\infty} = \varepsilon.$$

Thus the function $L$ is continuous. So, the linear functional $L$ defined on $F_{0,w}$ can be extended uniquely to a continuous linear functional on $A(G)$. Hence, there exists a pseudo-measure $\sigma \in A^*(G)$ such that $L(f) = Tf(0) = \langle f, \tilde{\sigma}\rangle$. Thus we find $Tf = \sigma * f$ for all $f \in F_{0,w}$. Now we will show the uniqueness of $\sigma$. Assume that $Tf = \sigma * f = \beta * f$,for $\alpha, \beta \in A^*(G)$. If we take the Fourier transformation of both sides, we have $\hat{\sigma}\hat{f} = \hat{\beta}\hat{f}$ and hence $\left(\hat{\sigma} - \hat{\beta}\right)\hat{f} = 0$. On the other hand, since $w$ satisfies (BD), then $F\left(L^1_w(G)\right) = F_w$ is a Wiener algebra (see [13]). From this, for given any $x \in \hat{G}$, we can find at least one $\hat{f} \in F_w$ such that $\hat{f}(x) \neq 0$.Thus from the equality $\left(\hat{\sigma} - \hat{\beta}\right)(x)\hat{f}(x) = 0$ we write $\left(\hat{\sigma} - \hat{\beta}\right)(x) = \hat{\sigma}(x) - \hat{\beta}(x) = 0$ and hence $\hat{\sigma}(x) = \hat{\beta}(x)$. Since this equality is true for all $x \in \hat{G}$, then we have $\hat{\sigma} = \hat{\beta}$. Finally from the uniqueness of Fourier transformation, it is obtained that $\sigma = \beta$.

Now, we shall show that the $Tf = \sigma * f$ is satisfied for all $f \in S_w(G)$. Let us take any $f \in S_w(G)$. Since $w$ satisfies (BD), then $L^1_w(G)$ has a bounded approximate identity $(e_\alpha), \alpha \in I$ with compactly supported Fourier transformation. Hence $e_\alpha * f \in F_{0,w}$ and $T(e_\alpha * f) = \sigma * (e_\alpha * f)$ for every $f \in S_w(G)$. From this result, we have

$$\|T(e_\alpha * f) - T(e_\beta * f)\|_{S_w} \leq \|T\| \, \|e_\alpha * f - e_\beta * f\|_{S_w}$$

$$\leq \|T\| \left\{\|e_\alpha * f - f\|_{S_w} + \|e_\beta * f - f\|_{S_w}\right\}$$

for all $\alpha, \beta \in I$. Since $S_w(G)$ is an essential ideal, the right side of the above inequality goes to 0 by Corollary 15.3 in [4]. Thus $\{\sigma * (e_\alpha * f)\}$ becomes a Cauchy net in the space $S_w(G)$. Also since the space $S_w(G)$ is a Banach space, there exists a function F $\in S_w(G)$ such that

$$(38) \qquad \lim_\alpha \|\sigma * (e_\alpha * f) - F\|_{1,w} \leq \lim_\alpha \|\sigma * (e_\alpha * f) - F\|_{S_w} = 0.$$

Moreover, if (38) and the fact that $(e_\alpha)_{\alpha \in I}$ is an approximate identity is used in the inequality

$$\|F - \sigma * f\|_{1,w} \leq \|F - \sigma * (e_\alpha * f)\|_{1,w} + \|\sigma * (e_\alpha * f) - \sigma * f\|_{1,w}$$

we obtain $F = \sigma * f$. Also we write

$$(39) \quad \|Tf - \sigma * (e_\alpha * f)\|_{S_w} = \|Tf - T(e_\alpha * f)\|_{S_w} \leq \|T\| \, \|f - e_\alpha * f\|_{S_w}.$$

Consequently, since $S_w(G)$ is an essential ideal from (39), we have

$$\lim_{\alpha} \|Tf - \sigma * (e_\alpha * f)\|_{S_w} = 0. \tag{40}$$

From (39), (40) and the uniqueness of limit, it is concluded that $Tf = F = \sigma * f$.

## 3. Multipliers from $L^1_w(G)$ to reflexive $S_w(G)$

By the techniques of proofs used in Lemma 1 and Lemma 2 in Ouyang [11], we easily proof the following two lemmas.

**Lemma 6.** *Let $S_w(G)$ be a reflexive on the locally compact Abelian group $G$. Suppose $T \in M\left(L^1_w(G), S_w(G)\right)$ and $(u_\alpha)_{\alpha\in I}$ is a bounded approximate identity of $L^1_w(G)$. Then:*

*1) For each $f \in L^1_w(G)$ we have $\lim_{\alpha} Tu_\alpha * f = Tf$ in $\left(S_w(G), \|.\|_{S_w}\right)$.*

*2) There exists a function $m \in S_w(G)$ such that for each $\sigma \in S^*_w(G)$ (the dual space of $S_w(G)$ ) we have*

$$\lim_{\beta} \langle Tu_\beta, \sigma\rangle = \langle m, \sigma\rangle,$$

*where $(Tu_\beta)$ is a subnet of $(Tu_\alpha)$.*

**Lemma 7.** *Let $S_w(G)$ be a reflexive on the locally compact Abelian group $G$. Assume $T \in M\left(L^1_w(G), S_w(G)\right)$ and $(u_\alpha)_{\alpha\in I}$ be a bounded approximate identity of $L^1_w(G)$. Then, for each $f \in C_C(G)$ and each $m \in S_w(G)$,*

$$\varphi : x \to \varphi(x) = \langle L_x Tu_\alpha(.) f(x), \sigma\rangle$$

$$\Psi : x \to \Psi(x) = \langle L_x m(.) f(x), \sigma\rangle, \qquad \sigma \in S^*_w(G)$$

*are the elements of $C_C(G)$.*

**Theorem 8.** *Let $S_w(G)$ be reflexive and $(u_\alpha)_{\alpha\in I}$ be bounded approximate identity in $L^1_w(G)$. Then for every $T \in M\left(L^1_w(G), S_w(G)\right)$, there exists a unique function $m \in S_w(G)$, such that $Tf = m * f$ for all $f \in L^1_w(G)$.*

*Moreover, the correspondence between $T$ and $m$ defines a homeomorpism from $M\left(L^1_w(G), S_w(G)\right)$ onto $S_w(G)$.*

Proof. The first part of the theorem is easily proved using Lemmas 6 and 7, by the technique of the proof used in Proposition 2.2, [11].

Next, we prove that $M\left(L_w^1(G), S_w(G)\right)$ and $S_w(G)$ are homeomorphic. Suppose $T \in M\left(L_w^1(G), S_w(G)\right)$. Since $S_w(G)$ is Banach ideal in $L_w^1(G)$, from the inequality

$$(41) \qquad \|Tf\|_{S_w} = \|m * f\|_{S_w} \leq \|m\|_{S_w} \|f\|_{1,w},$$

we obtain

$$\|T\| \leq \|m\|_{S_w}.$$

Also,

$$(42) \qquad \|m\|_{S_w} \leq \lim_{\beta} \|Tu_\beta\|_{S_w} \leq \lim_{\beta} \|T\| \|u_\beta\|_{1,w} \leq C.\|T\|.$$

Combining these results, we obtain

$$\|T\| \leq \|m\|_{S_w} \leq c\,\|T\|.$$

Thus $M\left(L_w^1(G), S_w(G)\right)$ and $S_w(G)$ are homeomorphic. ∎

**Example 1.** Let $G$ be a locally compact Abelian group (non-discrete, non-compact) and $w$ be a weight function (Beurling's weight function) on $G$. For $1 \leq p < \infty$, we set

$$(43) \qquad A = \left\{ f \in L_w^1(G) \;\middle|\; \hat{f} \in L_w^p\left(\hat{G}\right) \right\} \quad \text{and} \quad \|f\|_A = \|f\|_{1,w} + \left\|\hat{f}\right\|_{p,w}.$$

It is a Banach convolution algebra and $\|f\|_{1,w} \leq \|f\|_A$ for all $f \in A$, [5], [6]. If $w$ satisfies (BD), then $A$ is dense in $L_w^1(G)$, [5]. It is also known that $A$ is translation invariant and the function $y \to L_y f$ is continuous from $G$ into $A$, [5], [6]. Thus if $w$ satisfies (BD), $A$ is a $S_w(G)$ space.

**Example 2.** Let $G$ be a locally compact Abelian group (non-discrete, non-compact) and $w$ be a weight function (Beurling's weight function) on $G$. For $1 \leq p < \infty$, we set $B = L_w^1(G) \cap L_w^p(G)$ and $\|.\|_{L_w^1 \cap L_w^p} = \|.\|_{1,w} + \|.\|_{p,w}$. It is known that $B$ is a Banach convolution algebra, translation invariant and translation operator is continuous. Also, if $w$ satisfies (BD), it is known that $B$ is dense in $L_w^1(G)$ and $\|f\|_{1,w} \leq \|f\|_{L_w^1 \cap L_w^p}$ (Öztop-Gurkanli, [12]). Thus, if $w$ satisfies (BD), $B$ is a $S_w(G)$ space.

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