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Approximation of a Fixed Point of Some Nonself Mappings

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Let X be a Banach space, K a non-empty closed subset of X and let T be a (non-self) mapping of K into X. A fixed point theorem is proved for mappings which satisfy the contractive type condition (1) below, and a fixed point is approximated by an process. The result of this paper is a generalisation of the corresponding theorem of Assad, Rhoades and Ćirić.

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In many applications a function of a closed subset K of a Banach space X is not a selfmapping of K into K but into X, or test for $T(K) \subseteq K$ is complicate. So it is of interest to investigate functions for which $T(K) \subseteq K$ may be reduced to $T(\partial K) \subseteq K$ (∂K - the boundary of K). In this paper we shall prove a fixed point theorem for such mappings. The result proved in this paper is a generalisation of the corresponding theorem of Assad [1], Rhoades [3] and Ćirić [2].

Rhoades [3] investigated a class of mappings T of K into X which satisfy the following condition:

There exists a constant h, $0 < h < 1$, such that, for each $x, y \in K$,

$$(I) \quad d(Tx,Ty) \leq h \max\{d(x,y)/2, d(x,Tx), d(y,Ty), [d(x,Ty)+d(y,Tx)]/q\},$$

where q is any real number satisfying $q \geq 1 + 2h$.

He proved that if K is closed and if T on K satisfies (I) and in addition $T(\partial K) \subseteq K$, then T has a unique fixed point in K. Ćirić [2] slightly improved this result.

We remark that condition (I) implies the following condition:

$$d(Tx,Ty) \leq h_1 \max\{d(x,y)/2, d(x,Tx), d(y,Ty), [d(x,Ty)+d(y,Tx)]/3\},$$

with $h_1 = 3h/(1+2h)$.

The purpose of this paper is to extend the results of Rhoades [3] and Ćirić [2] to a class of non-self mappings of K which satisfy the following contractive type condition:

There exists a constant h, $0 < h < 1$, such that, for each $x, y \in K$,

$$(1) \qquad d(Tx,Ty) \leq h \max\{d(x,y)/2, d(x,Tx), d(y,Ty), m(x,y), M(x,y)/2\},$$

where

$$m(x,y) = \min\{d(x,Ty), d(y,Tx)\},$$

$$M(x,y) = \max\{d(x,Ty), d(y,Tx)\}.$$

In the proof of our theorem we shall use the fact that, if $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$ such that

$$d(x,z) + d(z,y) = d(x,y).$$

Our main result is the following theorem.

Theorem 1. *Let X be a Banach space, K a non-empty closed subset of X and $T : K \to X$ a mapping satisfying (1) on K and*

(a) for each $x \in \partial K$, $Tx \in K$.

Then T has a unique fixed point in K.

Proof. Let $x_0 \in \partial K$. We shall construct a sequence $\{x_n\}$ as follows. Since $Tx_0 \in K$ (by (a)), set $x_1 = Tx_0$. Consider now Tx_1. If $Tx_1 \in K$, set $x_2 = Tx_1$. If $Tx_1 \notin K$, choose $x_2 \in \partial K$ so that

$$d(x_1,x_2) + d(x_2,Tx_1) = d(x_1,Tx_1).$$

Now, if $Tx_2 \in K$, set $x_3 = Tx_2$. If not, choose $x_3 \in \partial K$ so that $d(x_2,x_3) + d(x_3,Tx_2) = d(x_2,Tx_2)$. Continuing in this manner, we obtain a sequence $\{x_n\}$ in K which can be devided in two separated classes A and B, where

$$A = \{x_i \in \{x_n\} : x_i = Tx_{i-1}\},$$

$$B = \{x_i \in \{x_n\} : x_i \neq Tx_{i-1}, x_i \in \partial K \text{ and } d(x_{i-1}, x_i) + d(x_i, Tx_{i-1}) = d(x_{i-1}, Tx_{i-1})\}.$$

Note that if $x_i \in B$, then x_{i+1} and x_{i-1} belong to A by condition (a). For if we suppose that $x_{i-1} \notin A$, then $x_{i-1} \in B \subset \partial K$, and so by (a), $Tx_{i-1} \in K$. By definition of $\{x_n\}$, we now have $Tx_{i-1} = x_i$ and hence $x_i \in A$, which is in contradiction with $x_i \in B$.

Now we will to estimate $d(x_n, x_{n+1})$. For simplicity, set $d_i = d(x_i, x_{i+1})$ and suppose that $d_i > 0$ for all $i \in N$ (otherwise, x_i is a fixed point of T). Actually, we have three cases.

Case I. $x_n, x_{n+1} \in A$. From (1) we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq h \max\{d_{n-1}/2, d_{n-1}, d_n, m(x_{n-1}, x_n), M(x_{n-1}, x_n)/2\}. \tag{2}$$

Since

$$m(x_{n-1}, x_n) \leq d(x_n, Tx_{n-1}) = d(x_n, x_n) = 0,$$

$$M(x_{n-1}, x_n) = d(x_{n-1}, x_{n+1}) \leq d_{n-1} + d_n \leq 2 \max\{d_{n-1}, d_n\},$$

from (2) we have $d_n \leq h \max\{d_{n-1}, d_n\}$ and hence

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n). \tag{3}$$

Case II. $x_n \in A$, $x_{n+1} \in B$. Then $d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, Tx_n)$ and hence $d_n < d(x_n, Tx_n) = d(Tx_{n-1}, Tx_n)$. From (1) and by the triangle inequality,

$$d(Tx_{n-1}, Tx_n) \leq h \max\{d_{n-1}/2, d_{n-1}, d_n, 0, (d_{n-1} + d_n)/2\}.$$

Hence

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n). \tag{4}$$

Case III. $x_n \in B$, $x_{n+1} \in A$. Since x_n is a convex combination of x_{n-1} and Tx_{n-1} it follows that

$$d(x_n, x_{n+1}) \leq \max\{d(Tx_{n-1}, x_{n+1}), d(x_{n-1}, x_{n+1})\}. \tag{5}$$

Since x_{n-1} and x_{n+1} belong to A, from (5) we have

$$d(x_n, x_{n+1}) \leq \max\{d(Tx_{n-1}, Tx_n), d(Tx_{n-2}, Tx_n)\}. \tag{6}$$

From (1),

$$d(Tx_{n-1}, Tx_n) \le h \max\{d_{n-1}/2, d(x_{n-1}, Tx_{n-1}), d_n,$$

$$d(x_n, Tx_{n-1}), \max\{d(x_n, Tx_{n-1}), d(x_{n-1}, Tx_n)\}/2. \tag{7}$$

Since $Tx_{n-2} = x_{n-1}$, similarly as in Case II, we obtain

$$d(x_{n-1}, Tx_{n-1}) = d(Tx_{n-2}, Tx_{n-1}) \le hd(x_{n-2}, x_{n-1}) = hd_{n-2}. \tag{8}$$

Further, using (8) and the triangle inequality, we have

$$d(x_n, Tx_{n-1}) = d(x_{n-1}, Tx_{n-1}) - d_{n-1} \le hd_{n-2}, \tag{9}$$

$$d(x_{n-1}, Tx_n) \le d(x_{n-1}, x_n) + d(x_n, Tx_n) = d_{n-1} + d_n \le 2\max\{d_{n-1}, d_n\}. \tag{10}$$

Thus, using (8), (9) and (10), from (7) we obtain

$$d(Tx_{n-1}, Tx_n) \le h \max\{d_{n-2}, d_{n-1}, d_n\}. \tag{11}$$

Again from (1) we have

$$d(Tx_{n-2}, Tx_n) \le h \max\{d(x_{n-2}, x_n)/2, d_{n-2}, d_n,$$

$$d_{n-1}, \max\{d_{n-1}, d(x_{n-2}, Tx_n)\}/2\}. \tag{12}$$

Since by the triangle inequality

$$d(x_{n-2}, x_n) \le d_{n-2} + d_{n-1} \le 2\max\{d_{n-2}, d_{n-1}\};$$

$$d(x_{n-2}, Tx_n) \le d(x_{n-2}, Tx_{n-2}) + d(Tx_{n-2}, Tx_n)$$
$$\le 2\max\{d_{n-2}, d(Tx_{n-2}, Tx_n)\},$$

from (12) we obtain

$$d(Tx_{n-2}, Tx_n) \le h \max\{d_{n-2}, d_{n-1}, d_n\}. \tag{13}$$

By (11) and (13), the inequality (6) becomes

$$d(x_n, x_{n+1}) \le h \max\{d_{n-1}, d_n, d_{n-2}\}.$$

Hence, as $d_n \le hd_n$ implies $d_n = 0$,

$$d(x_n, x_{n+1}) \le h \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}. \tag{14}$$

Since (3) and (4) imply (14), we conclude that in all cases (14) holds.

By induction it is easily shown that for $n \geq 1$,

$$d(x_n, x_{n+1}) \leq h^{(n-1)/2} \max\{d(x_0, x_1), d(x_1, x_2)\}.$$

Now, for $m > n > k$ we have

$$d(x_n, x_m) \leq \sum_{i=k}^{\infty} d(x_i, x_{i+1})$$

$$\leq (h^{(k-1)/2}/(1 - h^{1/2})) \max\{d(x_0, x_1), d(x_1, x_2)\}. \tag{15}$$

Hence we conclude that $\{x_n\}$ is a Cauchy sequence: hence convergent. Call the limit u. There exists an infinite subsequence $\{x_i\}$ of $\{x_n\}$ such that $x_{i+1} \in A$. By the triangle inequality and from (1) we obtain

$$d(u, Tu) \leq d(u, x_{i+1}) + d(Tx_i, Tu) \leq d(u, x_{i+1})$$

$$+ h \max\{d(u, x_i)/2, d_i, d(u, Tu), d(u, x_{i+1}),$$

$$\max\{d(x_i, Tu), d(u, x_{i+1}\}/2\}.$$

Taking the limit of the above as $n \to \infty$ we obtain $d(u, Tu) \leq hd(u, Tu)$, which implies $Tu = u$. From (15) we easy get an estimate of approximation of fixed point. Taking the limit of (15) as $m \to \infty$ we obtain

$$d(u, x_n) \leq (h^{(n-1)/2}/(1 - h^{1/2}) \max\{d(x_0, x_1), d(x_1, x_2)\}.$$

Condition (1) ensures that u is the unique fixed point of T. ■

Remark. Our theorem is a substantial generalization of the corresponding theorems of Assad [1], Ćirić [2] and Rhoades [3]. We observe that if a, b are non-negative reals, which for example, $a < b$, then

$$(a + b)/3 \approx 2/3 \min\{a, b\}, \quad \text{if} \quad a \approx b,$$

$$(a + b)/3 \approx 2/3[\max\{a, b\}/2], \quad \text{if} \quad a \approx 0.$$

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