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Expansion Formulas and Kelvin Principle for a Class of Partial Differential Equations

N. Özalp [1] *and A. Çetinkaya* [2]

Presented by P. Kenderov

We give some expansion formulas and Kelvin principle for solutions of a class of iterated elliptic equations including Laplace equation and its iterates.

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1. Introduction

In [1], Almansi gave an expansion formula for the solutions of the Laplace equation. In [2], Altın generalized the idea to a wide range of a class of singular partial differential equations and obtained a Lord Kelvin principle for this class of equations. Here we apply this idea to the equation

$$Lu = \sum_{i=1}^{n} \left(\frac{1}{x_i}\right)^p \left[x_i^2 \frac{\partial^2 u}{\partial x_i^2} + \alpha_i x_i \frac{\partial u}{\partial x_i}\right] + \frac{\gamma u}{r^p} = 0, \tag{1.1}$$

where γ, α_i $(i = 1, 2, ..., n)$ are real parameters, $p\,(> 0)$ is a real constant and r is defined by

$$r^p = x_1^p + x_2^p + ... + x_n^p. \tag{1.2}$$

The domain of the operator L is the set of all real valued functions $u(x)$ of the class $C^2(D)$, where $x = (x_1, x_2, ..., x_n)$ denotes points in R^n and D is a regularity domain of u in R^n.

2. Expansion formulas

We first give some properties of the operator L. By a direct computation, it can be shown that

$$L(r^m) = (m(m+\phi)+\gamma)\, r^{m-p}, \tag{2.1}$$

where

$$\phi = -p + n(p-1) + \sum_{i=1}^{n} \alpha_i. \tag{2.2}$$

The proof of the following lemma can be done easily by using induction arguments on k. For a special case of the lemma, see Altın [3].

Lemma 1. *For any real parameter m,*

$$L^k(r^m) = \prod_{j=0}^{k-1} [(m-pj)(m-pj+\phi)+\gamma]\, r^{m-pk}, \tag{2.3}$$

where the integer k is the iteration number.

Let $u, v \in C^2(D)$ be any two functions. It can be shown that

$$L(uv) = uLv + vLu - \frac{\gamma uv}{r^p} + 2\sum_{i=1}^{n} \left(\frac{1}{x_i}\right)^p \left(x_i^2 \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right). \tag{2.4}$$

By replacing v by r^m in (2.4) and by using (2.1), we get

$$L(r^m u) = r^{m-p} m\,(m+\phi+2T^*)\,u + r^m Lu \tag{2.5}$$

where

$$T^* = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}.$$

If u is a solution of the equation $Lu = 0$, then by (2.5),

$$L(r^m u) = r^{m-p} m\,(m+\phi+2T^*)\,u. \tag{2.6}$$

By a direct computation, one can show that

$$LT^* = (p+T^*)L. \tag{2.7}$$

In fact, for any integer k, we have, by induction on k,

$$L(T^*)^k = (p + T^*)^k L, \tag{2.8}$$

where $(T^*)^k$ denotes the successive iterations of the operator T^*, k times onto itself.

Now we are ready to give the following lemma.

Lemma 2. *Let u be a solution of the equation $Lu = 0$. Then for any positive integer k and for any real number m,*

$$L^k(r^m u) = r^{m-pk} \left\{ \prod_{j=0}^{k-1} (m - pj)(m - pj + \phi + 2T^*) \right\} u. \tag{2.9}$$

Proof. We give the proof by induction on k. It is clear by (2.6) that, the equality (2.9) is true for $k = 1$. Now, let us assume that the equality is hold for $k - 1$, that is,

$$L^{k-1}(r^m u) = r^{m-p(k-1)} \left\{ \prod_{j=0}^{k-2} (m - pj)(m - pj + \phi + 2T^*) \right\} u.$$

By applying the operator L on both sides of the above equality, we obtain

$$L^k(r^m u) = L \left[r^{m-p(k-1)} \left\{ \prod_{j=0}^{k-2} (m - pj)(m - pj + \phi + 2T^*) \right\} u \right].$$

Since u is a solution of $Lu = 0$, by using (2.8), it can be shown that, the function $\left\{ \prod_{j=0}^{k-2} (m - pj)(m - pj + \phi + 2T^*) \right\} u$ is also a solution of the same equation. Hence, by replacing m by $m - p(k - 1)$ and u by

$$\left\{ \prod_{j=0}^{k-2} (m - pj)(m - pj + \phi + 2T^*) \right\} u$$

in (2.6), we get

$$\begin{aligned} L^k(r^m u) &= L\left[r^{m-p(k-1)}\left\{\prod_{j=0}^{k-2}(m-pj)(m-pj+\phi+2T^*)\right\}u\right] \\ &= r^{m-pk}(m-p(k-1))(m-p(k-1)+\phi+2T^*) \\ &\quad\times\left\{\prod_{j=0}^{k-2}(m-pj)(m-pj+\phi+2T^*)\right\}u \\ &= r^{m-pk}\left\{\prod_{j=0}^{k-1}(m-pj)(m-pj+\phi+2T^*)\right\}u. \end{aligned}$$

Hence the proof is complete. ∎

Now, we give a generalization of Almansi's expansion.

Theorem 1. *Let $u_i(x)$, $i=0,1,...,k-1$ be any k solutions of the equation $Lu=0$. Then, the function*

$$w=\sum_{i=0}^{k-1} r^{ip}u_i(x) \tag{2.10}$$

gives a solution of the iterated equation $L^k u=0$.

Proof. By the hypothesis and by Lemma 2, we have

$$L^k\left[r^{pj}u_i(x)\right]=0, \quad i,j=0,1,...,k-1.$$

Hence, by the principle of superposition

$$L^k w=0.$$

∎

Remark 1. It is clear that, under the hypothesis of Theorem 1, in fact, the function

$$w=\sum_{i,j=0}^{k-1} r^{jp}u_i(x)$$

is also a solution of $L^k u=0$. In addition, if u is solution of the equation $Lu=0$, then, we conclude that, for any integer i, $r^{pi}u$ is also a solution of the iterated equation $L^k u=0$ $(k\geq 2)$.

The following theorem gives an expansion formula for the homogeneous solutions.

Theorem 2. *Let $u_\nu(x)$, $\nu = 0, 1, ..., k-1$ be homogeneous (of degree λ_ν, respectively) solutions of the equation $Lu = 0$. Then, the function*

$$w = \sum_{\nu=0}^{k-1} r^{\nu p - \phi - 2\lambda_\nu} u_\nu(x) \tag{2.11}$$

gives a solution of the iterated equation $L^k u = 0$.

Proof. Since $u_\nu(x)$ is a homogeneous function of degree λ_ν, then by the Euler theorem of homogeneous functions,

$$T^* u_\nu(x) = \sum_{i=1}^{n} x_i \frac{\partial u_\nu(x)}{\partial x_i} = \lambda_\nu u_\nu(x).$$

On the other hand, since u_ν satisfies the equation $Lu = 0$, by Lemma 2, for each ν, one has

$$L^k[r^m u_\nu(x)] = r^{m-pk} \prod_{j=0}^{k-1} (m - pj)(m - pj + \phi + 2\lambda_\nu)\, u_v(x)$$

which yields

$$L^k[r^{pj-\phi-2\lambda_\nu} u_\nu(x)] = 0, \quad j = 0, 1, ..., k-1.$$

Thus, by the principle of superposition

$$L^k w = 0.$$

■

3. Lord Kelvin principle

The Kelvin principle is studied by several authors (see, for example, Çelebi [4], Weinstein [5]). More recently, Altın [2] established the Kelvin principle for the solutions of the equation

$$\sum_{i=1}^{n} \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\alpha_i}{x_i} \frac{\partial u}{\partial x_i} \right) \pm \sum_{i=1}^{s} \left(\frac{\partial^2 u}{\partial y_i^2} + \frac{\beta_i}{y_i} \frac{\partial u}{\partial y_i} \right) + \frac{\gamma u}{R^2} = 0,$$

where α_i, β_i and γ are real constants and

$$R^2 = \sum_{i=1}^{n} x_i^2 \pm \sum_{i=1}^{s} y_i^2.$$

Here, in this section, we state a generalized Kelvin principle for the solutions of the equation (1.1) in the following theorem.

Theorem 3 (Kelvin principle). *Let $u(x) = u(x_1, x_2, ..., x_n)$ be any solution of the equation (1.1). Then, the function*

$$v = r^{p+n(1-p)-\sum_{i=1}^{n}\alpha_i} u\left(\frac{x_1}{r^2}, \frac{x_2}{r^2}, ..., \frac{x_n}{r^2}\right) \tag{3.1}$$

is also a solution of the same equation, where $r = (x_1^p + x_2^p + ... + x_n^p)^{1/p}$.

P r o o f. From (2.5), we already have

$$L(r^m u) = r^{m-p} m\,(m + \phi + 2T^*)\,u + r^m Lu.$$

Now, let $\xi = (\xi_1, \xi_2, ..., \xi_n)$, where $\xi_i = \frac{x_i}{r^2}$, $i = 1, 2, ...n$. Then for $\rho^p = \xi_1^p + \xi_2^p + ... + \xi_n^p$, clearly, $r^p \rho^p = 1$. By making a change of variables, a mess of computations yields

$$T^* u(\xi) = \sum_{i=1}^{n} x_i \frac{\partial u(\xi)}{\partial x_i} = -\sum_{i=1}^{n} \xi_i \frac{\partial u(\xi)}{\partial \xi_i} \tag{3.2}$$

and

$$\begin{aligned} Lu(\xi) &= \left\{\sum_{i=1}^{n}\left(x_i^{2-p}\frac{\partial^2}{\partial x_i^2} + \alpha_i x_i^{1-p}\frac{\partial}{\partial x_i}\right) + \frac{\gamma}{r^p}\right\} u(\xi) \\ &= \rho^{2p}\left\{\sum_{i=1}^{n}\left(\xi_i^{2-p}\frac{\partial^2}{\partial \xi_i^2} + \alpha_i \xi_i^{1-p}\frac{\partial}{\partial \xi_i}\right) + \frac{\gamma}{\rho^p}\right\} u(\xi) \\ &\quad -2\rho^p\left[-p + n(p-1) + \sum_{i=1}^{n}\alpha_i\right]\sum_{i=1}^{n}\xi_i\frac{\partial u(\xi)}{\partial \xi_i} \\ &= \rho^{2p} L_{(\xi)} u(\xi) - 2\rho^p \phi T^*_{(\xi)} u(\xi), \end{aligned} \tag{3.3}$$

where we use the notations $T^*_{(\xi)}$ and $L_{(\xi)}$ denoting, respectively, the operators T^* and L with x replaced by ξ.

Now substituting (3.2) and (3.3) in (2.5), we obtain

$$L\left[r^m u(\xi)\right] = r^{m-p} m(m + \phi - 2T^*_{(\xi)})u(\xi) + r^{m-2p} L_{(\xi)} u(\xi) - 2r^{m-p}\phi T^*_{(\xi)} u(\xi). \tag{3.4}$$

Since $u(x)$ is a solution of $Lu = 0$, the equation (3.4) becomes

$$L\left[r^m u(\xi)\right] = r^{m-p} m\left(m + \phi - 2T^*_{(\xi)}\right) u(\xi) - 2r^{m-p}\phi T^*_{(\xi)} u(\xi)$$

or, simply

$$L\left[r^m u(\xi)\right] = r^{m-p}(m+\phi)\left(m - 2T^*_{(\xi)}\right)u(\xi). \tag{3.5}$$

Hence, for $m = -\phi$, we get

$$L\left[r^{-\phi}u(\xi)\right] = 0$$

or, explicitly,

$$L\left[r^{p+n(1-p)-\sum_{i=1}^{n}\alpha_i}\, u\left(\frac{x_1}{r^2}, \frac{x_2}{r^2}, \ldots, \frac{x_n}{r^2}\right)\right] = 0,$$

which completes the proof. ∎

The Kelvin principle roughly states that if a solution of the equation (1.1) is known, then one can obtain another solution just by using the transformation mentioned above. The following simple example clears out the case.

An example of the inversion: Let, in (1.1), $n = 2$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\gamma = -5$, $p = 3$ and thus $r = (x_1^3 + x_2^3)^{1/3}$. By (2.1), we easily conclude that $u(x_1, x_2) = r = (x_1^3 + x_2^3)^{1/3}$ is a solution of (1.1) with the given parameters. Hence, by using the Kelvin principle, we obtain that

$$v = r^{-4}u\left(\frac{x_1}{r^2}, \frac{x_2}{r^2}\right) = r^{-4}\left[\left(\frac{x_1}{r^2}\right)^3 + \left(\frac{x_2}{r^2}\right)^3\right]^{1/3} = r^{-5}$$

is also a solution of the same equation. If we would use $u = r^{-5}$ as a solution, then we should get $v = r$ as the other solution under the inversion.

Lemma 3. *If $u(x)$ is any solution of the equation (1.1), then*

$$L^k\left[r^m u(\xi)\right] = r^{m-pk}\left\{\prod_{j=0}^{k-1}(m - pj + \phi)\left(m - pj - 2T^*_{(\xi)}\right)\right\}u(\xi), \tag{3.6}$$

where ξ is as given in the proof of Theorem 3.

Proof. The proof shall be done by induction arguments. For if, $k = 1$, (3.6) is reduced to (3.5) and hence the conclusion is hold for $k = 1$. Since the equality (2.8) holds for x replaced by ξ, the rest of the proof can be done by using the same arguments as in the proof of Lemma 2. ∎

By making use of the formula (3.6), and using the same arguments of Theorem 1 and Theorem 2, we obtain the following results which state that the Kelvin principle also holds for the iterated equation $L^k u = 0$.

Theorem 4. *Let $u_i(x)$, $i = 0, 1, ..., k-1$ be any k solutions of the equation $Lu = 0$. Then, the function*

$$w = r^{-\phi} \sum_{i=0}^{k-1} r^{ip} u_i(\xi) = r^{-\phi} \sum_{i=0}^{k-1} r^{ip} u_i \left(\frac{x_1}{r^2}, \frac{x_2}{r^2}, ..., \frac{x_n}{r^2} \right)$$

gives a solution of the iterated equation $L^k w = 0$.

Theorem 5. *Let $u_\nu(x)$, $\nu = 0, 1, ..., k-1$ be homogeneous (of degree λ_ν, respectively) solutions of the equation $Lu = 0$. Then, the function*

$$w = \sum_{\nu=0}^{k-1} r^{\nu p + 2\lambda_\nu} u_\nu(\xi)$$

gives a solution of the iterated equation $L^k w = 0$.

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Ankara University, Faculty of Sciences
Dept. of Mathematics
Beşevler, 06100 Ankara - TURKEY

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e-mails: [1] *nozalp@science.ankara.edu.tr*
[2] *cetinkay@science.ankara.edu.tr*