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A New Approach for Solving Equations with Interval Coefficients

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Provided is the definition of a new relation within the setting of real intervals. It is demonstrated that this relation resolves some difficulties, which arise when using the traditional equality of real intervals.

This concept leads to a novel algebraic structure, called the dynamic field. Basic properties and characterizations of the dynamic field are given and its application for solving interval equations in an extended sense is described.

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1. Introduction

Let $\Re$ denote the set of real numbers. The set of real closed intervals is abbreviated with $I\Re$. Intervals are either indicated with brackets $[a]$ or capital letters A. We assume that the reader is familiar with the basic facts of the interval arithmetic. Our concern is the solution of equations in $I\Re$, so we discuss first some typical problems which arise, when solving interval equations.

Example 1. An interval polynomial $IP_n(x)$ is a set of polynomials with real coefficients,

$$IP_n(x) := \sum_{i=0}^{n} A_i x^i = \{p_n(x) \quad | \quad p_n(x) = \sum_{i=0}^{n} a_i x^i, \quad a_i \in A_i, A_i \in I\Re\}.$$

The null set of IP_n is denoted with $N(IP_n)$:

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$$N\left(IP_n\right) = \{r \in \Re \quad | \quad \exists\, p_n(x) \in IP_n\left(x\right): \quad p_n(r) = 0\}.$$

Particularly, the interval polynomial $IP_1\left(x\right) = [2,3] + x$, has the null set

$$N\left(IP_1\right) = \{r \mid -3 \leq r \leq -2\}.$$

Note that it is not possible to compute this solution using interval arithmetic, since the usual definition of $=$ implies $[2,3] + [-3,-2] = [-1,1]$.

Example 2. As a second example we take the equation

$$[3,5] + e^x = [6,10].$$

With standard-interval analysis we obtain $x \in X = [\ln(3), \ln(5)]$, the special equation $3 + e^{x_0} = 10$ has $x_0 = \ln(7)$ as solution, which does not belong to X.

The problem in these examples comes from the interpretation of equality $=$ in $I\Re$ and not from the definition of the interval operations due to Moore [MOO 66], cf. also Ratschek [RAT 70]. To overcome this dilemma, several extensions of the classical interval arithmetic have been proposed, but the only providing an algberaic completion of interval arithmetic are those leading to the set of directed intervals. Basically developed by Kaucher (cf. [KAU 73], [KAU 80]) and Markov (cf. [MAR 92], [MAR 95]), the directed interval arithmetic is obtained as an extension of the set of normal intervals by improper intervals and a corresponding extension of the definition of the interval arithmetic operations. Directed interval arithmetic is among others useful for computation of inner and outer inclusion of functional ranges. Contrary to these attempts, we interpret equality $=$ in $I\Re$ in a weaker sense by introducing a new relation $\equiv$. This approach has the advantage that it fits within the normal interval analytical framework. We provide a thorough representation of the relation $\equiv$. The interplay of $\equiv$ with interval arithmetic leads to a novel algebraic structure, which is called dynamic field.

The paper is organized as follows: In Section 2 we introduce the relation $\equiv$ and derive some properties, the next Section 3 is concerned with the topic of the dynamic field, that is algebraic properties of $\equiv$. In the last Section 4 the solution of interval eqautions in the dynamic field is studied.

2. The relation $\equiv$

As already shown, classical interval-analysis has some shortcomings, when solving equations. For this reason we introduce a new relation $\equiv$ in $I\Re$, where the algebraic operations in $I\Re$ are explained in the usual manner (see Moore [MOO 66] and Alefeld [ALE 80]).

Definition 3. Let be $A \in I\Re$. For two elements $X_1, X_2 \in I\Re$ we define the following relation $\equiv_A$:

$$X_1 \equiv_A X_2 :\Leftrightarrow A \subseteq X_1 \quad \text{and} \quad A \subseteq X_2.$$

If $A = [a, a]$ is a point interval, we write $X_1 \equiv_a X_2$ instead of $X_1 \equiv_A X_2$, where sometimes the subscript $_a$ is omitted. For $a, b \in \Re$ we introduce the notation

$$X_1 \equiv_{a,b} X_2 :\Leftrightarrow X_1 \equiv_a X_2 \quad \text{and} \quad X_1 \equiv_b X_2.$$

Theorem 4. *Let $(I\Re, \subseteq)$ and $A, B, C \in I\Re$. Then $\equiv_x$ has the following properties:*

$A \equiv_x A, \quad x \in A$ *(x-reflexive),*
$A \equiv_x B,$ *then* $B \equiv_x A$ *(x-symmetric),*
$A \equiv_x B$ *and* $B \equiv_x C,$ *then* $A \equiv_x C$ *(x-transitive),*

that is, $\equiv_x$ is an equivalence relation.

Proof. The properties of an equivalence relation follow immediately from the definition of $\equiv$. ∎

Remark 1. The indexing element $\equiv_x$ can be interpreted as a common property that elements - which are related, with respect to x - share.

Next we establish a link between $\equiv_x$ and $=$.

Lemma 5. *Let $A, B \in I\Re$. Then,*

$$A \equiv_{a,b} B\,, \quad \textit{for all} \quad a \in A, b \in B \quad \textit{if and only if} \quad A = B.$$

Proof. Due to Definition 3, the relation $A \equiv_{a,b} B, a \in A, b \in B$ implies $A \subseteq B \wedge B \subseteq A$, which is only true if A and B coincide. ∎

From the following result we observe that the monotonicity law is valid.

Theorem 6. *Let $A, B, C, D \in I\Re$ and let $*$ denote one of the four arithmetic operations $+, -, \cdot, /$. Then,*

$$\textit{from} \quad A \equiv_x B \ \textit{and} \ C \equiv_y D \ \textit{it follows} \ A * C \equiv_{x*y} B * D,$$

where $0 \notin C, D$ *is assumed in the case of division.*

Proof. This assertion follows from the definition of $\equiv$, since $\{x\} \subseteq A$, $\{x\} \subseteq B$, $\{y\} \subseteq C$ and $\{y\} \subseteq D$ imply $\{x * y\} \subseteq A*C$ and $\{x * y\} \subseteq B*D$, $* \in \{+, -, \cdot, /\}$. ∎

Before we deal with the main algebraic properties of $(I\Re, \equiv_x)$, we study the evaluation of interval expressions. For readers' convenience, we sketch the corresponding foundations.

Enclosures for a function f are sets which contain its range. They play a vital role in the numerical analysis, since exact numerical values for function values are not computable in general. But we can derive results for computer assisted proofs of non-existence of solutions by showing that these enclosure sets do not contain the zero element.

Definition 7. Let $f : D \subseteq \Re \longrightarrow \Re$ be continuous and let X be an interval-subset of D. The interval range of f in X is explained as

$$I_f(X) = \{ f(x) \mid x \in X \} := [\min_{x \in X} f(x), \max_{x \in X} f(x)].$$

In the numerical computations, it is not feasible to determine $I_f(X)$ exactly, so in practice upper and lower bounds are calculated, by replacing constants and variables through according interval quantities, yielding an interval evaluation which is denoted by $f(X)$ (cf. Alefeld [ALE 98]).

Corollary 8. *Let* $I_f(X)$ *and* $f(X)$ *be as before. Then for all* $x \in X$,

$$f(x) \in I_f(X) \subseteq f(X).$$

Proof. Cf. [ALE 80]. ∎

The interval evaluations of algebraically equivalent but formally different arithmetic expressions yield in general different numerical results as demonstrated in the next example.

Example 9. For $X = [-1, 1]$, we consider two different evaluations f_1, f_2 of

$$f(x) = x \cdot (x + 1),$$

namely:

$$f_1(X) := X \cdot X + X = [-2, 2],$$

$$f_2(X) := X^2 + X = [-1, 2],$$

and obtain

$$I_f([-1,1]) \subseteq f_2(X) = [-1,2] \subseteq f_1(X) = [-2,2].$$

The relation $\equiv$ facilitates to compare various types of interval evaluations.

Corollary 10. *Let $f : D \subseteq \Re \longrightarrow \Re$ be continuous. If $f_1(X)$ and $f_2(X)$, $X \subseteq D$, are two different interval-evaluations of f in X, then, for each $y \in I_f(X)$, it holds*

$$f_1(X) \equiv_y f_2(X).$$

Additionally, for $x \in I_f(D)$ and $A, B \subseteq I_f(D)$, the implication

$$A \equiv_x B \Rightarrow f(A) \equiv_{f(x)} f(B),$$

is true.

P r o o f. Since

$$I_f(X) \subseteq f_1(X) \quad \text{and} \quad I_f(X) \subseteq f_2(X),$$

the first assertion holds. From $A \equiv_x B$, that is $x \in A$ and $x \in B$, we deduce $f(x) \in f(A)$ and $f(x) \in f(B)$, and the proof is complete. ■

3. The dynamic field

In this paragraph we study the interchange between the interval-operations and the equivalence relation $\equiv_x$.

Theorem 11. *For $A, B, C, D \in I\Re$, the following properties hold:*
a) $A + B \equiv_{a+b} B + A$, for all $a \in A, b \in B$;
b) $A \cdot B \equiv_{a \cdot b} B \cdot A$, for all $a \in A, b \in B$;
c) $A + (B + C) \equiv_{a+b+c} (A + B) + C$, for all $a \in A, b \in B, c \in C$;
d) $A \cdot (B \cdot C) \equiv_{abc} (A \cdot B) \cdot C$, for all $\in A, b \in B, c \in C$;
e) If $1 \in B$, then $A \cdot B \equiv_a A$, for all $A \in I\Re, a \in A$.

P r o o f. Assertions a) to e) are a consequences of Definition 3 and Theorems 1 and 6. ■

Theorem 12.
a) There exists an interval $\mathcal{O} \in I\Re$ such that: $A + \mathcal{O} \equiv_a A$, $A \in I\Re, a \in A$;
b) There exists an interval $\mathcal{I} \in I\Re$ such that: $\mathcal{I} \cdot A \equiv_a A$, $A \in I\Re, a \in A$;
c) To each $A \in I\Re$, there exists an $X \in I\Re$ such that $A + X \equiv_0 \mathcal{O}$;

d) To each $A \in I\Re$, *with* $0 \notin A$, *there exists* $X = A^{-1} \in I\Re$ *such that* $A \cdot X \equiv_1 \mathcal{I}$;

e) $A \cdot (B + C) \equiv_{a\cdot(b+c)} A \cdot B + A \cdot C$, *for all* $a \in A, b \in B, c \in C$.

Proof. a) and b) can be derived from Definition 3, Theorem 1 and Theorem 6; c) and resp. d), are consequences of $\{0\} \subseteq A - A$, resp. $\{1\} \subseteq A \cdot \frac{1}{A}$. e) Choosing in Corollary 10 especially $f_1 = A \cdot (B + C)$ and $f_2 = AB + AC$, the claimed result is a consequence of $a \cdot (b + c) \in I_f$. ■

Although the neutral and inverse elements are not unique, they are equivalent with respect to $\equiv_x$, which can be understood in a broader sense as uniqueness. The properties of the operations $+$ and $\cdot$ are similar to the properties of a field, this gives raise to the concept of the dynamic field.

Definition 13. The set of real intervals equipped with the algebraic opertions $+, \cdot$ and the relation $\equiv$, denoted with $(I\Re, +, \cdot, \equiv)$ is called dynamic field.

Remark 2. We can interpret $I\Re$ as the "error domain" of a field. It is also possible to understand the relation $\equiv_x$ as a weak equality relation, if A and B are error domains of a and b.

Next, we summarize some important properties of $(I\Re, +, \cdot, \equiv)$.

Theorem 14. *The following properties are true for* $A, B \in (I\Re, +, \cdot, \equiv)$:

(a) If $A \equiv_x B$ *and* $A \subseteq A'$, *then* $A' \equiv_x B$;

(b) If $B \subseteq A$, *then* $A \equiv_b B$, $b \in B$;

(c) $(A + B) \cdot (A - B) \equiv_c A^2 - B^2$, $c \in A^2 - B^2$;

(d) $A \cdot B \equiv_0 \mathcal{O}$, *if and only if,* $A \equiv_0 \mathcal{O}$ *or* $B \equiv_0 \mathcal{O}$;

(e) If $A \equiv_c B$, *then* $\pm\sqrt{A} \equiv_{\pm\sqrt{c}} \pm\sqrt{B}$, $c \in \Re$ *and* $A \geq 0$, $B \geq 0$.

Proof. (a), (b) follow by straightforward calculation; (c) The essential observation is $A^2 - B^2 \subseteq (A + B) \cdot (A - B)$; (d) $A \cdot B \equiv_0 \mathcal{O}$ is true if and only if $\{0\} \subseteq A \cdot B$, which implies $0 \in A \vee 0 \in B$; (e) Let $c \in A$ and $c \in B$, then $\sqrt{c} \in \sqrt{A}$ and $\sqrt{c} \in \sqrt{B}$. ■

Theorem 15. *Let* $Y, Z \in I\Re$, *then the following cancellation laws hold in* $(I\Re, +, \cdot, \equiv)$:

a) $A + Y \equiv_x A + Z$ *for all* $A \in I\Re \Leftrightarrow Y \equiv_x Z$;

b) $A - Y \equiv_x A - Z$ *for all* $A \in I\Re \Leftrightarrow Y \equiv_x Z$, *and* $Y - A \equiv_x Z - A$ *for all* $A \in I\Re \Leftrightarrow Y \equiv_x Z$;

c) For $0 \notin A$ *we have* $A \cdot Y \equiv_x A \cdot Z \Leftrightarrow Y \equiv_x Z$, *and* $Y/A \equiv_x Z/A \Leftrightarrow Y \equiv_x Z$.

Proof. Follows from the definition of $\equiv$. ■

Further, we will introduce an order relation in $(I\Re, \equiv)$. This entails not only the notation of positivity and negativity but also the possibility to deal with inequalities.

Definition 16. The nonvoid subset $P \subseteq I\Re$ characterized by the following properties:

i) $A, B \in P \Rightarrow A \cdot B \in P$;

ii) $A, B \in P \Rightarrow A + B \in P$;

iii) For $A \in I\Re$ exactly one of the assertions is true:

$$A \in P\ ,\ -A \in P\ ,\ A \equiv_0 \mathcal{O},$$

is called the set of strictly positive interval numbers. Elements $A \in P$ are called strictly positive intervals and we write $A > 0$; if $-A \in P$, we call A strictly negative and write $A < 0$.

The first two properties ensure the compatibility of the order relation with the arithmetic operations. Requirement iii) is also known as trichotomy relation and divides $I\Re$ into three disjoint sets.

4. Equations in the dynamic field

In this section we solve equations in the dynamic field. At the beginning we specify what we mean by a solution.

Definition 17. Let $x \in \Re$ and $X, Y \in I\Re$ and $f : \Re \longrightarrow \Re$. We call the set of all $x \in X$ which satisfy for prescribed $y \in Y$ the equation $f(x) = y$, its optimal solution X_o (or simply, solution). Any $X_e \in I\Re$ with $X_o \subseteq X_e$, is called external solution and an internal solution is any $X_i \in I\Re$ with $X_i \subseteq X_o$.

Theorem 18. *Let* $A, B \in I\Re$, *then* $X = B - A$ *is the optimal solution of*

$$A + X \equiv_b B.$$

Proof. It is

$$A + X \equiv_b B \Leftrightarrow A + X - A \equiv_{b-a} B - A.$$

Since

$$X \equiv_x A + X - A,$$

we have $X \equiv_x B - A$ for all $x = b - a, b \in B, a \in A$, hence $X = B - A$, is the solution set of all equations of the form $a + x = b, b \in B, a \in A$, and so X is the optimal solution of $A + X \equiv_b B$. ∎

Remark 3. Note that our purpose is to solve the whole family of equations $ax + b = c$, when $a \in A, b \in B$ and $c \in C$.

Theorem 19. *Let be given $A, B, C \in I\Re$, where it is supposed that $0 \notin A$, then*

$$X = \frac{C - B}{A}$$

is the optimal solution of

$$A \cdot X + B \equiv_c C.$$

Proof. From $A \cdot X + B \equiv_c C$ follows $A \cdot X \equiv_{ax} A \cdot X + B - B \equiv_{c-b} C - B$ and so, taking $ax = c - b$, $a \neq 0$, we get $A \cdot X \equiv_{c-b} C - B$, hence

$$X \equiv_{\frac{c-b}{a}} \frac{C - B}{A}, \quad a \in A, b \in B \quad \text{and} \quad c \in C,$$

and the proof is complete. ∎

Remark 4. The classical interval solution of the equation

$$A \cdot X + B = C \cdot X + D\,, \quad A, B, C, D \in I\Re,$$

does not solve the whole set of equations $ax + b = cx + d$, when a, b, c, d belong to the corresponding intervals; in this case the optimal solution is given by $X = \frac{D-B}{A-C}$ provided $0 \notin A - C$.

We continue and treat quadratic equations

$$ax^2 + bx + c = 0,$$

where a, b, c are elements of $A, B, C \in I\Re$, respectively. For simplicity we consider the case of real roots.

Theorem 20. *Let $A, B, C \in I\Re$ with $0 \notin A$, and*

$$B^2 - 4A \cdot C = [d_1, d_2]\,, \quad d_1 \geq 0.$$

Then

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2 \cdot A}$$

is an external solution of

$$A \cdot X^2 + B \cdot X + C \equiv_o \mathcal{O}.$$

Proof. When a, b and c are intervals, we apply interval analytical methods to compute zeros of the polynomial $f(x) = ax^2 + bx + c$. We can also write

$$\mathcal{O} \equiv_o AX^2 + BX + C \equiv_{I_f} (2AX + B)^2 - B^2 + 4AC,$$

thus

$$(2AX + B)^2 \equiv_{b^2-4ac} B^2 - 4AC,$$

consequently

$$2AX + B \equiv_{\sqrt{b^2-4ac}} \sqrt{B^2 - 4AC},$$

and finally

$$X \equiv_{\frac{-b\pm\sqrt{b^2-4ac}}{2a}} \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

■

Note that X is not the optimal solution, since in interval-analysis only subdistributivity holds. As a consequence, instead of the exact range we determine a larger interval evaluation.

Example 21. Consider the problem

$$ax^2 + bx + c = 0,$$

for $a \in [1,2]$, $b \in [3,5]$ and $c \in [0,1]$. According to Theorem 20, we find

$$X \equiv \frac{-[3,5] \pm \sqrt{[9,25] - [0,8]}}{[2,4]},$$

that is $X_1 \equiv [-2,1]$ and $X_2 \equiv [-5,-1]$. Due to 'Descartes' rule none of the real polynomials has any positive zero. But employing the product rule for the roots:

$$X_1 \cdot X_2 = \frac{[0,1]}{[1,2]} = [0,1],$$

and taking $X_2 = [-5, -1]$, we arrive at $X_1 = [-1, 0]$. Although we have not obtained the optimal solution, we have got a much better result than the previous one. For the example at hand, we get the optimal solution $X_1 = [-0.5\,,\,0]$, if we rewrite

$$X_1 = \frac{-2C}{B + \sqrt{B^2 - 4AC}} .$$

The optimal solution is

$$X_o = [-5\,,\,-1] \cup [-0.5\,,\,0],$$

because for $p(x) = [1,2]x^2 + [3,5]x + [0.1]$ we have

$$p(-5) = [0, 36]\ ,\ p(-1) = [-4, 0]\ ,\ p(-0.5) = [-2.25, 0]\ ,\ p(0) = [0, 1].$$

Another standard problem in numerical analysis is to estimate the range of a function. It is well known that the range can be hardly computed exactly, therefore one could be tempted to use interval evaluations. The following example illustrates the drawbacks of this process.

Example 22. The polynomial $f(x) = x^2 - 5x + 6$, has 2 and 3 as roots. Taking $X = [0, 1.5]$ and the interval evaluations $f_1(X) = X^2 - 5X + 6$ and $f_2(X) = X \cdot (X - 5) + 6$, yields

$$f_1([0, 1.5]) = [-1.5, 8.25],$$
$$f_2([0, 1.5]) = [-1.5, 6],$$
$$I_f([0, 1.5]) = [0.75, 6].$$

This observation leads to the attendant rule: In the search for a solution x^* of an equation $f(x^*) = 0$, it is not possible to obtain an interval X containing the root by simply requiring $\{0\} \subseteq f(X)$.

Nevertheless, the technique we are using bears an important result, since theorems guaranteeing numerically the non-existence of solutions are rare in numerical analysis. The range of a function is contained in each of its interval evaluations, for this reason the following theorem is useful when treating root-finding-problems.

Theorem 23. *Let $f(X)$ and $I_f(X)$ be range and interval evaluation of a function f respectively. If for a given interval $Y \subseteq X$, holds $0 \notin f(Y)$, then no root of f lies within Y.*

Proof. The assumption that an $x^* \in Y$ with $f(y^*) = 0$ exists, contradicts $0 \notin f(Y)$. ■

Final Remark. The relation $\equiv$ is a powerful tool for the the development of enclosure schemes in interval analysis. This process is still in its initial stage but recently first attempts have been made in this direction (see Dobner [DOB 01]). Another topic of future investigations will be the comprehensive study of applications of $\equiv$ in domain theory.

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