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On Estimating of a Fourth-Order Differential Equations

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We find the optimal upper bound of the principal eigenvalue for fourth-order self-adjoint two points boundary eigenvalue problems. The estimates depend on the coefficients of differential equation as they vary over a given class of equimeasurable functions. The results here develop the ideas introduced in [4], where second-order problems are considered.

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1. Introduction

Consider the following fourth-order eigenvalue problem

(1) $$y^{(4)} - (p(x)y')' - \lambda r(x)y = 0 \ , \quad x \in (0,l),$$

(2) $$y(0) = y'(0) = y(l) = y'(l) = 0,$$

where $p(x)$ and $r(x)$ are determined below in the functional sets U and W.

The eigenvalue problem (1), (2) presents the natural frequencies of a long bar of length l which is supported at its endpoints. The solution of the eigenvalue problem of this kind yields the natural frequencies for free vibration and the critical load for the stability analysis of the vibrating systems. The most important influence of this considerations has the first eigenvalue (frequency) and the corresponding shape function for a mode of vibration.

The aim of the study described here is to investigate some variational extremal properties of the first eigenvalue. Banks [2] has determined upper and lower bounds for the eigenvalues of the problem

$$y^{(4)} - \lambda r(x)y = 0 \ , \quad x \in (0,l),$$

$$y(0) = y''(0) = y(l) = y''(l) = 0.$$

S. Karaa [4] has presented the optimal bounds for the eigenvalues of some second-order problems.

Let A, B and M be positive numbers such that $Ml > B$, and let $H^m(0,l)$ be the usual Sobolev space of order m on the interval $(0,l)$ (see for example [1]). We define the functional sets:

$$U = \left\{ p(x) \in H^1(0,l), \quad p(x) \geq 0, \int_0^l p(x)\,dx = A \right\},$$

$$W = \left\{ r(x) \in L_\infty(0,l), \quad 0 \leq r(x) \leq M, \int_0^l r(x)\,dx = B \right\}.$$

We denote the set of functions belonging to $H^2(0,l)$ and satisfying the boundary conditions (2) by V.

For any functions $y, z \in V$ we define the inner product

$$(y,z) = \int_0^l ryz\,dx, \quad r(x) \in W,$$

and the bilinear a–form

$$a(y,z) = \int_0^l [y''z'' + py'z']\,dx, \quad p(x) \in U.$$

We shall use the fact that the lowest eigenvalue is the minimum of the Rayleigh quotient [3] corresponding to the given eigenvalue problem:

$$R(y) = \frac{a(y,y)}{(y,y)}.$$

Then, when $p(x) \in U$ we have $R(y) \leq G[r,y]$, where

$$G[r,y] = \frac{\int_0^l y''^2\,dx + A \max_{x \in (0,l)} |y'|^2}{\int_0^l ry^2\,dx}.$$

Therefore, the problem of maximizing the first eigenvalue $\lambda(p,r)$, subject to $(p,r) \in U \times W$, is reduced to the problem of maximizing $\mu(r)$, $r(x) \in W$, where

$$\mu(r) = \inf_{y \in V} G[r,y].$$

Now we shall solve the following problems:

Problem 1. *Find a function* $r_0 \in W$ *such that*

$$\mu(r) \leq \mu(r_0), \quad \forall r(x) \in W.$$

Problem 2. *Find a pair of functions* $(p_0, r_0) \in U \times W$ *for which*

$$\lambda(p, r) \leq \lambda(p_0, r_0), \quad \forall (p(x), r(x)) \in U \times W.$$

For this purpose, first we prove the following lemma.

Lemma 1. *If* $(p, r) \in U \times W$, *then there exist functions* $\overline{p} \in U$ *and* $\overline{r} \in W$ *which are symmetric with respect to* $\frac{l}{2}$ *and such that*

$$\lambda(p, r) \leq \lambda(\overline{p}, \overline{r}).$$

Proof. For any $p \in U$ and $r \in W$ we define the functions:

$$\overline{p}(x) = \frac{p(x) + p(l - x)}{2}, \quad r(x) = \frac{r(x) + r(l - x)}{2}.$$

Having in mind that

$$\int_0^l p(x)\,dx = \int_0^l p(l - x)\,dx, \quad \int_0^l r(x)\,dx = \int_0^l r(l - x)\,dx,$$

we deduce that $\overline{p} \in U$ and $\overline{r} \in W$.

Let $\overline{y}$ be the first eigenfunction corresponding to $\lambda(\overline{p}, \overline{r})$. Evidently $\overline{y}$ is symmetric with respect to $\frac{l}{2}$.

Then,

$$\int_0^l p(x)\overline{y}'(x)^2\,dx = \int_0^l p(l - x)\overline{y}'(l - x)^2\,dx = \int_0^l \overline{p}(x)\overline{y}'(x)^2\,dx$$

and, similarly,

$$\int_0^l r(x)\overline{y}(x)^2\,dx = \int_0^l \overline{r}(x)\overline{y}(x)^2\,dx.$$

Finally,

$$\lambda(p, r) \leq R(\overline{y}) = \lambda(\overline{p}, \overline{r}).$$

■

2. Optimal conditions

In this section we will give some sufficient conditions for the fulfillement of the optimal solutions of the setting problems. Like in [4], we need the following optimization problem:

Problem 3. *Minimize the functional* $\int_0^l F_0(x, r(x))\,dx$, *where* $m \leq r(x) \leq M$,
$0 \leq m \leq M$, *subject to the constraints:*

$$\int_0^l F_i(x, r(x))\,dx = K_i\ , \quad i = 1, \ldots, N,$$

and F_0, $F_1, \ldots,$ F_N *are given continuous functions in* $[0, l] \times [m, M]$, *and* K_0, $K_1, \ldots,$ K_N *are given constants.*

The following theorem maximizes the above functional [5]:

Theorem 1. *Let* $r_0(x)$ *be the solution of Problem* 3. *Then there are constants (Lagrange multipliers)* $\nu_0 \geq 0$, $\nu_1, \ldots,$ ν_N, *not all zero, such that for all* $x \in [0, l]$:

$$\min_{r \in [m,M]} \{\nu_0 F_0(x, r) + \nu_1 F_1(x, r) + \cdots + \nu_N F_N(x, r)\} \tag{3}$$

$$= \nu_0 F_0(x, r_0(x)) + \nu_1 F_1(x, r_0(x)) + \cdots + \nu_N F_N(x, r_0(x)).$$

Conversely, if $r_0(x)$, ν_0, $\nu_1, \ldots,$ ν_N *satisfy* (3) *with* $\nu_0 \geq 0$ *and the conditions*

$$\int_0^l F_i(x, r(x))\,dx = K_i\ , \quad i = 1, \ldots, N,$$

hold for $r(x) \equiv r_0(x)$ *then* $r_0(x)$ *is a solution of Problem* 3.

We shall use the second part of Theorem 1. We look for the function $r_0(x) \in W$ that fulfills the conditions of the theorem. The following two lemmas give the relation between Theorem 1 and the optimal solutions of Problem 1 and Problem 2. These two lemmas will give us the sufficient optimal conditions for which any given function r_0 (a couple of functions (p_0, r_0)) is a solution of Problem 1 (Problem 2).

Lemma 2. *Let $r_0(x) \in W$ and $y_0(x) \in V$ be a minimizer of functional $G[r_0, y]$ over V. If*

$$\int_0^l r_0(x) y_0^2(x)\, dx \leq \int_0^l r(x) y_0^2(x)\, dx, \quad \forall r(x) \in W, \tag{4}$$

then $r_0(x)$ is a solution of Problem 1.

Proof. For any $r(x) \in W$ we have, according to (4):

$$\mu(r) = \inf_{y \in V} G[r, y] \leq G[r, y_0] \leq G[r_0, y_0] = \inf_{y \in V} G[r_0, y] = \mu(r_0).$$

∎

Lemma 3. *Let $(p_0(x), r_0(x)) \in U \times W$ and $y_0(x) \in V$ be any eigenfunction of the problem (1), (2) with $p(x) \equiv p_0(x), r(x) \equiv r_0(x)$ corrresponding to the first eigenvalue. If*

$$\int_0^l r_0(x) y_0^2(x)\, dx \leq \int_0^l r(x) y_0^2(x)\, dx, \tag{5}$$

$$\int_0^l p_0(x) y_0'^2(x)\, dx \geq \int_0^l p(x) y_0'^2(x)\, dx, \tag{6}$$

for every couple $(p, r) \in U \times W$, then $(p_0(x), r_0(x))$ is a solution of Problem 2.

Proof. From the inequalities (5),(6) we obtain:

$$\lambda(p, r) = \inf_{y \in V} R(y) \leq R(y_0) = \frac{\int_0^l y_0''^2\, dx + \int_0^l p y_0'^2\, dx}{\int_0^l r y_0^2\, dx}$$

$$\leq \frac{\int_0^l y_0''^2\, dx + \int_0^l p_0 y_0'^2\, dx}{\int_0^l r_0 y_0^2\, dx} = \lambda(p_0, r_0).$$

Remark 1. The conditions (5),(6) are necessarry in order to (p_0, r_0) be a solution of the Problem 2. In fact, let (p_0, r_0) be a solution of the Problem 2. Then p_0 is a maximizer of the first eigenvalue of (1),(2) with $r(x) \equiv r_0(x) \in W$. But the set U is convex, consequently

$$\int_0^l [p(x) - p_0(x)] y_0'^2\, dx \leq 0, \quad \forall p(x) \in U.$$

Similarly, since $r_0(x)$ maximizes the first eigenvalue of (1),(2) with $p(x) \equiv p_0(x) \in U$ and the set W is convex, we get:

$$\int_0^l [r(x) - r_0(x)] y_0^2 \, dx \geq 0, \quad \forall r(x) \in W.$$

■

3. Solutions of the setting problems

Problem 1 will be solved by finding a function $r_0(x) \in W$ and $y_0(x) \in V$ fulfilling the conditions of Lemma 2. Respectively, Problem 2 will be solved by finding a couple of functions $(p_0(x), r_0(x)) \in U \times W$ and $y_0(x) \in V$, satisfying the conditions of Lemma 3. We define the function $r_0(x) \in W$:

$$r_0(x) = \begin{cases} M, & x \in [0, a_0], \\ 0, & x \in (a_0, l - a_0), \\ M, & x \in [l - a_0, l], \end{cases}$$

where $a_0 < \frac{l}{2}$. Because of the definitions of B, M and $r_0(x) \in W$, $a_0 = \frac{B}{2M}$.

Theorem 2. *Let*

$$m_0 = \inf_{y \in V} G[r, y] = \inf_{y \in V} \frac{\int_0^l y''^2 \, dx + A \max_{x \in (0,l)} |y'|^2}{\int_0^l r_0 y^2 \, dx}.$$

Then, $m_0 = G[r_0, y_0]$, where $y_0 \in V$ is a nonnegative function, which is convex for $x \in [0, l]$ and symmetric with respect to $\frac{l}{2}$. In addition, $b_0 = \frac{k}{\sqrt[4]{m_0 M}}$, where k is the smallest positive solution of the equation

$$cos k . ch k + 1 = 0.$$

and m_0 is the solution of the equation

$$\left(a_0 \sqrt[4]{m_0 M} - k\right)^4 + \left(k + \frac{3}{c}\right) \left(a_0 \sqrt[4]{m_0 M} - k\right)^3$$

$$+ \frac{3}{c} \left(k + \frac{1}{c}\right) \left(a_0 \sqrt[4]{m_0 M} - k\right)^2 + \frac{3k}{c^2} \left(a_0 \sqrt[4]{m_0 M} - k\right) - \frac{3A}{2} = 0,$$

where $c = \cot k + \coth k$.

P r o o f. (i) First we shall prove that $y_0(x)$ is nonnegative and symmetric function with respect to $\frac{l}{2}$.

Let $y_0(x) \in V$ be such that $m_0 = G[r_0, y_0]$.

Then if we put $z_1(x) = |y_0(x)|\ \forall x \in [0, l]$, it is easy to see that $G[r_0, y_0] = G[r_0, z_1]$. Hence, we assume without loss of generality that $y_0(x) \geq 0\ \forall x \in [0, l]$.

Let us consider $z_2(x) = y_0(l - x)$. Obviously, $G[r_0, y_0] = G[r_0, z_2]$. Then $G[r_0, y_0] = G[r_0, \overline{y}_0]$, where $\overline{y}_0(x) = \frac{1}{2}[y_0(x) + y_0(l - x)]$. Therefore we can conclude that $y_0(x)$ is a symmetric function with respect to $\frac{l}{2}$.

(ii) Next we prove that y_0 is convex and we determine for which $x \in [0, l]$ $\max_{x\in[0,l]} |y_0'|^2$ is accessible.

We suppose that there exists an interval $(\xi_1, \xi_2) \subset [0, l]$ in which y_0 is concave. Then, consider the function

$$z_3(x) = \begin{cases} y_0(\xi_1) + y_0'(\xi_1)(x - \xi_1), & x \in (\xi_1, \xi], \\ y_0(\xi_2) + y_0'(\xi_2)(x - \xi_2), & x \in (\xi, \xi_2), \\ y_0(x), & otherwise, \end{cases}$$

and ξ is a cross point of the two lines $p_1 : y = y_0(\xi_1) + y_0'(\xi_1)(x - \xi_1)$ and $p_2 : y = y_0(\xi_2) + y_0'(\xi_2)(x - \xi_2)$. We can easy verify that $G[r_0, y_0] \geq G[r_0, z_3]$, which is a contradiction with the relation $m_0 = G[r_0, y_0]$. This reasonings are true for every interval $(\xi_1, \xi_2) \subset [0, l]$ in which y_0 is concave. Therefore, $y_0(x)$ is convex for $x \in [0, l]$.

Moreover, a similar argument gives an inequality $G[r_0, y_0] \geq G[r_0, z_4]$ for the symmetric about $\frac{l}{2}$ function $z_4(x)$, where

$$z_4(x) = \begin{cases} y_0(x), & x \in [0, a_0], \\ y_0(a_0) + y_0'(a_0)(x - a_0), & x \in (a_0, \frac{l}{2}], \\ z_4(l - x), & x \in (\frac{l}{2}, l], \end{cases}$$

with $a_0 \in (0, \frac{l}{2})$.

Then, without loss of generality, we consider that $y_0(x)$ is symmetric with respect to $\frac{l}{2}$ and is linear in the intervals $(a_0, \frac{l}{2})$ and $(\frac{l}{2}, l - a_0)$.

Let $b_0 \in (0, \frac{l}{2})$ be such that

$$y_0'(b_0) = \max_{x\in[0,l]} |y_0\prime(x)| \ , \quad y_0'(x) < \max_{[0,l]} |y_0'(x)| \ \ \forall x \in [0, b_0).$$

Having in mind that y_0 is convex in $[0, \frac{l}{2}]$, i.e. y_0' is nondecreasing in this interval, we conclude that the set of arguments where $\max_{x\in[0,l]} |y_0'(x)|$ is accessible is the interval $[b_0, l - b_0]$, because $y_0(x)$ is symmetric about $\frac{l}{2}$. Evidently $0 < b_0 \leq a_0$.

Remark 2. In order to assure the belonging of y_0 to V we define a smoothness procedure in the following way: For any $\varepsilon > 0$ approximating zero and $x \in [0,l]$, the function

$$y_\varepsilon(x) = \begin{cases} a(x-\frac{l}{2})+b, & x \in (\frac{l}{2}-\varepsilon, \frac{l}{2}+\varepsilon), \\ y_0(x), & otherwise, \end{cases}$$

is such that $y'_\varepsilon(x)$ is continuous because of the suitable choice of the parameters a and b. Moreover, $y_\varepsilon(x) \in V$ and we easily calculate that

$$y_\varepsilon(x) = -\frac{y_0'(b_0)}{2\varepsilon}\left(x-\frac{l}{2}\right)^2 + y_0'(b_0)\left(\frac{l-\varepsilon}{2}-b_0\right) + y_0'(b_0), \quad x \in (\frac{l}{2}-\varepsilon, \frac{l}{2}+\varepsilon).$$

Evidently $\lim_{\varepsilon\to 0} y_\varepsilon(x) = y_0(x)$, $\forall x \in [0,l]$.

This procedure is applicable to every continuous and partualy linear function (for example to the function $z_4(x)$).

(iii) Finally we shall determine b_0 and m_0. Consider the set

$$\mathcal{O} = \left\{ x \in (0,l)\ ,\quad |y_0'(x)| < \max_{[0,l]} |y_0'| \right\}.$$

Then the functional $G[r_0, y_0]$ has the Gateaux derivative $G'[r_0,y_0][z]$ in y_0 for all $z \in V$ and such that $supp\ z \subseteq \mathcal{O}$.

More detailed but simple calculations give us

$$G'[r_0,y_0][z] = 2\left(\int_0^l r_0 y_0^2\,dx\right)^{-1} \cdot \left(\int_0^l y_0'' z''\,dx - G[r_0,y_0]\int_0^l r_0 y_0 z\,dx\right).$$

From the relations $G[r_0,y_0] = m_0$ and $\int_0^l r_0 y_0 z\,dx = 2M\int_0^{a_0} y_0 z\,dx$, we obtain that $y_0(x)$ satisfies the equation

$$y_0^{(4)} - m_0 M y_0 = 0,\ x \in \mathcal{O}, \tag{7}$$

if $y_0(b_0)y_0'''(b_0) - y_0'(b_0)y_0''(b_0) = 0$.

After integration we have (see Remark 2):

$$y_0(x) = \begin{cases} C_1(\cos\sqrt[4]{m_0M}x - \cosh\sqrt[4]{m_0M}x) & \\ +C_2(\sin\sqrt[4]{m_0M}x - \sinh\sqrt[4]{m_0M}x), & x \in [0,b_0], \\ y_0(b_0) + y_0'(b_0)(x-b_0), & x \in (b_0, \frac{l}{2}], \\ y_0(l-x), & x \in (\frac{l}{2}, l]. \end{cases}$$

Besides that y_0 is convex and $\max|y_0'|$ is attained in b_0, consequently

$$y_0''(b_0) = 0\ ,\quad C_1 = -C_2\frac{\sin\sqrt[4]{m_0M}b_0 + \sinh\sqrt[4]{m_0M}b_0}{\cos\sqrt[4]{m_0M}b_0 + \cosh\sqrt[4]{m_0M}b_0)}$$

and $\sqrt[4]{m_0 M} b_0$ is equal to the first positive solution $k = 1.87510407$ of the equation $\cos k . \cosh k = -1$, i.e.

$$b_0 = \frac{k}{\sqrt[4]{m_0 M}}.$$

Plugging y_0 in $m_0 = G[r_0, y_0]$, we get the equation

$$(8) \qquad (a_0 - b_0)^3 + 3\frac{y_0(b_0)}{y_0'(b_0)}(a_0 - b_0)^2 + 3\frac{y_0^2(b_0)}{y_0'^2(b_0)}(a_0 - b_0) = \frac{3A}{2m_0 M},$$

which is equivalent to

$$(a_0 t - k)^4 + (k + \frac{3}{c})(a_0 t - k)^3 + \frac{3}{c}(k + \frac{1}{c})(a_0 t - k)^2 + \frac{3k}{c^2}(a_0 t - k) - \frac{3A}{2} = 0,$$

where $t = \sqrt[4]{m_0 M}$, $c = \cot k + \coth k$.

The last equation has a unique positive solution and we infer that m_0 is uniquely determined. ■

Theorem 3. *The function $r_0(x)$ is a solution of Problem 1.*

Proof. Setting $\nu_0 = 1$ and $\nu_1 = -y_0^2(a_0)$ we get for all $x \in [0, l]$ that $r_0(x)$ satisfies the sufficiency part of Theorem 1:

$$\min_{0 \le r \le M} \{y_0^2(x) r - y_0^2(a_0) r\} = \{y_0^2(x) - y_0^2(a_0)\} r_0(x).$$

Consequently, integrating the last equality we obtain:

$$\int_0^l r(x) y_0^2(x)\, dx \ge \int_0^l r_0(x) y_0^2(x)\, dx, \quad \forall r(x) \in W.$$

Then, in view of Lemma 2, it follows that r_0 is a solution of Problem 1.

Next, let us introduce the function

$$\varphi(x) = \frac{1}{2}(a_0 - b_0)^2 + \frac{3y_0(b_0)}{2y_0'(b_0)}(a_0 - b_0) + \frac{y_0^2(b_0)}{y_0'^2(b_0)} - \frac{y_0(b_0)}{y_0'(b_0)}(x - b_0) - \frac{1}{2}(x - b_0)^2$$

and prove the next theorem:

Theorem 4. *The pair of functions $(p_0(x), r_0(x)) \in U \times W$, where*

$$p_0(x) = \begin{cases} \varphi(x), & b_0 \le x \le a_0, \\ p_0(l - x), & l - a_0 < x < l - b_0, \\ 0, & \textit{otherwise}, \end{cases}$$

is a solution of Problem 2.

Proof. Using (8), it is easily verified that $p_0(x) \in U$. At that, the condition (6) from Lemma 3 is fulfilled:

$$\int_0^l p(x) y_0'^2(x)\, dx \le A \max_{x\in[0,l]} |y_0'(x)|^2 = A y_0'^2(b_0)$$

$$= y_0'^2(b_0) \int_0^l p_0(x)\, dx = \int_0^l p_0(x) y_0'^2(x)\, dx, \quad \forall p(x) \in U.$$

In order to apply Lemma 3, it remains to show that y_0 is an eigenvalue of the problem

$$y^{(4)} - (p_0 y')' - \lambda r_0 y = 0 \ , \quad x \in (0,l),$$
$$y(0) = y'(0) = y(l) = y'(l) = 0,$$

corresponding to the first eigenvalue, i.e. y_0 satisfies the equation

$$y^{(4)} - (p_0 y')' - m_0 r_0 y = 0 \ , \quad x \in (0,l). \tag{9}$$

But when $x \in [b_0, a_0] \bigcup [l - a_0, l - b_0]$, it is easy to see that (9) holds because of (8). Taking account that for $x \in [0, b_0) \bigcup (l - b_0, l]$ y_0 satisfies the equation (7), we obtain the result we wanted. ■

Remark 3. Let us mention that the method used here holds if the boundary conditions (2) are replaced with certain other boundary conditions. For example, if we consider the problem

$$y^{(4)} - (p(x)y')' - \lambda r(x) y = 0 \ , \quad x \in (0,l),$$

$$y(0) = y''(0) = y(l) = y''(l) = 0,$$

completely analogous considerations are valid.

Notice that when the boundary conditions are symmetric, the functions $p_0(x)$ and $r_0(x)$ are symmetric. By contrast, when the boundary conditions are not symmetric, the functions $p_0(x)$ and $r_0(x)$ are not in general symmetric.

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