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A Note on Universal and Non-Universal Invertible Selfmappings [1]

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In this note it is shown that a sequence of left quasi-inverses of a densely universal sequence of selfmappings on a metrizable Baire space without isolated points is also densely universal. Furthermore, a pair of strong results on universal and non-universal invertible linear operators on a Banach space due to Herrero and Kitai are extended to a nonlinear setting.

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1. Introduction and notations

In the last few years a number of relevant results about universal operators on a topological vector space (specially on a Banach/Hilbert space) has been proved. The aim of this work is to extend some of those results to universal selfmappings or universal sequences of mappings in a setting which is not necessarily linear. Specifically, we study in Section 2 the inversion of universal sequences and selfmappings; we also consider sequences of quasi-inverses as well as invertible selfmappings T such that neither T nor T are universal. In particular, two strong results due to Herrero and Kitai are extended. Before this, we need to fix some notations and recall some elementary topological notions and facts.

Throughout this paper, **N** will be the set of positive integers and the symbol *card*(A) will stand for the cardinality of a set A. If X is a topological

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space and $A \subset X$, then A^{o} = the interior of A and $\overline{A}$ = the closure of A. $\mathbb{C}$ denotes the complex plane.

Let us consider the following rather general notion of universality, which can be found in [4]. Let X and Y be topological spaces and $(T_n) \subset C(X,Y) :=$ {continuous mappings from X into Y} be a sequence of continuous mappings. Then an element $x \in X$ is called universal for (T_n) if the set $\{T_n x : n \in \mathbb{N}\}$ is dense in Y. We restrict ourselves to countable families of mappings. The set of universal elements for (T_n) is denoted by $U((T_n))$. If this set is not empty then (T_n) is called universal. If this is so then Y is, trivially, separable. (T_n) is said to be densely universal whenever $U((T_n))$ is dense in X. Assume that $T \in C(X) := C(X,X) =$ {continuous selfmappings of X}. Then an element $x \in X$ is called universal for T, if the orbit $O(x,T) := \{T^n x : n \in \mathbb{N}\}$ (where $T^1 = T$, $T^2 = T \circ T, \ldots$) is dense, i.e., if x is universal for the sequence of iterates (T^n). Note that x is universal for T if and only if there is no closed T-invariant proper subset in X containing x. Denote $U(T) = U((T^n))$. T is called universal (densely universal) whenever (T^n) is universal (densely universal, resp.). If T is universal, then it is densely universal because each T^m ($m \in \mathbb{N}$) has dense range and $O(x,T) \subset U(T)$.

If X and Y are topological vector spaces and $(T_n) \subset L(X,Y) := \{T \in C(X,Y) : T \text{ is linear}\}$ –and, mainly, if X is a topological space and $(T_n) = (T^n)$ for a single $T \in L(X) := L(X,X) = \{T \in C(X) : T \text{ is linear}\} =$ {operators on X} – then the word "hypercyclic" is preferred to "universal" (see, for instance, [1] and [3]). In order to cause no confusion, we keep the denomination "universal" in the remaining of the paper. We refer the reader to [5] for a good general overview of the knowledge on universality up to date.

A topological space X is T_1 if and only if for each pair of points $x, y \in X$ there is a open subset U with $x \in U$ and $y \notin U$. Then X is T_1 if and only if each singleton set $\{x\}$ is closed. We will make use of the following four elementary lemmas.

Lemma 1. *Let X be a topological space and A, B, C three subsets of X such that $C = A \cup B$, A is closed and $A^{\text{o}} = \emptyset = B^{\text{o}}$. Then $C^{\text{o}} = \emptyset$.*

Lemma 2. *Let X be a topological space and A, $B \subset X$, in such a way that B is closed with empty interior. Then $(\overline{A})^{\text{o}} = (\overline{A \setminus B})^{\text{o}}$. In particular, if A is dense in X and B is closed with empty interior, then $A \setminus B$ is dense in X.*

Lemma 3. *Let X be a topological space. The following conditions are equivalent:*

(a) *X is a T_1-space without isolated points.*

(b) *Each singleton set* $\{x\}$ *is closed and has empty interior.*

(c) *Every finite subset of* X *is closed and has empty interior.*

Lemma 4. *If* V *is a nonempty open subset of a* T_1*–space* X *without isolated points, then* V *has infinitely many points.*

We point out here that the inclusion of the point x in the orbit of a selfmapping is irrelevant in the definition of universality for topological vector spaces (and this is the usual manner to do it in this setting), but could cause some trouble in a nonlinear setting. The key is to consider T_1–spaces without isolated points. Indeed, it is trivial if X is such a space then a point $x \in X$ is universal for a selfmapping $T \in C(X)$ if and only if the "completed" orbit $\{x\} \cup O(x,T) = \{x, Tx, T^2, ...\}$ is dense. It is not difficult to find examples of topological spaces where the latter equivalence does not hold.

2. Universal inverses, quasi-inverses and invariant subsets

If X is a topological space and $T \in C(X)$, then T is said to be *invertible* whenever it is one-one and onto and $T^{-1} \in C(X)$. In a linear setting, the inverse of a universal operator is universal. Specifically, it is well known that if X is an F–space (= complete linear metric space) and $T \in L(X)$ is invertible, then T is universal if and only if T^{-1} is; moreover, in such a case, there is a vector z such that $\overline{O(z,T)} = X = \overline{O(z,T^{-1})}$ (see [1, III.5.3], [2, 2.5] [3, Section 1], [5] and [8, 2.2]). Note that X is a Baire T_1–space without isolated points, and the fact "T (or T^{-1}) is universal" forces X to be separable, hence second-countable. Does the same property hold without linearity? To start with, we show in Theorem 6 that, as a consequence of the following universality criterion due to Grosse-Erdmann (see [4, Satz 1.2.2 and its proof] and also [7, Lemma 1.2], [3, 1.2], [5, Theorem 1]), the answer is affirmative under certain hypotheses, even for sequences.

Lemma 5. *Suppose that* X *is a Baire space and* Y *is second-countable. Assume that* $(T_n) \subset C(X,Y)$*. Then the following assertions are equivalent:*

(a) $U((T_n))$ *is residual in* X*.*

(b) (T_n) *is densely universal.*

(c) *To every pair of nonempty open subsets* U *of* X *and* V *of* Y *there exists some* $n \in \mathbb{N}$ *with* $T_n(U) \cap V \neq \emptyset$*.*

If one of these conditions holds, then $U((T_n))$ *is a dense* G_δ *(hence residual) subset of* X*.*

Theorem 6. *Suppose that X and Y are second-countable Baire spaces. Assume that $(T_n) \subset C(X,Y)$ and that every T_n has continuous inverse $T_n^{-1} : Y \to X$. Assume also that $T \in C(X)$ is invertible. Then we have:*

(a) *(T_n) is densely universal if and only if (T_n^{-1}) is densely universal. In such a case, there is a point $x \in X$ such that $\overline{\{T_n x : n \in N\}} = Y = \overline{\{T_n^{-1} x : n \in N\}}$.*

(b) *T is universal if and only if T^{-1} is universal. In such a case, there is a point $x \in X$ such that $\overline{O(x,T)} = X = \overline{O(x,T^{-1})}$.*

Proof. As for (a), just apply Lemma 5 and the fact that, for every $n \in \mathbb{N}$ and every pair of subsets $A \subset X$ and $B \subset Y$, $T_n(A) \cap B \neq \emptyset$ if and only if $A \cap T_n^{-1}(B) \neq \emptyset$. The second assertion of (a) comes from the fact that the intersection of the residual sets $U((T_n))$ and $U((T_n^{-1}))$ is also residual. Part (b) is derived by taking into account that a continuous selfmapping is universal if and only if it is densely universal. ∎

We will be able to establish an improvement in the case that (T_n) be a sequence of selfmappings of a metrizable space. In fact, if (T_n) is densely universal and (S_n) is a sequence of "left quasi-inverses" of (T_n), then (S_n) is densely universal, see Theorem 8 below. Before this, we need a definition and the next refinement of Lemma 5.

Lemma 7. *Suppose that X is a Baire space and that Y is a second-countable T_1-space without isolated points. Then the following assertions are equivalent:*

(a) *(T_n) is densely universal.*

(b) *To every pair of nonempty open subsets U of X and V of Y there exist infinitely many $n \in \mathbb{N}$ with $T_n(U) \cap V \neq \emptyset$.*

Proof. By Lemma 5, (b) implies (a). Fix now open subsets U of X and V of Y. If (a) holds, then there exists $n_1 \in \mathbb{N}$ such that $T_{n_1}(U) \cap V \neq \emptyset$. By Lemmas 2-3, the sequence $\{T_n : n > n_1\}$ is densely universal, hence there exists $n_2 > n_1$ such that $T_{n_2}(U) \cap V \neq \emptyset$. Continuing this process gives (b). ∎

If X is a topological space, we say that a family $\mathcal{V}$ of nonempty open subsets is a *quasi-basis* for (the topology of) X if and only if for each nonempty open subset A of X, there is $V \in \mathcal{V}$ with $V \subset A$. For instance, if $\{q_j : j \in \mathbb{N}\}$ is an enumeration of all rational numbers in the real line $X = \mathbb{R}$, then the family $\mathcal{V} = \{(q_j - 2^{-j}, q_j + 2^{-j}) : j \in \mathbb{N}\}$ is a quasi-basis for X but not a basis, because $\bigcup_{V \in \mathcal{V}} V \neq \mathbb{R}$. In fact, a family $\mathcal{V}$ of subsets of a topological space X is a quasi-basis if and only if each member $V \in \mathcal{V}$ is a nonempty open subset and the set $\{x_V : V \in \mathcal{V}\}$ is dense for each choice of points $x_V \in V$ $(V \in \mathcal{V})$.

Theorem 8. *Suppose that X is a metrizable Baire space without isolated points and that (T_n) is a densely universal sequence in $C(X)$. Assume that $(S_n) \subset C(X)$ is a sequence satisfying the following property: There is a quasi-basis $\mathcal{V}$ for X and a distance d compatible with the topology of X such that*

$$\lim_{n\to\infty} \sup_{x\in V} d(x, S_n T_n x) = 0 \quad \textit{for all} \quad V \in \mathcal{V}.$$

Then (S_n) is also densely universal.

Proof. Since (T_n) is densely universal, it is universal, so X is separable. Hence X is a second-countable Baire T_1–space without isolated points. Fix a pair of nonempty open subsets A, B of X. By Lemma 5, it is enough to show that there exists $m \in \mathbb{N}$ with $S_m(A) \cap B \neq \emptyset$. Pick a set $V \in \mathcal{V}$ with $V \subset B$ and take a ball $B(b,r) := \{x \in X : d(x,b) < r\} \subset V$. From Lemma 7 there are a sequence $n_1 < n_2 < n_3 < \ldots$ of positive integers and a sequence of points $(a_j) \subset A$ such that $a_j \in T_{n_j}(B(b,r/2))$, so $a_j = T_{n_j} b_j$ with $d(b_j, b) < r/2$ $(j \in \mathbb{N})$. By hypothesis,

$$\lim_{j\to\infty} \sup_{x\in B(b,r/2)} d(x, S_{n_j} T_{n_j} x) = 0.$$

Then there is $j_0 \in \mathbb{N}$ such that

$$d(b_{j_0}, S_{n_{j_0}} T_{n_{j_0}} b_{j_0}) < r/2,$$

whence

$$d(S_m T_m c, b) \leq d(S_m T_m c, c) + d(c, b) < r/2 + r/2 = r,$$

where $m = n_{j_0}$, $c = b_{j_0}$. Hence $S_m(T_m c) \in B(b,r) \subset V \subset B$ and $T_m c \in A$, so $S_m(A) \cap B \neq \emptyset$, as required. ∎

As a consequence, we derive from Theorem 8 that, under the same hypotheses on X, if $T \in C(X)$ is universal and there is $S \in C(X)$ such that $ST = I =$ the identity operator on X, then S is also universal. Nevertheless, Theorem 8 does not extend to right inverses. Indeed, the differentiation operator $T = D : f \mapsto f'$ on the Fréchet space $X = H(\mathbb{C})$ of entire functions is universal due to MacLane's theorem [9], and the integral operator $S : H(\mathbb{C}) \to H(\mathbb{C})$ given by

$$(Sf)(z) = \int_0^z f(t)\, dt$$

satisfies $TS = I$, but S is clearly non-universal because $(S^n f)(0) = 0$ for all $n \in \mathbb{N}$ and all $f \in H(\mathbb{C})$.

In [6, Theorem 1] it is shown the following complementary result on inverses: If T is an invertible operator acting on a Banach space X such that there is $x \in X$ satisfying $\overline{O(x,T) \cup O(x,T^{-1})} = X$, then T (and, consequently, T^{-1}) is universal. We next show how this can be translated to a nonlinear setting. It will be done after recalling the notion of "limit point" of a sequence in a general topological space. Assume that $\sigma = (x_n)$ is a sequence in a topological space X and that $x \in X$. Then x is called a limit point of σ whenever the following is fulfilled: Given an open neighbourhood V of x, there exist infinitely many $n \in \mathbb{N}$ with $x_n \in V$. Let us denote by $LP(\sigma)$ the set of limit points of σ. It is clear that $LP(\sigma) \subset \overline{\{x_n : n \in \mathbb{N}\}}$, $LP(\sigma)$ is closed and $LP_1(\sigma) := \{x \in X :$ there exists a strictly increasing sequence $n_1 < n_2 < n_3 < \ldots$ in $\mathbb{N}$ such that $x_{n_k} \to x\ (k \to \infty)\} \subset LP(\sigma)$. If X is first-countable then $LP(\sigma) = LP_1(\sigma)$.

Lemma 9. *Assume that X is a topological space and $T \in C(X)$. If $x \in X$ and $\sigma = (T^n x)$ then $LP(\sigma)$ is invariant under T. If, in addition, T is invertible, then $LP(\sigma)$ is also invariant under T^{-1}.*

Proof. Suppose that $z = Ty$ with $y \in LP(\sigma)$ and that V is an open neighbourhood of z. Since T is continuous, $T^{-1}(V)$ is an open neighbourhood of y, so there are positive integers $n_1 < n_2 < n_3 < \ldots$ such that $T^{n_j}x \in T^{-1}(V)$ $(j \in \mathbb{N})$. Hence there are infinitely many positive integers n (namely, $n = n_1 + 1, n_2 + 1, \ldots$) with $T^n x \in V$, that is, $z \in LP(\sigma)$, as required. The second part is analogous: Consider this time $z = T^{-1}y$ with $y \in LP(\sigma)$ and take into accout that $T(V)$ is open and contains y. The required numbers n are now $n_1 - 1, n_2 - 1, \ldots$ because n_1 may trivially be chosen not less than 2. ∎

Theorem 10. *Assume that T is an invertible continuous selfmapping of a T_1-space X without isolated points and that there is $x \in X$ with $\overline{O(x,T) \cup O(x,T^{-1})}$ $= X$. Then either T or T^{-1} is universal. If, in addition, X is Baire and second-countable, then T and T^{-1} are universal.*

Proof. The second part is a consequence of Theorem 6. Let us prove the first part. By hypothesis,

$$X = \overline{O(x,T) \cup O(x,T^{-1})},$$

so from Lemma 1 we get that either $C := \overline{O(x,T)}$ or $D := \overline{O(x,T^{-1})}$ has nonempty interior. If $C^{\circ} = \emptyset$ then Lemma 2 says us that $X \setminus C$ $(\subset D)$ is dense, hence D is dense and, therefore, T^{-1} is universal.

Assume finally that $C^{\mathrm{o}} \neq \emptyset$. Consider $F := LP((T^n x))$. Then F is closed and invariant under T and T^{-1} by Lemma 9. In addition, $F \subset \overline{O(x,T)}$. If we prove that $x \in F$ then we would obtain that $O(x,T) \subset F$ and $O(x,T^{-1}) \subset F$ due to the invariance of F. But F is closed, so we would arrive to $X = \overline{O(x,T) \cup O(x,T^{-1})} \subset F \subset \overline{O(x,T)}$, hence $\overline{O(x,T)} = X$ and T would be universal. Thus, it suffices to show that $x \in F$. For this, take a nonempty open subset $V \subset C = \overline{O(x,T)}$. Fix any $z \in V$. Then $z \in \overline{O(x,T)}$ and there exists $m \in \mathbb{N}$ with $T^m x \in V$. If we are able to prove that $T^m x \in F$, then the T^{-1}-invariance of F would yield $x \in F$ and we are done.

Let us see that $T^m x \in F$. Fix an open subset U of X with $T^m x \in U$. Since $U \cap V \subset (\overline{O(x,T)})^{\mathrm{o}}$, an application of Lemma 2 on $A = O(x,T)$, $B = \{Tx, T^2 x, ..., T^{n_1} x\}$ (where $n_1 = m$) gives that $U \cap V \subset \overline{\{T^n x : n > n_1\}}$ (note that B is closed with empty interior by Lemma 3). There is therefore a positive integer $n_2 > n_1$ such that $T^{n_2} x \in U \cap V$, because $U \cap V$ is an open subset containing, for instance, the point $T^{n_1} x$. A new application of Lemma 2, this time with $B = \{Tx, T^2 x, ..., T^{n_2} x\}$, drives us to the existence of a positive integer $n_3 > n_2$ such that $T^{n_3} x \in U \cap V$. This process furnishes infinitely many positive integers $n_1 < n_2 < n_3 < ...$ with $T^{n_j} x \in U \cap V$, so $T^{n_j} x \in U$ $(j \in \mathbb{N})$. This tells us that $T^m x$ is a limit point of $(T^n x)$, as required. ∎

Finally, we study the opposite situation: T is not universal. In [6, Theorem 2] it is established the following assertion: If T is an invertible operator on a Banach space X and T is not universal, then T and T^{-1} have a common nontrivial invariant closed subset. That a subset $F \subset X$ is "nontrivial" means here that $F \neq \emptyset$, $\{0\}$, X. Since "$\{0\}$" makes no sense in a general topological space, we redefine this concept. If X is a set with $card(X) \geq 3$, we will say that a subset $F \subset X$ is *nontrivial* whenever $card(F) \geq 2$ and $F \neq X$. Note that for linear spaces this is stronger than the former notion. We obtain again that the above assertion holds for rather general topological spaces.

Theorem 11. *Let X be a T_1-space without isolated points and that $T \in C(X)$ is universal. We have:*

(a) *If neither T nor T^{-1} are universal, then there is a nontrivial closed subset F which is invariant under T and T^{-1}.*

(b) *If X is Baire second-countable and either T or T^{-1} is not universal, then there is a nontrivial closed subset F which is invariant under T and T^{-1}.*

Proof. Part (b) follows from (a) and Theorem 6. Let us prove (a). Firstly, X has infinitely many points by Lemma 4. If $T = I$, it suffices to take $F = \{x, y\}$ with $x \neq y$ (F is closed by Lemma 3). Assume that $T \neq I$. Then there is $x \in X$ with $Tx \neq x$. Consider the set

$$Z = \overline{O(x,T) \cup \{x\} \cup O(x,T^{-1})}.$$

Note that Z is closed, $card(Z) \geq 2$ (since x, $Tx \in Z$) and Z is invariant under T and T^{-1} (because $O(x,T) \cup \{x\} \cup O(x,T^{-1})$ is invariant under T and T^{-1}, and these mappings are continuous). If $Z \neq X$ then it suffices to take $F = Z$.

Finally, assume that $Z = X$. Then from Lemma 1 it is derived that at least one of the sets $\overline{O(x,T)}$, $\overline{O(x,T^{-1})}$, $\{x\}$ has nonempty interior (the last case is not possible). Thus, we can leave from the fact that $\overline{O(x,T)}$ has nonempty interior (the case $(\overline{O(x,T^{-1})})^{\mathrm{o}} \neq \emptyset$ is analogous), so there is a nonempty open subset $V \subset \overline{O(x,T)}$. Take

$$F := LP((T^n x)).$$

Then we have:

- F is closed.
- F is invariant under T and T^{-1} by Lemma 9.
- $F \neq X$ because $F \subset \overline{O(x,T)}$, and $\overline{O(x,T)} \neq X$ since T is not universal.
- $card(F) \geq 2$ because V is infinite (by Lemma 4) and, as we will see immediately, $V \subset F$.

To see that $V \subset F$, fix $z \in V$ and an open subset U with $z \in U$. Then the same argument of the final part of the proof of Theorem 10 furnishes infinitely many positive integer $n_1 < n_2 < n_3 < \ldots$ such that $T^{n_j}x \in U \cap V$ $(j \in \mathbb{N})$, hence $T^n x \in U$ for infinitely many $n \in \mathbb{N}$, that is, $z \in F$. But this holds for any $z \in V$. The proof is finished. ■

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