Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 15, 2001, Fasc. 3-4

Homogeneous Primary Components in Abelian Group Rings

Peter Danchev

Presented by P. Kenderov

Let $S(RG)$ be the p-primary component in a group ring RG of an abelian group G over an abelian ring R with 1 of prime characteristic p. In the present paper a criterion is founded for $S(RG)$ to be homogeneous. The isomorphic class of $S(RG)$ in these cases is also provided. The established necessary and sufficient condition improves, in some instance, a result due to T. Mollov published in Publ. Math. Debrecen (1971).

AMS Subj. Classification: Primary 20C07; Secondary 20K10, 20K21, 16U60

Key words: homogeneous abelian groups, group rings, isomorphic type

1. Introduction and known facts

Throughout this work, let p be a fixed but arbitrary prime and RG be an abelian group ring of a group G over a ring R with identity of characteristic p. Denote by $N(R) = \bigcup_{n=1}^{\infty} R(p^n)$ the nilradical (a Baer's radical) of R, where $R(p^n) = \{x \in R : x^{p^n} = 0\}$ are its p^n-socles, and $G_p = \bigcup_{n=1}^{\infty} G[p^n]$ denotes the p-component of G where $G[p^n]$ are its p^n-socles, i.e. $G[p^n] = \{y \in G : y^{p^n} = 1\}$. For the most part, the notations and terminology of [2] will be followed.

In the present paper research a criterion (i.e. a necessary and sufficient condition) is established for $S(RG)$ to be homogeneous when R and G are absolute arbitrary commutative algebraic systems. Now, for the sake of completeness and for convenience of the reader, we shall formulate below the following three statements in this direction, given by the author in [1].

First, we start with the next definition: The ring R is called perfect if $R = R^p$, where $R^p = \{r^p : r \in R\}$. And so, we can state the following theorem.

Theorem. ([1]) *Suppose G is an abelian group and R is an unitary commutative ring with 1 and* $\operatorname{char} R = p$. *Then $S(RG)$ is divisible, if and only if $G_p \neq 1$, G is p-divisible and R is perfect; or $G_p = 1$ and $N(R) = 0$; or $G_p = 1$, $1 \neq G$ is p-divisible and $0 \neq N(R)$ is perfect; or $G = 1$.*

Next, we continue with the following theorem.

Theorem. ([1]) *The group $S(RG)$ is bounded if and only if there is a natural number m with the property that $N(R^{p^m}) = 0$ and $G_p^{p^m} = 1$; or $N(R^{p^m}) \neq 0$ and $G^{p^m} = 1$.*

In the other words, it is easy to see that the last theorem could be restated thus: $S(RG)$ is bounded if and only if $N(R^{p^i}) = 0$ for some $i \in N$ and G_p is bounded; or $N(R^{p^i}) \neq 0$ for every $i \in N$ and G is a bounded p-group.

Any abelian p-group A is called elementary if $A = A[p]$, i.e. $A^p = 1$, i.e. A is a direct sum of cyclics of the same order p.

So, applying the above theorem we derive the next corollary.

Corollary. *$S(RG)$ is elementary if and only if $N(R^p) = 0$ and $G_p^p = 1$; or $N(R^p) \neq 0$ and $G^p = 1$.*

2. Main results

Following [6], we give the next definition.

Definition. A p-primary abelian group will be said to be homogeneous if it is either divisible or a direct sum of cyclic groups of a fixed order p^t where t is a natural.

For the specific notation and terminology, we follow the monograph of L. Fuchs [2]. Our central conclusions are based on the following results of T. Szele (see [5] or [2], p.109, Exercise 7), namely:

Criterion. (Szele, 1951) *Suppose A is an abelian p-group and $A^{p^t} = 1$ for any fixed positive integer t. Then A is homogeneous, i.e. A is a direct sum of cyclics of the same order p^t if and only if for each element a of A is fulfilled: $a \in A^{p^t/o(a)}$, where $o(a)$ is the order of a.*

Now we proceed by proving the following main assertion, which is our goal here.

Theorem. $1 \neq S(RG)$ *is reduced homogeneous no elementary, i.e. it is a direct sum of cyclics of the same order p^t for $2 \leq t \in N$ if and only if the* *following are valid*

$$(*)\qquad G_p = 1,\ G = G^p \ \textit{and}\ R^{p^t}(p) = 0 \ \textit{plus}\ 0 \neq R(p^i) = R^{p^{t-i}}(p^i)$$

for all $i = 1, \ldots, t-1$.

Proof. **"Necessity"**. Assume $G_p \neq 1$. Take $1 \neq g \in G_p$ such that $o(g) = p^t$ and besides choose $1 \neq g_p \in G_p$ so that $o(g_p) = p$. Hence the element $x_g = 1 + g(1 - g_p)$ has order p and moreover lies in $S(RG)$. Indeed, $x_g^p = 1 + g^p(1 - g_p^p) = 1$. Further, by virtue of the Szele theorem and [1] we deduce, $x_g \in S^{p^{t-1}}(RG) = S(R^{p^{t-1}}G^{p^{t-1}})$. But since $g \neq g_p^{-1}$, then x_g is a canonical element and so $g \in G^{p^{t-1}}$, i.e. $g \in G_p^{p^{t-1}}$. Observe that $S^{p^t}(RG) = 1$ obviously implies $G_p^{p^t} = 1$ and thus $g^p \in G_p^{p^t} = 1$, i.e. $g^p = 1$ immedeately. This contradiction yields automatically that $G_p = G[p]$, i.e. $G_p^p = 1$. Apparently $N(R^p) \neq 0$, otherwise the above corollary leads us to $S(RG)$ is elementary, contradicting with the hypothesis.

Let now $1 \neq g \in G$ and $0 \neq r \in R(p)$. Therefore $1 + r(1 - g)$ has order p and it belongs to $S(R^{p^{t-1}}G^{p^{t-1}}) = S^{p^{t-1}}(RG) \neq 1$. Consequently, $g \in G^{p^{t-1}}$ and $r \in R^{p^{t-1}}$, i.e. $G = G^p$ and $R(p) = R^{p^{t-1}}(p)$. Furthermore the first equality implies $G_p = 1$, and the second preliminary theorem ensure $N(R^{p^t}) = 0$, i.e. $R^{p^t}(p) = 0$. By a reason of symmetry we obtain $R(p^i) = R^{p^{t-1}}(p^i)$ for every $i = 1, \ldots, t-1$ using the element $1 + r(1 - a)$, where $0 \neq r \in R(p^i) \setminus R(p^{i-1})$ and $1 \neq a \in G$. The choosing of r is correct since $R(p^i) \neq R(p^{i-1})$; otherwise $R^{p^{i-1}}(p) = 0$ that is false. This proves the first half.

"Sufficiency". For the converse, because $G_p = 1$, we observe that each element of $S(RG)$ can be written in the form $1 + \sum r_g(1 - g)$, where $0 \neq r_g \in N(R)$ and $g, 1 \in \prod$ is the complete set of representatives (i.e. the transversal) of G with respect to $< 1 >$, which simple fact we leave to the reader. Evidently since $S^{p^t}(RG) = 1$ (see [1]) and whence $S(RG) \subseteq S(RG)[p^t]$, then we have $r_g \in R(p^t)$. Moreover, the order of such an element in $S(RG)$ is equal to the maximal exponent of some r_g, when $g \in \prod$ certainly. If all $r_g \in R(p^t) \setminus R(p^{t-1})$, then clearly $y_g = 1 + \sum r_g(1 - g) \in S(RG) = S^{p^t/o(y_g)}(RG)$ because $o(y_g) = p^t$. But now, if $0 \neq r_{g1} \in R(p)$, $0 \neq r_{g2} \in R(p^2) \setminus R(p), \ldots, 0 \neq r_{gj} \in R(p^j) \setminus R(p^{j-1})$ for some $j < t$, then we conclude $o(1 + \sum_{l=1}^{j} r_{gl}(1 - g_l)) = p^j$ for $g_l \in G$ $(1 \leq l \leq j)$. Next, since $r_{gl} \in R^{p^{t-j}}$ for all $1 \leq l \leq j$, we get that $1 + \sum_{l=1}^{j} r_{gl}(1 - g_l) \in S^{p^{t-j}}(RG)$, and thus the second part is verified in accordance with the Szele criterion. Then, the theorem is proved in general. ∎

The following remarks hold.

R e m a r k 1. It is a simple matter to see that $N(R) \neq N^p(R)$; in the remaining case $S(RG)$ must be itself divisible by making use of the first preliminary theorem. Thus $S(RG) = 1$, contrary to our hypothesis.

R e m a r k 2. According to our main result presented above and in view of [3, 4], we may establish the structural type of the homogeneous (reduced or not) group $S(RG)$.

3. Concluding discussion

Our aim in the present article, to derive a criterion when $S(RG)$ is homogeneous by the above minimal restrictions on R and G, however, is finished. That is why, the study of $S(RG)$ in this way is completely exhausted. Moreover, our central theorem extends the result for direct sums of cyclic p-groups proved by Mollov (1971) in Publ. Math. Debrecen (see the bibliography in [0]).

References

[0] P. V. D a n c h e v, Units in abelian group rings of prime characteristic, *Compt. Rend. Acad. Bulg. Sci.*, **48**, No 8 (1995), 5-8.

[1] P. V. D a n c h e v, Topologically pure and basis subgroups in commutative group rings, *Compt. Rend. Acad. Bulg. Sci.*, **48**, No 9-10 (1995), 7-10.

[2] L. F u c h s, *Infinite Abelian Groups*, **I**, Mir, Moscow (1974).

[3] N. A. N a c h e v, Invariants of the Sylow p-subgroup of the unit group of commutative group ring of characteristic p, *Compt. Rend. Acad. Bulg. Sci.*, **47**, No 1 (1994), 9-12.

[4] N. A. N a c h e v, Invariants of the Sylow p-subgroup of the unit group of a commutative group ring of characteristic p, *Commun. Algebra*, **23**, No 7 (1995), 2469-2489.

[5] T. S z e l e, On direct sums of cyclic groups, *Publ. Math. Debrecen*, **2** (1951), 76-78.

[6] D. T a r w a t e r, Homogeneous primary abelian groups, *Proc. Amer. Math. Soc.*, **24** (1970), 154-155.

Plovdiv State University
Department of Mathematics
4000 Plovdiv, BULGARIA

Received: 31.08.2000