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Complete Lifts of Derivations of Special Types to the Tensor Bundle

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The main purpose of this paper is to study the complete lifts of derivations of special types, that is, those of Lie derivations, covariant differentiations and derivations determined by a tensor field of type (1,1).

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1. Introduction

Let M_n be an n-dimensional manifold of class C^∞. Consider the tensor bundle $T^p_q(M_n) = \bigcup_{p\in M_n} T^p_q(P)$ and denote the natural projection $T^p_q(M_n) \to M_n$ by π. Let $x^j, j = 1, ..., n$ be local coordinates in a neighborhood U of a point p of M_n. Then a tensor t of type (p,q) at $p \in M_n$ which is an element of $T^p_q(M_n)$ is expressible in the form $(x^j, t^{i_1\dots i_p}_{j_1\dots j_q}) = (x^j, x^{\overline{j}}), x^{\overline{j}} = t^{i_1\dots i_p}_{j_1\dots j_q}$, $\overline{j} = n+1, ..., n+n^{p+q}$, whose $t^{i_1\dots i_p}_{j_1\dots j_q}$ are components of t with respect to the natural frame ∂_j. We may consider $(x^j, x^{\overline{j}})$ as local coordinates in a neighborhood $\pi^{-1}(u)$ of $T^p_q(M_n)$.

To a transformation of local coordinates of M_n : $x^{j'} = x^{j'}(x^j)$, there corresponds in $T^p_q(M_n)$ the coordinates transformation

$$(1) \qquad \begin{cases} x^{j'} = x^{j'}(x^j), \\ x^{\overline{j}'} = t^{i'_1\dots i'_p}_{j'_1\dots j'_q} = A^{i'_1}_{i_1}\dots A^{i'_p}_{i_{1p}} A^{j_1}_{j'_1}\dots A^{j_q}_{j'_q} t^{i_1\dots i_p}_{j_1\dots j_q} = A^{(i')}_{(i)} A^{(j)}_{(j')} x^{\overline{j}}, \end{cases}$$

where

$$A^{(i')}_{(i)} A^{(j)}_{(j')} = A^{i'_1}_{i_1}\dots A^{i'_p}_{i_{1p}} A^{j_1}_{j'_1}\dots A^{j_q}_{j'_q}, A^{i'}_i = \frac{\partial x^{i'}}{\partial x^i}, A^j_{j'} = \frac{\partial x^j}{\partial x^{j'}}.$$

The Jacobian of (1) is given by the matrix

$$(2)\qquad \left(\frac{\partial x^{J'}}{\partial x^{J}}\right)=\begin{pmatrix}\frac{\partial x^{j'}}{\partial x^{j}} & \frac{\partial x^{j'}}{\partial x^{\bar{j}}}\\ \frac{\partial x^{\bar{j}'}}{\partial x^{j}} & \frac{\partial x^{\bar{j}'}}{\partial x^{\bar{j}}}\end{pmatrix}=\begin{pmatrix}A_{j}^{j'} & 0\\ t_{(k)}^{(i)}\partial_j(A_{(i)}^{(i')}A_{(j')}^{(k)}) & A_{(i)}^{(i')}A_{(j')}^{(j)}\end{pmatrix},$$

where $J=(j,\bar{j})$, $J=1,\dots n+n^{p+q}$, $t_{(k)}^{(i)}=t_{k_1\dots k_q}^{i_1\dots i_p}$.

We denote by $\Im_q^p(M_n)$ the module over $F(M_n)$ of C^∞ tensor fields of type (p,q) ($F(M_n)$ is ring of real-valued C^∞ functions on M_n).

If $\alpha\in\Im_p^q(M_n)$, it is regarded, in a natural way, by contraction, as a function in $T_q^p(M_n)$, which we denote by $\imath\alpha$. If α has the local expression $\alpha=\alpha_{i_1\dots i_p}^{j_1\dots j_q}\partial_{j_1}\otimes\dots\otimes\partial_{j_q}\otimes dx^{i_1}\otimes\dots\otimes dx^{i_p}$ in a coordinate neighborhood $U(x^i)\subset M_n$, then $\imath\alpha$ has the local expression

$$\imath\alpha=\alpha_{i_1\dots i_p}^{j_1\dots j_q}t_{j_1\dots j_q}^{i_1\dots i_p}$$

with respect to the coordinates $(x^i,x^{\bar{i}})$ in $\pi^{-1}(U)$.

We first prove a lemma which will be of use later.

Lemma 1. *Let $\widetilde{X}$ and $\widetilde{Y}$ be vector fields in $T_q^p(M_n)$ such that $\widetilde{X}(\imath\alpha)=\widetilde{Y}(\imath\alpha)$, for any $\alpha\in\Im_p^q(M_n)$. Then, $\widetilde{X}=\widetilde{Y}$, i.e. $\widetilde{X}\in\Im_0^1(T_q^p(M_n))$ is completely determined by its action on functions of type $\imath\alpha$.*

Proof. It is sufficient to show that if $\widetilde{Z}(\imath\alpha)=(\widetilde{X}-\widetilde{Y})(\imath\alpha)=0$, then $\widetilde{Z}$ is zero. If $\widetilde{Z}$ has components $\widetilde{Z}^J$ with respect to the coordinates $(x^j,x^{\bar{j}})$ in $\pi^{-1}(U)$, then we have

$$(3)\qquad \begin{aligned}\widetilde{Z}(\imath\alpha)&=\widetilde{Z}^j\partial_j(\alpha_{i_1\dots i_p}^{j_1\dots j_q}t_{j_1\dots j_q}^{i_1\dots i_p})+\widetilde{Z}^{\bar{j}}\partial_{\bar{j}}(\alpha_{i_1\dots i_p}^{j_1\dots j_q}t_{j_1\dots j_q}^{i_1\dots i_p})\\ &=\widetilde{Z}^j t_{j_1\dots j_q}^{i_1\dots i_p}\partial_j\alpha_{i_1\dots i_p}^{j_1\dots j_q}+\widetilde{Z}^{\bar{j}}\alpha_{i_1\dots i_p}^{j_1\dots j_q}=0\,.\end{aligned}$$

If this holds for any $\imath\alpha\in\Im_p^q(M_n)$, $\partial_j\alpha_{i_1\dots i_p}^{j_1\dots j_q}$ taking any preassigned values at a fixed point, from (3) (homogeneous system of linear equations), we have

$$(4)\qquad \widetilde{Z}^j t_{j_1\dots j_q}^{i_1\dots i_p}=0,\quad \widetilde{Z}^{\bar{j}}=0.$$

From the first equation of (4), it follows that $\widetilde{Z}^j=0$ at all points of $T_q^p(M_n)$ except possibly those at which all the components $t_{j_1\dots j_q}^{i_1\dots i_p}$ are zero: that is, at points of the base space. However the components of $\widetilde{Z}$ are continuous and so $\widetilde{Z}^j$ is also zero at points of the base space.

Thus $\widetilde{Z}^j = 0$ at all points of $\pi^{-1}(U)$. Therefore, taking account of the second equation of (4), we see that $\widetilde{Z}$ is zero in $\pi^{-1}(U)$ and so $\widetilde{Z} = 0$ in $T^p_q(M_n)$. Thus Lemma 1 is proved. ∎

2. Vertical lifts of tensor fields and γ-operator

Let $A \in \Im^p_q(M_n)$. Then there is a unique (see Lemma 1) vector field ${}^V A \in \Im^1_0(T^p_q(M_n))$ such that for $\alpha \in \Im^q_p(M_n)$,

$$ {}^V A(\imath\alpha) = \alpha(A) \circ \pi = {}^V(\alpha(A)), \tag{5} $$

where ${}^V(\alpha(A))$ is the vertical lift of the function $\alpha(A) \in F(M_n)$. We note that, the vertical lift ${}^V f = f \circ \pi$ of the arbitrary function $f \in F(M_n)$ is constant along each fibre $\pi^{-1}(p) = T^p_q(p)$. We call ${}^V A$ the vertical lift of $A \in \Im^p_q(M_n)$ to $T^p_q(M_n)$ (see [1]).

If ${}^V A = {}^V A^k \partial_k + {}^V A^{\overline{k}} \partial_{\overline{k}}$, $x^{\overline{k}} = t^{l_1 \dots l_p}_{k_1 \dots k_q}$, then we have from (5)

$$ {}^V A^k t^{i_1 \dots i_p}_{j_1 \dots j_q} \partial_k \alpha^{j_1 \dots j_q}_{i_1 \dots i_p} + {}^V A^{\overline{k}} \alpha^{k_1 \dots k_q}_{l_1 \dots l_p} = \alpha^{k_1 \dots k_q}_{l_1 \dots l_p} A^{l_1 \dots l_p}_{k_1 \dots k_q}. $$

Using the argument similar to that used in the proof of Lemma 1, we see that the vertical lift ${}^V A$ has components of the form

$$ {}^V A = \begin{pmatrix} {}^V A^j \\ {}^V A^{\overline{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A^{i_1 \dots i_p}_{j_1 \dots j_q} \end{pmatrix} \tag{6} $$

with respect to the coordinates $(x^j, x^{\overline{j}})$ in $T^p_q(M_n)$.

Let $\varphi \in \Im^1_1(M_n)$ and $\xi \in \Im^p_q(M_n)$:

$$ \varphi = \varphi^i_j \frac{\partial}{\partial x^i} \otimes dx^j, $$

$$ \xi = \xi^{i_1 \dots i_p}_{j_1 \dots j_q} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}. $$

We define a vector field $\gamma_\xi \varphi$ in $\pi^{-1}(U) \subset T^p_q(M_n)$ by

$$ \left.\begin{aligned} \gamma_\xi \varphi &= \left(\sum_{\lambda=1}^{p} \xi^{i_1 \dots m \dots i_p}_{j_1 \dots j_q} \varphi^{i_\lambda}_m \right) \frac{\partial}{\partial x^{\overline{j}}}, \quad (p \geq 1, q \geq 0), \\ \gamma_\xi \varphi &= \left(\sum_{\mu=1}^{q} \xi^{i_1 \dots i_p}_{j_1 \dots m \dots j_q} \varphi^m_{j_\mu} \right) \frac{\partial}{\partial x^{\overline{j}}}, \quad (p \geq 0, q \geq 1), \end{aligned} \right\} \tag{7} $$

and a vector field $\gamma\varphi$ in $\pi^{-1}(U)$ by

$$(8)\quad \left.\begin{aligned} \gamma\varphi &= \left(\sum_{\lambda=1}^{p} t_{j_1\ldots j_q}^{i_1\ldots m\ldots i_p}\varphi_m^{i_\lambda}\right)\frac{\partial}{\partial x^{\bar{J}}}, \quad (p\geq 1, q\geq 0),\\ \tilde{\gamma}\varphi &= \left(\sum_{\mu=1}^{q} t_{j_1\ldots m\ldots j_q}^{i_1\ldots i_p}\varphi_{j_\mu}^{m}\right)\frac{\partial}{\partial x^{\bar{J}}}, \quad (p\geq 0, q\geq 1),\end{aligned}\right\}$$

with respect to the coordinates $(x^j, x^{\bar{J}})$ in $T_q^p(M_n)$. From (2) we easily see that the vector fields $\gamma_\xi\varphi$, $\tilde{\gamma}_\xi\varphi$ and $\gamma\varphi$, $\tilde{\gamma}\varphi$ defined in each $\pi^{-1}(U)$ determine respectively global vector fields in $T_q^p(M_n)$.

From (6) and (7), we have

$$\gamma_\xi\varphi = {}^V(\varphi_\xi) \ \ (\tilde{\gamma}_\xi\varphi = {}^V(\tilde{\varphi}_\xi)),$$

where $\varphi_\xi \in \Im_q^p(M_n)$ $(\tilde{\varphi}_\xi \in \Im_q^p(M_n))$ and

$$\begin{aligned} \varphi_\xi &= \left(\sum_{\mu=1}^{q} \xi_{j_1\ldots j_q}^{i_1\ldots m\ldots i_p}\varphi_m^{i_\lambda}\right)\frac{\partial}{\partial x^{i_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{i_p}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_q}\\ \tilde{\varphi}_\xi &= \left(\sum_{\mu=1}^{q} \xi_{j_1\ldots m\ldots j_q}^{i_1\ldots i_p}\varphi_{j_\mu}^{m}\right)\frac{\partial}{\partial x^{i_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{i_p}}\otimes dx^{j_1}\otimes\ldots\otimes dx^{j_q}\end{aligned}$$

are respectively, local expressions of φ_ξ and $\tilde{\varphi}_\xi$, with respect to the coordinates x^j in $U\subset M_n$.

From (7) and (8), we see that, $\gamma_\xi\varphi$, $\tilde{\gamma}_\xi\varphi$ and $\gamma\varphi$, $\tilde{\gamma}\varphi$ have respectively components

$$(9)\quad \gamma_\xi\varphi = \begin{pmatrix} 0 \\ \sum_{\lambda=1}^{p} \xi_{j_1\ldots j_q}^{i_1\ldots m\ldots i_p}\varphi_m^{i_\lambda}\end{pmatrix},$$

$$(10)\quad \tilde{\gamma}_\xi\varphi = \begin{pmatrix} 0 \\ \sum_{\mu=1}^{q} \xi_{j_1\ldots m\ldots j_q}^{i_1\ldots i_p}\varphi_{j_\mu}^{m}\end{pmatrix},$$

$$(11)\quad \gamma\varphi = \begin{pmatrix} 0 \\ \sum_{\lambda=1}^{p} t_{j_1\ldots j_q}^{i_1\ldots m\ldots i_p}\varphi_m^{i_\lambda}\end{pmatrix},$$

(12)
$$\gamma\ \varphi = \begin{pmatrix} 0 \\ \sum_{\mu=1}^{q} t^{i_1 \dots i_p}_{j_1 \dots m \dots j_q} \varphi^m_{j_\mu} \end{pmatrix}$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $\pi^{-1}(U) \subset T^p_q(M_n)$.

3. Complete lifts of derivations and γ-operator

We now put $\Im(M_n) = \sum_{p,q=0}^{\infty} \Im^p_q(M_n)$, which is the direct sum of all tensor modules in M_n. A map $D : \Im(M_n) \to \Im(M_n)$ is a derivation in M_n, if, [1]:

a) D is linear with respect to constant coefficients,
b) for all p, q, $D\Im^p_q(M_n) \subset \Im^p_q(M_n)$,
c) for all tensor fields T_1 and T_2 in M_n,

$$D(T_1 \otimes T_2) = (DT_1) \otimes T_2 + T_1 \otimes (DT_2),$$

d) D commutes with contraction.

From the definition of the tensor derivation, we have

(13)
$$DI = 0,$$

where $I = (\delta^i_j)$ denotes the identity tensor field of type (1,1) in M_n.

For a derivation D in M_n, there exists a vector field p in M_n such that

(14)
$$pf = Df\,, \quad f \in F(M_n).$$

If we put

(15)
$$D(\partial_i) = Q^h_i \partial_h$$

in each coordinate neighborhood U of M_n, then from (13) and $(dx^i)(\partial_j) = \delta^i_j$. We have in U

(16)
$$D(dx^h) = -Q^h_i dx^i.$$

Let α be an element of $\Im^q_p(M_n)$ with local expression $\alpha = \alpha^{j_1 \dots j_q}_{i_1 \dots i_p} \partial_{j_1} \otimes \dots \otimes \partial_{j_q} \otimes dx^{i_1} \otimes \dots \otimes dx^{i_p}$. Then, from (15) and (16) we see that $D\alpha$ have components of the form

(17)
$$D\alpha : (p^m \partial_m \alpha^{j_1 \dots j_q}_{i_1 \dots i_p} + \sum_{\mu=1}^{q} \alpha^{j_1 \dots m \dots j_q}_{i_1 \dots i_p} Q^{j_\mu}_m - \sum_{\lambda=1}^{p} \alpha^{j_1 \dots j_q}_{i_1 \dots m \dots i_p} Q^m_{i_\lambda})$$

in M_n, p^h being the components of $p \in \Im_0^1(M_n)$ given by (14). The pair (p^h, Q_i^h) is called the components of the derivation D in U, [2, p.26].

Let D be a derivation in M_n. Then there is a unique (see Lemma 1) vector field ${}^C D \in \Im_0^1(T_q^p(M_n))$ such that for $\alpha \in \Im_p^q(M_n)$, [1],

$$(18) \qquad {}^C D(\imath\alpha) = \imath(D\alpha).$$

We call ${}^C D$ the complete lift of D to $T_q^p(M_n)$.

From (17) and (18), we have

$$\begin{aligned} &{}^C D^j \partial_j \left(t_{j_1\ldots j_q}^{i_1\ldots i_p} \alpha_{i_1\ldots i_p}^{j_1\ldots j_q}\right) + {}^C D^{\bar{j}} \partial_{\bar{j}} \left(t_{j_1\ldots j_q}^{i_1\ldots i_p} \alpha_{i_1\ldots i_p}^{j_1\ldots j_q}\right) \\ &= t_{j_1\ldots j_q}^{i_1\ldots i_p} (p^m \partial_m \alpha_{i_1\ldots i_p}^{j_1\ldots j_q} + \sum_{\mu=1}^{q} \alpha_{i_1\ldots i_p}^{j_1\ldots m\ldots j_q} Q_m^{j_\mu} - \sum_{\lambda=1}^{p} \alpha_{i_1\ldots m\ldots i_p}^{j_1\ldots j_q} Q_{i_\lambda}^m), \end{aligned}$$

or

$$(19) \qquad \begin{aligned} &{}^C D^j (\partial_j \alpha_{i_1\ldots i_p}^{j_1\ldots j_q}) t_{j_1\ldots j_q}^{i_1\ldots i_p} + {}^C D^{\bar{j}} \alpha_{i_1\ldots i_p}^{j_1\ldots j_q} \\ &= t_{j_1\ldots j_q}^{i_1\ldots i_p} p^m \partial_m \alpha_{i_1\ldots i_p}^{j_1\ldots j_q} + \alpha_{i_1\ldots i_p}^{j_1\ldots j_q} (\sum_{\mu=1}^{q} t_{j_1\ldots m\ldots j_q}^{i_1\ldots i_p} Q_{j_\mu}^m - \sum_{\lambda=1}^{p} t_{j_1\ldots j_q}^{i_1\ldots m\ldots i_p} Q_m^{i_\lambda}). \end{aligned}$$

By the same token as in the proof of Lemma 1, it follows from (19) that, ${}^C D$ has components

$$(20) \qquad {}^C D = \begin{pmatrix} p^j \\ \sum_{\mu=1}^{q} t_{j_1\ldots m\ldots j_q}^{i_1\ldots i_p} Q_{j_\mu}^m - \sum_{\lambda=1}^{p} t_{j_1\ldots j_q}^{i_1\ldots m\ldots i_p} Q_m^{i_\lambda} \end{pmatrix}$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $T_q^p(M_n)$.

When a derivation D in M_n satisfies the condition $Df = 0$ for any $f \in F(M_n)$, D determines an element $\varphi \in \Im_1^1(M_n)$ in such a way that $DX = \varphi X$, $\forall X \in \Im_0^1(M_n)$. In such a case, D is denoted by $D\varphi$ and called the derivation determined by φ. From $D_\varphi f = 0$ and $D_\varphi X = \varphi X$, we easily verify that the $D\varphi$ has local components

$$(21) \qquad D_\varphi : (0, \varphi_i^h), p^h = 0, \ Q_i^h = \varphi_i^h,$$

where φ_i^h are local components of φ in M_n.

From (11), (12), (20) and (21), we have

$${}^C D = \begin{pmatrix} 0 \\ \sum_{\mu=1}^{q} t_{j_1\ldots m\ldots j_q}^{i_1\ldots i_p} Q_{j_\mu}^m - \sum_{\lambda=1}^{p} t_{j_1\ldots j_q}^{i_1\ldots m\ldots i_p} Q_m^{i_\lambda} \end{pmatrix} = \tilde{\gamma}\varphi - \gamma\varphi,$$

or

$$(22) \qquad {}^C(D\varphi) = \tilde{\gamma}\varphi - \gamma\varphi.$$

Let $t_q^p(M_n)$ be a pure tensor subbundle with respect to $\varphi \in \Im_1^1(M_n)$, [3] (see also [4]), i.e.

(23)

$$\varphi_{j_1}^r t_{r j_2 \dots j_q}^{i_1 \dots i_p} = \dots = \varphi_{j_q}^r t_{j_1 \dots j_{q-1} r}^{i_1 \dots i_p} = \varphi_r^{i_1} t_{j_1 \dots j_q}^{r i_2 \dots i_p} = \dots = \varphi_r^{i_p} t_{j_1 \dots j_q}^{i_1 \dots i_{p-1} r} \overset{def}{=} (t(\varphi))_{j_1 \dots j_q}^{i_1 \dots i_p}.$$

From (22) and (23), we see that along the pure tensor subbundle ${}^C(D\varphi)$ reduces to

$${}^C(D\varphi) = (q-p)^V(t(\varphi)).$$

4. Complete lifts of Lie derivations

Suppose that $V \in \Im_0^1(M_n)$. We define the complete lift ${}^CV$ of V to $T_q^p(M_n)$ by [5],

$$(24) \qquad {}^CV = \begin{pmatrix} {}^CV^j \\ {}^CV^{\bar{j}} \end{pmatrix} = \begin{pmatrix} V^j \\ \sum\limits_{\lambda=1}^{p} t_{j_1 \dots j_q}^{i_1 \dots m \dots i_p} \partial_m V^{i_\lambda} - \sum\limits_{\mu=1}^{q} t_{j_1 \dots m \dots j_q}^{i_1 \dots i_p} \partial_{j_\mu} V^m \end{pmatrix}.$$

Let L_V denote Lie derivation with aspect to V, i.e. L_V is a tensor derivation such that

$$L_V f = Vf, \forall f \in F(M_n),$$

$$L_V W = [V, W], \forall V, W \in \Im_0^1(M_n),$$

where $[V, W]$ is the Lie bracket of V and W. From $[V, W]^h = V^i \partial_i W^h - W^i \partial_i V^h$, we see that the Lie derivation in M_n having components

$$(25) \qquad L_V = (V^h, -\partial_i V^h).$$

Using (20), (24) and (25), we have

$${}^C(L_V) = {}^C V.$$

5. Complete lifts of covariant derivations

Let ∇ be an affine connection in M_n and ∇_V denote covariant derivation with respect to V, i.e. ∇_V is a tensor derivation such that

$$\nabla_V f = Vf,$$

$$\nabla_{fV+gW} X = f\nabla_V X + g\nabla_W X, \quad \forall f, g \in F(M_n), \forall V, W, X \in \Im_0^1(M_n).$$

From $(\nabla_V X)^h = V^j \partial_j X^h + V^j \Gamma_{ji}^h X^i$ (Γ_{ji}^h are local components of ∇ in M_n), we see that the covariant derivation ∇_V is a derivation in M_n having components

$$(26) \qquad \nabla_V = (V^h, V^j \Gamma_{ji}^h).$$

Using (20) and (26), we can easily verify that ${}^C(\nabla_V)$ have components of the form

$$(27) \qquad {}^C(\nabla_V) = \begin{pmatrix} V^j \\ -\sum_{\lambda=1}^{p} t_{j_1 \ldots j_q}^{i_1 \ldots m \ldots i_p} V^j \Gamma_{jm}^{i_\lambda} + \sum_{\mu=1}^{q} t_{j_1 \ldots m \ldots j_q}^{i_1 \ldots i_p} V^j \Gamma_{j j_\mu}^{m} \end{pmatrix}.$$

We introduce now an affine connection $\check{\nabla}$ in M_n by

$$(28) \qquad \check{\nabla}_X Y = \nabla_X Y - S(X,Y) = \nabla_X Y - (\nabla_X Y - \nabla_Y X - [X,Y])$$
$$= \nabla_Y X + [X,Y], \forall X, Y \in \Im_0^1(M_n),$$

where $S(X,Y)$ is the torsion tensor of the given affine connection ∇. We have from (28)

$$(29) \qquad \check{\Gamma}_{ji}^h = \Gamma_{ij}^h,$$

where $\check{\Gamma}_{ji}^h$ are components of the new affine connection $\check{\nabla}$.

Now taking account of the formulas (20), (24), (27) and (29), we find

$$
\begin{aligned}
{}^{C}V - {}^{C}(\nabla_V) \\
&= \begin{pmatrix} 0 \\ \sum_{\lambda=1}^{p} t_{j_1\ldots j_q}^{i_1\ldots m\ldots i_p}(\partial_m V^{i_\lambda} + \Gamma_{jm}^{i_\lambda} V^j) - \sum_{\mu=1}^{q} t_{j_1\ldots m\ldots j_q}^{i_1\ldots i_p}(\partial_{j_\mu} V^m + \Gamma_{jj_\mu}^{m} V^j) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \sum_{\lambda=1}^{p} t_{j_1\ldots j_q}^{i_1\ldots m\ldots i_p}(\partial_m V^{i_\lambda} + \check{\Gamma}_{mj}^{i_\lambda} V^j) - \sum_{\mu=1}^{q} t_{j_1\ldots m\ldots j_q}^{i_1\ldots i_p}(\partial_{j_\mu} V^m + \check{\Gamma}_{j_\mu j}^{m} V^j) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \sum_{\lambda=1}^{p} t_{j_1\ldots j_q}^{i_1\ldots m\ldots i_p}\check{\nabla}_m V^{i_\lambda} - \sum_{\mu=1}^{q} t_{j_1\ldots m\ldots j_q}^{i_1\ldots i_p}\check{\nabla}_{j_\mu} V^m \end{pmatrix} \\
&= \gamma(\check{\nabla}_V) - \tilde{\gamma}(\check{\nabla}_V)\,.
\end{aligned}
$$

Thus, we have

$$
{}^{C}(\nabla_V) = {}^{C}V - \gamma(\check{\nabla}_V) + \tilde{\gamma}(\check{\nabla}_V). \tag{30}
$$

As a corollary to (30), we have

$$
{}^{C}(\nabla_V) = {}^{C}V = {}^{C}(L_V),
$$

if $\check{\nabla}_V = 0$.

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