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One Class of Submanifolds of Permutation Products of Complex Manifolds

Samet Kera

Presented by Bl. Sendov

In the permutation product $M^{(n)}$ on arbitrary 2-dimensional differentiable manifold M, it is defined a subset

$$A_n^{n_1,n_2,\cdots,n_k} = \{(\underbrace{x_1,\cdots,x_1}_{n_1},\underbrace{x_2,\cdots,x_2}_{n_2},\cdots,\underbrace{x_k,\cdots,x_k}_{n_k})^{\approx} : \ x_i \in M, \ i=1,\cdots,k\},$$

for arbitrary decomposition $n = n_1 + n_2 + \cdots + n_k$ of the positive integer n. It is defined when such a decomposition of n is a "good decomposition" and further it is proved that the subset $A_n^{n_1,n_2,\cdots,n_k}$ is a differentiable manifold, if and only if the decomposition $n = n_1 + n_2 + \cdots + n_k$ of n is a good decomposition.

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1. Introduction

Let M be a nonvoid set and let n be a positive integer. In the Cartesian product M^n we define a relation $\approx$ such that $(x_1,\cdots,x_n) \approx (y_1,\cdots,y_n)$, if there exists a permutation $\vartheta : \{1,2,\cdots,n\} \to \{1,2,\cdots,n\}$ such that $y_i = x_{\vartheta(i)}$ $(1 \leq i \leq n)$. This is an equivalence relation. The class represented by $(x_1,\cdots,x_n)$ will be denoted by $(x_1,\cdots,x_n)/\approx$ and the set $M^n/\approx$ will be denoted by $M^{(n)}$. The set $M^{(n)}$ is called a *permutation product* of M and it was mainly studied in [4].

If M is a topological space, then $M^{(n)}$ is also a topological space. The space $(R^m)^{(n)}$ $(n \geq 2)$ is a manifold only for $m = 2$. If $m = 2$, then $(R^2)^{(n)} =$

$C^{(n)}$ is homeomorphic to C^m. Indeed, using the fact that the field C is algebraically closed, the mapping $\varphi : C^{(n)} \to C^n$ defined by

$$\varphi((z_1, \cdots, z_n)/\approx) = (\sigma_1, \sigma_2, \cdots, \sigma_n)$$

is a bijection, where $\sigma_i (1 \leq i \leq n)$ is the i-th symmetric function of $z_1, \cdots, z_n$. The mapping φ is also a homeomorphism. Moreover, if M is a 2-dimensional manifold, then $M^{(n)}$ is a manifold. In [2] it is proved that if M is orientabled, i.e. if M is an 1-dimensional complex manifold, then $M^{(n)}$ is a complex manifold. If $dim M > 2$, then $M^{(n)}$ is not a manifold for $n > 1$.

Now let $dim M = 2$ and let us consider a subgroup G of the permutation group S_n. Then define a relation $\approx$ in M^n by

$$(x_1, x_2, \cdots, x_n) \approx (x_{\tau(1)}, x_{\tau(2)}, \cdots, x_{\tau(n)}),$$

if and only if $\tau \in G$. The factor space $M^n/\approx$ will be denoted by M^n/G. In [3] it was given the following conjecture and later it was proved in [1].

Conjecture. *Let $G \leq S_n$ and M be a 2-dimensional real manifold. Then M^n/G is a manifold if and only if $G = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_r}$, where $S_{m_1}, \cdots, S_{m_r}$ are permutation groups of a partition of $\{1, 2, \cdots, n\}$ on r subsets with cardinalities $m_1, \cdots, m_r$.*

In this paper will be considered some kinds of subsets of the permutation product. Especially, answer will be given to the following question. Let us consider the set of polynomials of n-th degree with complex coefficients of type

$$P(z) = (z - x_1)^{n_1}(z - x_2)^{n_2} \cdots (z - x_k)^{n_k}.$$

Is this set a manifold or not?

2. Main results

Let M be a 2-dimensional orientabled manifold. We are going to consider the class of subsets of $M^{(n)}$ of type

$$A = \{(\underbrace{x_1, \cdots, x_1}_{n_1}, \underbrace{x_2, \cdots, x_2}_{n_2}, \cdots, \underbrace{x_k, \cdots, x_k}_{n_k})^{\approx} : x_i \in M, \ i = 1, \cdots, k\},$$

where $n_1 + n_2 + \cdots + n_k = n$ and the elements $x_1, \cdots, x_k \in M$ are not necessary distinct. We are going to give answers to the following questions:

1. Which conditions should be satisfied such that the set above is manifold?

2. Are the corresponding manifolds of new kind, or they can be obtained via known manifolds using Cartesian products of manifolds of smaller dimensions?

Before we consider these questions, we will consider the topology of the above topological subspaces.

Let d be any metric on the manifold M. The topology of the permutation product $M^{(n)}$ is induced by the following metric on $M^{(n)}$

$$D(x^{\approx}, y^{\approx}) = \min_{\pi}\{\max\{d(x_1, y_{\pi(1)}), d(x_2, y_{\pi(2)}), \cdots, d(x_n, y_{\pi(n)})\}\}$$

where the minimum is over the set of all permutations π of the set $\{1, 2, \cdots, n\}$ and $x^{\approx} = (x_1, \cdots, x_n)^{\approx}, y^{\approx} = (y_1, \cdots, y_n)^{\approx} \in M^{(n)}$.

Proof that D is a metric. (i) Using that $d(x_i), y_{\pi(i)} \geq 0$ it follows that $D(x^{\approx}, y^{\approx}) \geq 0$. Since $d(x_i, x_i) = 0$ it is obvious that $D(x^{\approx}, x^{\approx}) = 0$. Further, let us assume that $D(x^{\approx}, y^{\approx}) = 0$. Then there exists a permutation π such that $\max\{d(x_i, y_{\pi(i)}\} = 0$ which means that $d(x_i, y_{\pi(i)}) = 0$ for any $i \in \{1, \cdots, n\}$. Hence $x_i = y_{\pi(i)}$ for $i = 1, \cdots, n$ and thus $x^{\approx} = y^{\approx}$.

(ii) Note that for any $x^{\approx}, y^{\approx} \in M^{(n)}$ there exist a permutation π and index i such that $D(x^{\approx}, y^{\approx}) = d(x_i, y_{\pi(i)})$. Since $d(x_i, y_{\pi(i)}) = d(x_{\pi'(j)}, y_j)$ for $\pi' = \pi^{-1}$ and $j = \pi(i)$, we get $D(x^{\approx}, y^{\approx}) = D(y^{\approx}, x^{\approx})$.

(iii) According to the inequalities

$$\max\{d(x_i, z_{\pi(i)})\} + \max\{d(z_j, y_{\sigma(j)})\}$$

$$\geq d(x_k, z_{\pi(k)}) + d(z_{\pi(k)}, y_{\sigma(\pi(k))}) \geq d(x_k, y_{\sigma(\pi(k))}),$$

we obtain that for any index k and any permutations σ and π

$$\max\{d(x_i, z_{\pi(i)})\} + \max\{d(z_j, y_{\sigma(j)})\} \geq d(x_k, y_{\sigma(\pi(k))}).$$

Since k is arbitrary, we get

$$\max\{d(x_i, z_{\pi(i)})\} + \max\{d(z_j, y_{\sigma(j)})\} \geq \max\{d(x_k, y_{\sigma(\pi(k))})\}.$$

Using that $D(x^{\approx}, y^{\approx}) \leq \max\{d(x_k, y_{\sigma(\pi(k))})\}$ and the arbitrariness of π and σ we obtain

$$D(x^{\approx}, y^{\approx}) \leq \max\{d(x_i, z_{\pi(i)})\} + \max\{d(z_j, y_{\sigma(j)})\} \leq D(x^{\approx}, z^{\approx}) + D(z^{\approx}, y^{\approx}).$$

■

The subset

$$A = \{(\underbrace{x_1, \cdots, x_1}_{n_1}, \underbrace{x_2, \cdots, x_2}_{n_2}, \cdots, \underbrace{x_k, \cdots, x_k}_{n_k})^{\approx} \in M^{(n)} : x_i \in M, \; i = 1, \cdots, k\}$$

of the permutation product $M^{(n)}$ can be considered as a topological subspace induced by the metric D.

First we are going to answer to the following question: If M is a 2-dimensional differentiable manifold, for which n and for which decompositions $n_1, \cdots, n_k$ of n, the subset A is differentiable manifold? Note that if the subset A is a manifold, then $dim(A) = k$. First we will consider some simple examples.

Example 1. If $n_1 = n_2 = \cdots = n_k = 1$ and $k = n$, then $A = M^{(n)}$ and it is a differentiable manifold.

Example 2. If $n_1 = n$ and $k = 1$, then A is manifold homeomorphic to M.

Note that the answer of the previous theorem does not depend of the topology or the shape of the 2-dimensional manifold, but depends on choice of the decomposition of $n = n_1 + \cdots + n_k$. Thus without loss of generality we assume that $M = C$ is the set of complex numbers.

Example 3. If $n = 3$, $n_1 = 1$ and $n_2 = 2$, then $A = A_3^{1,2} = \{(a, b, b)^{\approx} : a, b \in C\}$ and we will prove that it is manifold diffeomorphic to C^2. Indeed, we will prove that the mapping $f : A_3^{1,2} \to C^2 = C \times C$ defined by $f((a, b, b)^{\approx}) = (a, b)$ is the required diffeomorphism. Form the definition of the metric it is clear that f is bicontinuous. The class $(a, b, b)^{\approx}$ is close to the class $(x, y, x)^{\approx}$ iff y is close to a and x is close to b, i.e. iff the coordinate which appears once in the first class is close to the coordinate which appears once in the second class, and analogously to the coordinate which appears twice in both classes. It is equivalent that the pair (a, b) is close to the pair (y, x). Assume that $f((a, b, b)^{\approx}) = f((a', b', b')^{\approx})$ then $(a, b) = (a', b')$. Hence, $a = a'$, $b = b'$ and thus $f((a, b, b)^{\approx}) = f((a', b', b')^{\approx})$, i.e. f is injection. Note that naturally $f((a, a, a)^{\approx}) = (a, a)$ and this degenerate case does not make any complications in this example. Finally, if $(a, b) \in C^2$ then $(a, b, b)^{\approx}$ maps onto (a, b). Thus f is surjection.

Example 4. Let $n = 7, n_1 = 1, n_2 = 2, n_3 = 4$. Then we will verify that $A_7^{1,2,4} = \{(a, b, b, c, c, c, c)^{\approx} : a, b, c \in C\}$ is homeomorphic to $C \times C \times C = C^3$.

The required homeomorphism $f : A_7^{1,2,4} \to C^3$ is given by $f((a,b,b,c,c,c,c)^{\approx}) = (a,b,c)$. Note that this mapping f is well defined and analogously as in Example 3, f is bijective mapping and f is bicontinuous.

Note that one of the most important argument in Example 4 is that $1+2 \neq 4$. Thus if some of the elements a,b,c are equal, the mapping f is still well defined. Indeed, if $n = 6$, $n_1 = 1, n_2 = 2$ and $n_3 = 3$, then the mapping $f((a,b,b,c,c,c)^{\approx}) = (a,b,c)$ is not well defined, because if $a = b$ we do not know whether $f((b,b,b,c,c,c)^{\approx}) = (b,b,c)$ or $f((b,b,b,c,c,c)^{\approx}) = (c,c,b)$.

According to the above examples and the previous discussion, we come to the following definition.

Definition. Let n be a positive integer and the decomposition $n = n_1 + \cdots + n_k$ has the following property: if

$$n_{i_1} + n_{i_2} + \cdots + n_{i_s} = n_{j_1} + n_{j_2} + \cdots + n_{j_p} \,,$$

where $1 \leq i_1 < i_2 < \cdots < i_s \leq n$, $1 \leq j_1 < j_2 < \cdots < j_p \leq n$, then $p = s$ and $(n_{j_1}, n_{j_2}, \cdots, n_{j_p})$ is a permutation of $(n_{i_1}, n_{i_2}, \cdots, i_{j_s})$. The decomposition $n = n_1 + \cdots + n_k$ satisfying this property is called *good decomposition* and otherwise, we say that the composition is not good.

Lemma 1. *If n is positive integer and $n = n_1 + n_2 + \cdots + n_k$ is a good decomposition such that $n_i \neq n_j$ for $i \neq j$, then the subset A is homeomorphic to C^k.*

Proof. The mapping $f : A \to C^k$ defined by

$$f((\underbrace{x_1,\cdots,x_1}_{n_1},\underbrace{x_2,\cdots,x_2}_{n_2},\cdots,\underbrace{x_k,\cdots,x_k}_{n_k})^{\approx}) = (x_1,x_2,\cdots,x_k)$$

and analogously to Examples 3 and 4, it is a bijection. Since $n = n_1 + \cdots + n_k$ is a good decomposition, it follows that two classes of the corresponding space A are close with respect to the above metric iff any number p of type $p = n_{i_1} + \cdots + n_{i_s}$, exactly p coordinates of the first class are close to p coordinates of the second class. This is true iff for any $i = 1,2,\cdots,k$, the coordinate of the first class which repeats n_i times is close to the coordinate of the second class which repeats n_i times. This is equivalent to the argument that the images of $(x_1,x_2,\cdots,x_k)$ and $(x'_1,x'_2,\cdots,x'_k)$ of these two classes upon the mapping f are close. This means that f is homeomorphism. Since C is differentiable manifold, the space

A heredities the differentiable structure and moreover f is diffeomorphism with respect to the differentiable structures of A and C^k. ∎

Note that Lemma 1 also holds if C is changed by arbitrary 2-dimensional manifold M. In that case A is homeomorphic to M^k. Note that in Lemma 1 we assumed that $n_i \neq n_j$ for $i \neq j$. In order to avoid this assumption, we will prove the next Lemma 2. Previously, we consider an example.

Example 5. The decomposition $7 = 2 + 2 + 3$ is a good decomposition and the corresponding set A is homeomorphic to $C^{(2)} \times C \cong C^3$. Indeed, in this case the corresponding diffeomorphism $f : A \to C^{(2)} \times C$ is defined by $f((a, a, b, b, c, c, c)^{\approx}) = ((a, b)^{\approx}, c)$.

Lemma 2. *Let $n'_1, n'_2, \cdots, n'_t \in \{n_1, \cdots, n_k\}$ be distinct summands of some good decomposition $n = n_1 + n_2 + \cdots + n_k$ of n. Let n'_i repeat s_i times, where $s_1 + s_2 + \cdots + s_t = k$ and $n = n_1 + n_2 + \cdots + n_k = s_1 n'_1 + s_2 n'_2 + \cdots + s_t n'_t$, then A is homeomorphic (diffeomorphic) to $M^{(s_1)} \times M^{(s_2)} \times \cdots \times M^{(s_t)}$.*

Proof. We define a mapping $F : A \to M^{(s_1)} \times M^{(s_2)} \times \cdots \times M^{(s_t)}$ by

$$F((x_1, \cdots, x_1, x_2, \cdots, x_2, \cdots, x_{s_1}, \cdots, x_{s_1}, y_1, \cdots, y_1, y_2, \cdots, y_2, \cdots, y_{s_2}, \cdots,$$

$$y_{s_2}, \cdots, w_1, \cdots, w_1, w_2, \cdots, w_2, \cdots, w_{s_t}, \cdots, w_{s_t})^{\approx}) =$$

$$= ((x_1, x_2, \cdots, x_{s_1})^{\approx}, (y_1, y_2, \cdots, y_{s_2})^{\approx}, \cdots, (w_1, w_2, \cdots, w_{s_t})^{\approx}),$$

where each of the coordinates denoted by x (and the same index) repeats n'_1 times, each coordinate denoted by y (and the same index) repeats n'_2 times, and so on.

Using that $n = n_1 + \cdots + n_k$ is a good decomposition, it follows that F is well defined and also it is a bijective mapping. Again, using the good decomposition of n, analogously to the proof of Lemma 1 we can prove that f is bicontinuous. ∎

Corollary 1. *Let M be a differential manifold of arbitrary dimension and let $n = n_1 + \cdots + n_k$ be a good decomposition of n such that any n_i appears exactly once. Then the corresponding subset A is a manifold diffeomorphic to M^k.*

Proof. Since $M^{(1)} = M$, the mapping F like in Lemma 2 yields to a homeomorphism $A \cong M^{(1)} \times M^{(1)} \times \cdots \times M^{(1)} \cong M^k$. ∎

According to Lemmas 1 and 2, we obtain the following theorem.

Theorem 1. *Let M be a two dimensional manifold and $n = n_1 + n_2 + \cdots + n_k$ be a good decomposition of n. Then A is $2k$-dimensional manifold. If M is 1-dimensional complex manifold, then A is k-dimensional complex manifold.*

Lemma 3. *Let $B \subset M^{(n)}$ be a union of sets of type A with respect to distinct good decompositions $n = n_1 + n_2 + \cdots + n_k$ of n for fixed number k. Then B is a manifold iff B is union of only one set of type A.*

Note that two decompositions $n = n_1 + n_2 + \cdots + n_k$ and $n = n'_1 + n'_2 + \cdots + n'_k$ we consider to be different, if the k-tuple $(n_1, n_2, \cdots, n_k)$ is not a permutation of $(n'_1, n'_2, \cdots, n'_k)$.

Proof. If B is union of only one set of type A, then it is a manifold according to Theorem 1.

Assume that B is union of at least two sets of A. For the sake of simplicity we assume that B is union of two subsets A_1 and A_2 corresponding to distinct good decompositions $\{n_1, n_2, \cdots, n_k\}$ and $\{n'_1, n'_2, \cdots, n'_k\}$. The general case can be considered analogously.

The set $M^{\approx} = \{(a, a, \cdots, a)^{\approx} : \ a \in M\}$ is a submanifold of $M^{(n)}$ homeomorphic to M. The corresponding homeomorphism is given by $a \to (a, a, \cdots, a)^{\approx} := \mathbf{a}^{\approx}$. Obviously, $M^{\approx} \subset A_1 \cap A_2$. If $\mathbf{a}^{\approx}$ is an arbitrary element of $M^{\approx}$, we will prove that in each neighborhood in A_2 of this point and each $\epsilon > 0$, there is a point of A_1 which does not belong to A_2 and which is on a D-distance of $\mathbf{a}^{\approx}$ smaller than ϵ. This will imply that an arbitrary neighborhood of $\mathbf{a}^{\approx}$ in $B = A_1 \cup A_2$ is not homeomorphic to C^{2k}.

Since the decompositions are distinct, there exists $n_i \notin \{n'_1, n'_2, \cdots, n'_k\}$, if b is such that $d(a, b) < \epsilon$, then the point $(a, a, \cdots, a, b, b, \cdots, b, a, a, \cdots, a)^{\approx}$, where b appears n_i times, belongs to $A_1 \setminus A_2$ and it is on a D-distance smaller than ϵ from the point $\mathbf{a}^{\approx}$ in the space B. ∎

The neighborhood of the point $\mathbf{a}^{\approx}$ is union of two neighborhoods homeomorphic to C^{2k}.

Further will use the well known statement: If X and Y are compact manifolds, X is submanifold of Y and $dim X = dim Y$, then $X = Y$. Hence we obtain that if M is a compact manifold of dimension 2, then $M^{(n)}$ is unique submanifold of $M^{(n)}$ of dimension $2n$.

Next we will prove that if $n = n_1 + n_2 + \cdots + n_k$ is not a good decomposition of n, then the corresponding subset A is not a manifold. Before we prove this statement, we will consider some examples as special cases.

Example 6. Let $n = 4 = 1 + 1 + 2$ and let $M = C$. The corresponding set A in this case is $A = \{(a, b, c, c)^{\approx} : a, b, c, \in C\}$. We will prove that if $a \neq c$, then the point $(a, a, c, c)^{\approx}$ does not have a neighborhood homeomorphic to C^3. Let U_a and U_c be nonintersecting neighborhoods of the points a and c which are homeomorphic to C^3. An arbitrary neighborhood of the point $(a, a, c, c)^{\approx}$ contains sufficiently small neighborhood (in the space A), which is generated by two nonintersecting neighborhoods of the type of U_a and U_c. Thus, it is sufficient to prove that the corresponding neighborhood G generated by U_a and U_c is not homeomorphic to C^3.

The set G, i.e. the neighborhood of $(a, a, c, c)^{\approx}$ in A, can be represented as $G = W \cup V$, where $W = \{(a', a', c', c'')^{\approx} : a' \in U_a, c', c'' \in U_c\}$ and $V = \{(a', a'', c', c')^{\approx} : a', a'' \in U_a, c' \in U_c\}$.

First we will prove that V and W are homeomorphic to C^3. It is clear that V and W are homeomorphic, so it is sufficient to prove that W is homeomorphic to C^3.

The mapping $f : W \to U_a^{(2)} \times U_c (\cong C^{(2)} \times C \cong C^3)$ defined by $f((a', a', c', c'')^{\approx}) = ((c', c'')^{\approx}, a')$ is homeomorphism. Indeed, f is well defined because the second coordinate (of the image) a' is chosen from U_a which does not have a common point with U_c. Indeed, this argument enables uniqueness of the image. It is easy to verify that f is a bijection. Thus we obtain that V and W are homeomorphic to C^3.

According to this discussion, an arbitrary sufficiently small neighborhood of the point $(a, a, c, c)^{\approx}$ is union of two sets W and V homeomorphic to C^3. But, $W \cap V = \{(a', a', c', c')^{\approx} : a' \in U_a, c' \in U_c\}$ and according to lemma 2 it is homeomorphic to C^2. Hence W and V are not equal and thus G is not homeomorphic to C^3 and A is not a manifold.

Note that in Example 6 the condition $M = C$ has no role and can be neglected.

This example can be generalized into the following theorem.

Theorem 2. *If $n = n_1 + \cdots + n_k$ is not a good decomposition, then the corresponding set A obtained by 2-dimensional manifold M is not a manifold.*

P r o o f. Assume that $n = n_1 + \cdots + n_k$ is not a good decomposition of n. It means that there are summands $n'_1, n'_2, \cdots, n'_s \in \{n_1, \cdots, n_k\}$ and

$n''_1, n''_2, \cdots, n''_p \in \{n_1, \cdots, n_k\}$, such that $s \geq 1$ and $p > 1$, or $p \geq 1$ and $s > 1$, such that $n'_1 + n'_2 + \cdots + n'_s = n''_1 + n''_2 + \cdots + n''_p$ and moreover, if $p = s$, then the s-tuple $(n'_1, n'_2, \cdots, n'_s)$ is not a permutation of $(n''_1, n''_2, \cdots, n''_p)$.

Let $\mathbf{c}^{\approx} = (a, a, \cdots, a, b, b, \cdots, b, x, x, \cdots, x, \cdots, y, y, \cdots, y)^{\approx}$ be a point of the set A, where $a, b, x, \cdots, y$ are different points such that a appears $n'_1 + n'_2 + \cdots + n'_s$ times and the point b appears $n'_1 + n'_2 + \cdots + n'_s = n''_1 + n''_2 + \cdots + n''_p$ times. The other points $x, \cdots, y$ appear respectively $n_{i_x}, \cdots, n_{i_y}$ times, where $n_{i_x}, \cdots, n_{i_y}$ are summands of decomposition of n, different from $n'_1, n'_2, \cdots, n'_s$ and $n''_1, n''_2, \cdots, n''_p$.

Let $U_a, U_b, U_x, \cdots, U_y$ be nonintersecting neighborhoods corresponding to the points $a, b, x, \cdots, y$ and let G is a neighborhood of $\mathbf{c}^{\approx}$ in A, generated by the previous neighborhoods. Since any neighborhood of $\mathbf{c}^{\approx}$ contains neighborhoods of the type as G, it is sufficient to prove that G is not euclidean.

Note that if the neighborhoods $U_a, U_b, U_x, \cdots, U_y$ are sufficiently small, then the corresponding neighborhood of the permutation product generated of them is of the form

$$G \cong (U_a \times \cdots \times U_a \times U_b \times \cdots \times U_b)^{\approx} \times (U_x)^{(n_{i_x})} \times \cdots \times (U_y)^{(n_{i_y})}.$$

Now it is clear that the neighborhood G is euclidean iff the neighborhood $(U_a \times \cdots \times U_a \times U_b \times \cdots \times U_b)^{\approx}$ is euclidean. Thus it is sufficient to consider the case when $n = n'_1 + n'_2 + \cdots + n'_s + n''_1 + n''_2 + \cdots + n''_p = 2(n'_1 + n'_2 + \cdots + n'_s)$ and to prove that the neighborhood $(U_a \times \cdots \times U_a \times U_b \times \cdots \times U_b)^{\approx}$ is not euclidean.

In this case it is $G = W \cup V$, where

$$W = \{(a', a', \cdots, a', b_1, b_2, \cdots, b_{n/2})^{\approx} : a' \in U_a, b_i \in U_b, i = 1, 2, \cdots, n/2\}$$

and

$$V = \{(b', b', \cdots, b', a_1, a_2, \cdots, a_{n/2})^{\approx} : b' \in U_b, a_i \in U_a, i = 1, 2, \cdots, n/2\}.$$

Next we will prove that V and W are homeomorphic to $C^{(n/2)+1}$. Indeed, it is sufficient to prove that W is homeomorphic to $C^{(n/2)+1}$, because V and W are homeomorphic.

The mapping $f : W \to U_a \times (U_b)^{(n/2)}$ $(\cong C \times C^{(n/2)} \cong C^{(n/2)+1})$ defined by

$$f((a', a', \cdots, a', b_1, b_2, \cdots, b_{n/2})^{\approx} = (a', (b_1, b_2, \cdots, b_{n/2})^{\approx})$$

is a homeomorphism. The mapping f is well defined because the first coordinate of the image a' is a point of the set U_a, while the second coordinate (the class $(b_1, \cdots, b_{n/2})^{\approx}$) is obtained of points of U_b. Indeed, an unique problematic case is $b_1 = b_2 \cdots = b_{n/2}$. But it is clear if we remember that $U_a \cap U_b = \emptyset$.

Directly can be verified that f is a bijection. Hence we proved that V and W are homeomorphic to $C^{n/2+1}$.

According to the previous discussion, an arbitrary small neighborhood of the point $(a, a, \cdots, b, b, \cdots, b)^{\approx}$ is union of two sets W and V homeomorphic to $C^{n/2+1}$.

Note that the neighborhood G is not homeomorphic to $C^{(n/2)+1}$ if $W \cap V$ has (real) dimension smaller that $n+2$. But

$$W \cap V = \{(a', a', \cdots, a', c', c', \cdots, c')^{\approx} : a' \in U_a, c' \in U_c\}$$

according to Lemma 1 is homeomorphic to C^2. Hence, if $n > 3$ we get that $dim(W \cap V) = 4 < n+2$, and hence in this case A is not a manifold. Additionally, we should consider the cases $n = 2$ and $n = 3$. The case $n = 2 = 1+1$ is trivial. If $n = 3$, two decompositions are possible: I. $3 = 1+1+1$ is trivial case; II. $3 = 1+2$ is good decomposition of 3. ■

For $M = C$, the set A is given by

$$A = \{(\underbrace{x_1, \cdots, x_1}_{n_1}, \underbrace{x_2, \cdots, x_2}_{n_2}, \cdots, \underbrace{x_k, \cdots, x_k}_{n_k})^{\approx} : x_i \in C, i = 1, \cdots, k\}$$

and it has the following interpretation. Let us consider the set of polynomials of n-th degree with complex coefficients of type

$$P(z) = (z - x_1)^{n_1}(z - x_2)^{n_2} \cdots (z - x_k)^{n_k}.$$

This set of polynomials is indeed the previous set A and it is a manifold iff the $n = n_1 + n_2 + \cdots + n_k$ is good decomposition.

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Institute of Mathematics
P.O. Box 162, 1000 Skopje, MACEDONIA
e-mail: sametasa@iunona.pmf.ukim.edu.mk

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