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Argument Estimates of Certain Meromorphic Functions

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The object of the present paper is to derive some argument properties of certain meromorphic functions in the punctured open unit disk. Furthermore, we investigate their integral-preserving property in a sector.

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1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which are analytic in the punctured open unit disk $\mathcal{D} = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$. We denote by $\Sigma^*(\gamma)$ the subclasses of Σ consisting of all functions which is meromorphic starlike of order γ in $\mathcal{U} = \mathcal{D} \cup \{0\}$ $(0 \leq \gamma < 1)$. For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h, if there exists an analytic function w such that $w(0) = 0, |w(z)| < 1$ $(z \in \mathcal{U})$, and $g(z) = h(w(z))$. We denote this subordination by $g \prec h$ or $g(z) \prec h(z)$. Let

$$\Sigma^*[A, B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \;\; (z \in \mathcal{U} \; ; \; -1 \leq B < A \leq 1) \right\}. \quad (1.1)$$

In particular, we note that $\Sigma^*[1-2\gamma,-1] = \Sigma^*(\gamma)$ $(0 \leq \gamma < 1)$. Furthermore, from (1.1), we observe [5] that a function f is in $\Sigma^*[A,B]$ if and only if

$$\left|\frac{zf'(z)}{f(z)} + \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \qquad (-1 < B < A \leq 1\ ;\ z \in \mathcal{U}). \tag{1.2}$$

A function $f \in \Sigma$ is said to be in the class $\Sigma_c(\gamma,\beta)$, if there is a meromorphic starlike function g of order γ such that

$$-\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \beta \qquad (0 \leq \beta < 1\ ;\ z \in \mathcal{U}).$$

Libera and Robertson [2] showed that $\Sigma_c(0,0)$, the class of meromorphic close-to-convex functions, is not univalent. Also, $\Sigma_c(\gamma,\beta)$ provides an interesting generalization of the class of meromorphic close-to-convex functions [6].

In the present paper, we give some argument properties of the aforementioned classes of meromorphic functions in the open unit disk. An application of a certain integral operator is also considered.

2. Main results

In proving our main results, we need the following lemmas.

Lemma 2.1. ([1]) *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and* Re $(\beta h(z) + \gamma) > 0(\beta, \gamma \in \mathbb{C})$. *If q is analytic in $\mathcal{U}$ with $q(0) = 1$, then*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$q(z) \prec h(z) \quad (z \in \mathcal{U}).$$

Lemma 2.2. ([3]) *Let h be convex univalent in $\mathcal{U}$ and λ be analytic in $\mathcal{U}$ with* Re $\lambda(z) \geq 0$. *If q is analytic in $\mathcal{U}$ and $q(0) = h(0)$, then*

$$q(z) + \lambda(z)zq'(z) \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$q(z) \prec h(z) \quad (z \in \mathcal{U}).$$

Lemma 2.3. ([4]) *Let q be analytic in U with $q(0) = 1$ and $q(z) \neq 0$ in U. Suppose that there exists a point $z_0 \in U$ such that*

$$\left|\arg\ q(z)\right| < \frac{\pi}{2}\eta \ \text{for}\ |z| < |z_0| \tag{2.1}$$

and

$$\left|\arg\ q(z_0)\right| = \frac{\pi}{2}\eta \quad (0 < \eta \leq 1). \tag{2.2}$$

Then we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\eta, \tag{2.3}$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \ \arg\ q(z_0) = \frac{\pi}{2}\eta \tag{2.4}$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{when} \ \arg\ q(z_0) = -\frac{\pi}{2}\eta \tag{2.5}$$

and

$$q(z_0)^{\frac{1}{\eta}} = \pm ia \ (a > 0). \tag{2.6}$$

By using the above lemmas, we now derive the following theorem.

Theorem 2.1. *Let $f \in \Sigma$ and suppose that*

$$1 + B > \alpha(2 + A + B) \quad \left(-1 < B < A \leq 1\ ;\ 0 < \alpha < \frac{1}{2}\right).$$

If

$$\left|\arg\left(-\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zg'(z) + (1-\alpha)g(z)} - \beta\right)\right| < \frac{\pi}{2}\delta \quad (0 \leq \beta < 1\ ;\ 0 < \delta \leq 1)$$

for some $g \in \Sigma^*[A, B]$, then

$$\left|\arg\left(-\frac{zf'(z)}{g(z)} - \beta\right)\right| < \frac{\pi}{2}\eta,$$

where η $(0 < \eta \leq 1)$ is the solution of the equation:

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{\eta\sin[\frac{\pi}{2}\{1 - t(A,B,\alpha)\}]}{\left(\frac{1+A}{1+B} + \frac{1}{\alpha} - 1\right) + \eta\cos[\frac{\pi}{2}\{1 - t(A,B,\alpha)\}]}\right) \tag{2.7}$$

and

$$t(A,B,\alpha)=\frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{(\frac{1}{\alpha}-1)(1-B^2)-(1-AB)}\right). \tag{2.8}$$

Proof. Let

$$q(z)=-\frac{1}{1-\beta}\left(\frac{zf'(z)}{g(z)}+\beta\right) \quad \text{and} \quad r(z)=-\frac{zg'(z)}{g(z)}.$$

Then, by a simple calculation, we have

$$-\frac{1}{1-\beta}\left(\frac{\alpha z(zf'(z))'+(1-\alpha)zf'(z)}{\alpha zg'(z)+(1-\alpha)g(z)}+\beta\right)$$

$$=q(z)+\frac{zq'(z)}{-r(z)+(\frac{1}{\alpha}-1)}.$$

Since $g\in\Sigma^*[A,B]$, from (1.2), we have

$$r(z)\prec\frac{1+Az}{1+Bz}\quad(z\in\mathcal{U}).$$

If we let

$$-r(z)+(\frac{1}{\alpha}-1)=\rho e^{i\frac{\pi\phi}{2}}\quad(z\in\mathcal{U}),$$

then it follows from (1.1) and (1.2) that

$$\begin{cases}\frac{(\frac{1}{\alpha}-1)(1+B)-(1+A)}{1+B}<\rho<\frac{(\frac{1}{\alpha}-1)(1-B)-(1-A)}{1-B},\\ -t(A,B,\alpha)<\phi<t(A,B,\alpha),\end{cases}$$

where $t(A,B,\alpha)$ is defined by (2.8). Let h be a function which maps $\mathcal{U}$ onto the angular domain $\{w:|\arg w|<\frac{\pi}{2}\delta\}$ with $h(0)=1$. Applying Lemma 2.2 for this h with $\lambda(z)=\frac{1}{-r(z)+\frac{1}{\alpha}-1}$, we see that $\operatorname{Re} q(z)>0$ in $\mathcal{U}$ and hence $q(z)\neq 0$ in $\mathcal{U}$.

If there exists a point $z_0\in\mathcal{U}$ such that the conditions (2.1) and (2.2) are satisfied, then (by Lemma 1) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

First, we suppose that

$$\{q(z_0)\}^{\frac{1}{\eta}}=ia\quad(a>0).$$

Then we obtain

$$\arg\left(-\frac{\alpha z_0(z_0f'(z_0))'+(1-\alpha)z_0f'(z_0)}{\alpha z_0g'(z_0)+(1-\alpha)g(z_0)}-\beta\right)$$

$$=\arg\left(q(z_0)+\frac{z_0q'(z_0)}{-r(z_0)+(\frac{1}{\alpha}-1)}\right)$$

$$=\arg\left\{q(z_0)\left(1+\frac{z_0q'(z_0)}{q(z_0)}\frac{1}{-r(z_0)+(\frac{1}{\alpha}-1)}\right)\right\}$$

$$=\arg\{q(z_0)\}+\arg\left(1+i\eta k(\rho e^{i\frac{\pi\phi}{2}})^{-1}\right)$$

$$=\frac{\pi}{2}\eta+\tan^{-1}\left(\frac{\eta k\sin[\frac{\pi}{2}(1-\phi)]}{\rho+\eta k\cos[\frac{\pi}{2}(1-\phi)]}\right)$$

$$\geq\frac{\pi}{2}\eta+\tan^{-1}\left(\frac{\eta\sin[\frac{\pi}{2}\{1-t(A,B,\alpha)\}]}{\left(\frac{(\frac{1}{\alpha}-1)(1-B)-(1-A)}{1-B}\right)+\eta\cos[\frac{\pi}{2}\{1-t(A,B,\alpha)\}]}\right)=\frac{\pi}{2}\delta,$$

where δ and $t(A,B,\alpha)$ are given by (2.7) and (2.8), respectively. This evidently contradicts the assumption of Theorem 2.1.

Next, we suppose that

$$q(z_0)^{\frac{1}{\eta}}=-ia\quad(a>0).$$

Applying the same method as above, we have

$$\arg\left(-\frac{\alpha z_0(z_0f'(z_0))'+(1-\alpha)z_0f'(z_0)}{\alpha z_0g'(z_0)+(1-\alpha)g(z_0)}-\beta\right)$$

$$\leq-\frac{\pi}{2}\eta-\tan^{-1}\left(\frac{\eta\sin[\frac{\pi}{2}\{1-t(A,B,\alpha)\}]}{\left(\frac{(\frac{1}{\alpha}-1)(1-B)-(1-A)}{1-B}\right)+\eta\cos[\frac{\pi}{2}\{1-t(A,B,\alpha)\}]}\right)=-\frac{\pi}{2}\delta,$$

where δ and $t(A,B,\alpha)$ are given by (2.7) and (2.8), respectively. This also contradict the assumption of Theorem 2.1. Therefore, we complete the proof of Theorem 2.1. ■

Letting $A=1$, $B=0$ and $\delta=1$ in Theorem 2.1, we have

Corollary 2.1. *Let $f\in\Sigma$. If*

$$-Re\left\{\frac{\alpha z(zf'(z))'+(1-\alpha)zf'(z)}{\alpha zg'(z)+(1-\alpha)g(z)}\right\}>\beta\quad\left(0<\alpha<\frac{1}{3};\ 0\leq\beta<1\right)$$

for some $g \in \Sigma$ *satisfying the condition:*

$$\left|\frac{zg'(z)}{g(z)} + 1\right| < 1 \quad (z \in \mathcal{U}),$$

then

$$-\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \beta \quad (0 \leq \beta < 1).$$

If we put $g(z) = \frac{1}{z}$ in Theorem 2.1, then by letting $B \to A$ $(A < 1)$, we obtain

Corollary 2.2. *Let* $f \in \Sigma$. *If*

$$\left|\arg\left(-\frac{z^2(f'(z) + \alpha z f''(z))}{1 - 2\alpha} - \beta\right)\right| < \frac{\pi}{2}\delta \quad \left(0 < \alpha < \frac{1}{2};\ 0 \leq \beta < 1;\ 0 < \delta \leq 1\right)$$

then

$$\left|\arg\left\{-z^2 f'(z) - \beta\right\}\right| < \frac{\pi}{2}\eta,$$

where η $(0 < \eta \leq 1)$ *is the solution of the equation:*

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}(\alpha\eta). \tag{2.9}$$

The proof of Theorem 2.2 below is much akin to that of Theorem 2.1. The details may be omitted.

Theorem 2.2. *Let* $f \in \Sigma$ *and suppose that*

$$1 + B > \alpha(2 + A + B) \quad \left(-1 < B < A \leq 1;\ 0 < \alpha < \frac{1}{2}\right).$$

If

$$\left|\arg\left(\beta + \frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha z g'(z) + (1-\alpha)g(z)}\right)\right| < \frac{\pi}{2}\delta \quad (\beta > 1;\ 0 < \delta \leq 1)$$

for some $g \in \Sigma^*[A, B]$, *then*

$$\left|\arg\left(\beta + \frac{zf'(z)}{g(z)}\right)\right| < \frac{\pi}{2}\eta,$$

where η $(0 < \eta \leq 1)$ *is the solution of the equation (2.7).*

For a function f belonging to the class Σ, we define the integral operator F_α as follows:

$$F_\alpha(f) := F_\alpha(f)(z) = \frac{1-2\alpha}{\alpha z^{\frac{1}{\alpha}-1}} \int_0^z t^{\frac{1}{\alpha}-2} f(t)dt \tag{2.10}$$

$$(0 < \alpha < \frac{1}{2};\ z \in \mathcal{D}).$$

The following lemma will be required for the proof of Theorem 2.3 below.

Lemma 2.4. *Let* $f \in \Sigma$ *and let* h *be a convex (univalent) function in* $\mathcal{U}$ *with* $h(0) = 1$ *and* $Re\{h(z)\} > 0$ *in* $\mathcal{U}$. *If*

$$-\frac{zf'(z)}{f(z)} \prec h(z) \quad (z \in \mathcal{U}),$$

then

$$-\frac{zF'_\alpha(f)}{F_\alpha(f)} \prec h(z) \quad (z \in \mathcal{U}),$$

for $\max_{z\in\mathcal{U}} \operatorname{Re} h(z) < \frac{1}{\alpha} - 1$ $(0 < \alpha < \frac{1}{2})$, *where* F_α *is defined by (2.10).*

Proof. From the definition (2.10), we get

$$\alpha z F'_\alpha(f)(z) + (1-\alpha)F_\alpha(f)(z) = (1-2\alpha)f(z) \tag{2.11}$$

Let

$$q(z) = -\frac{zF'_\alpha(f)}{F_\alpha(f)}.$$

Then (2.11) yields

$$q(z) - \left(\frac{1}{\alpha} - 1\right) = -\left(\frac{1}{\alpha} - 2\right)\frac{f(z)}{F_\alpha(f)}. \tag{2.12}$$

Taking logarithmic derivatives in (2.12) and multiplying by z, we get

$$q(z) + \frac{zq'(z)}{-q(z) + \frac{1}{\alpha} - 1} = -\frac{zf'(z)}{f(z)} \prec h(z) \ (z \in \mathcal{U}).$$

Therefore, by Lemma 2.1, we have that $q(z) \prec h(z)$ for $\max_{z\in\mathcal{U}} \operatorname{Re} h(z) < \frac{1}{\alpha} - 1$ $(0 < \alpha < \frac{1}{2})$. This evidently completes the proof of Lemma 2.4. ■

Next, we prove the following theorem.

Theorem 2.3. *Let $f \in \Sigma$ and suppose that*

$$1 + B > \alpha(2 + A + B) \quad \left(-1 < B < A \leq 1;\ 0 < \alpha < \frac{1}{2}\right).$$

If

$$\left|\arg\left(-\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zg'(z) + (1-\alpha)g(z)} - \beta\right)\right| < \frac{\pi}{2}\delta \quad (0 < \alpha \leq 1;\ \beta > 1;\ 0 < \delta \leq 1)$$

for some $g \in \Sigma^*[A, B]$, then

$$\left|\arg\left(-\frac{\alpha z(zF'_\alpha(f))' + (1-\alpha)zF'_\alpha(f)}{\alpha zF'_\alpha(g) + (1-\alpha)F_\alpha(g)} - \beta\right)\right| < \frac{\pi}{2}\eta, \tag{2.13}$$

where F_α is given by (2.10) and η $(0 < \eta \leq 1)$ is the solution of the equation (2.7).

Proof. Since $g \in \Sigma^*[A, B]$, by applying Lemma 2.4, the function $F_\alpha(g)$ belongs to the class $\Sigma[A, B]$. Then, from (2.11), we get

$$-\frac{\alpha z(zF'_\alpha(f))' + (1-\alpha)zF'_\alpha(f)}{\alpha zF'_\alpha(g) + (1-\alpha)F_\alpha(g)} = -\frac{zf'(z)}{g(z)}.$$

Hence, by the hypothesis and Theorem 2.1, we have (2.13), which completes the proof of Theorem 2.3. ■

Taking $A = 1$, $B = 0$ and $\delta = 1$ in Theorem 2.3, we have

Corollary 2.3. *Let $f \in \Sigma$. If*

$$-Re\left\{\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zg'(z) + (1-\alpha)g(z)}\right\} > \beta \quad (0 \leq \beta < 1)$$

for some $g \in \Sigma$ satisfying the condition:

$$\left|\frac{zg'(z)}{g(z)} + 1\right| < 1 \ (z \in \mathcal{U}),$$

then

$$-Re\left\{\frac{\alpha z(zF'_\alpha(f))' + (1-\alpha)zF'_\alpha(f)}{\alpha zF'_\alpha(g) + (1-\alpha)F_\alpha(g)}\right\} > \beta \quad (0 \leq \beta < 1).$$

Putting $g(z) = \frac{1}{z}$ in Theorem 2.3, and then by letting $B \to A$ $(A < 1)$, we obtain

Corollary 2.4. *Let $f \in \Sigma$. If*

$$\left|\arg\left(-\frac{z^2(f'(z)+\alpha z f''(z)}{1-2\alpha} - \beta\right)\right| < \frac{\pi}{2}\delta \quad \left(0 < \alpha < \frac{1}{2};\ 0 \le \beta < 1;\ 0 < \delta \le 1\right),$$

then

$$\left|\arg\left(-\frac{z^2(F'_\alpha(f)+\alpha z F''_\alpha(f)}{1-2\alpha} - \beta\right)\right| < \frac{\pi}{2}\eta \quad \left(0 < \alpha < \frac{1}{2};\ 0 \le \beta < 1;\ 0 < \delta \le 1\right),$$

where η $(0 < \eta \le 1)$ is the solution of the equation (2.9).

By a similar method of the proof in Theorem 2.3, we get the next theorem.

Theorem 2.4. *Let $f \in \Sigma$ and suppose that*

$$1 + B > \alpha(2 + A + B) \quad \left(-1 < B < A \le 1;\ 0 < \alpha < \frac{1}{2}\right).$$

If

$$\left|\arg\left(\beta + \frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha z g'(z) + (1-\alpha)g(z)}\right)\right| < \frac{\pi}{2}\delta \quad (0 < \alpha \le 1;\ \beta > 1;\ 0 < \delta \le 1)$$

for some $g \in \Sigma^*[A, B]$, then

$$\left|\arg\left(\beta + \frac{\alpha z(zF'_\alpha(f))' + (1-\alpha)zF'_\alpha(f)}{\alpha z F'_\alpha(g) + (1-\alpha)F_\alpha(g)}\right)\right| < \frac{\pi}{2}\eta, \tag{2.13}$$

where F_α is given by (2.10) and η $(0 < \eta \le 1)$ is the solution of the equation (2.7).

Finally, we prove

Theorem 2.5. *Let $f \in \Sigma$. If*

$$\left|\arg\left[-\left(\alpha\frac{(zf'(z))'}{g'(z)} + (1-\alpha)\frac{zf'(z)}{g(z)}\right) - \beta\right]\right| < \frac{\pi}{2}\delta \quad (\alpha < 0;\ 0 \le \beta < 1;\ 0 < \delta \le 1)$$

for some $g \in \Sigma^*[A, B]$, then

$$\left|\arg\left(-\frac{zf'(z)}{g(z)} - \beta\right)\right| < \frac{\pi\eta}{2},$$

and η $(0 < \eta \leq 1)$ is the solution of the equation:

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{-\alpha\eta\sin[\frac{\pi}{2}\{1-\sin^{-1}(\frac{A-B}{1-AB})\}]}{\frac{1+A}{1+B} - \alpha\eta\cos[\frac{\pi}{2}\{1-\sin^{-1}(\frac{A-B}{1-AB})\}]}\right).$$

Proof. Setting

$$q(z) = -\frac{1}{1-\beta}\left(\frac{zf'(z)}{g(z)} + \beta\right) \quad \text{and} \quad r(z) = -\frac{zg'(z)}{g(z)},$$

we have

$$-\frac{1}{1-\beta}\left(\alpha\frac{(zf'(z))'}{g'(z)} + (1-\alpha)\frac{zf'(z)}{g(z)} + \beta\right) = q(z) + \frac{\alpha z q'(z)}{-r(z)}.$$

The remaining part of the proof of Theorem 2.5 is similar to that of Theorem 2.1, and so we omit it. ■

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