On the Orthogonal Projection and the Best Approximation of a Vector in a Quasi-Inner Product Space

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Let $X$ be a quasi-inner product space ([7]), and $x, y \in X \setminus \{0\}$. We resolve the problem of the relations between the three vectors: the so-called $g$-orthogonal projection of the vector $y$ on the subspace $[x]$ ($-a(x, y)x$, Lemma 2), the best approximation of the vector $y$ with vector from $[x]$ ($-b(x, y)x$, Lemma 1), and the vector $\frac{-g(x, y)}{\|x\|^2}x$. The equality

$$a(x, y) = b(x, y) = \frac{-g(x, y)}{\|x\|^2},$$

is valid if and only if $X$ is an inner product space (i.p. space) $^1$

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Let $X$ be a real normed space and

$$g(x, y) := \frac{\|x\|}{2} \left( \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} + \lim_{t \to +0} \frac{\|x + ty\| - \|x\|}{t} \right) \quad (x, y \in X).$$

$^1$If $X$ is a i.p. space then the vector $\frac{-g(x, y)}{\|x\|^2}x$ is the orthogonal projection of the vector $y$ to the vector $x$.

$^2$The functional $g$ always exists.
We say that $X$ is a quasi-inner product space (q.i.p. space), if the equality
\begin{equation}
||x + y||^4 - ||x - y||^4 = 8 \left[ ||x||^2 g(x, y) + ||y||^2 g(y, x) \right] \quad (x, y \in X)^3
\end{equation}
holds ([7]).

The equality (1) holds in the space $l^4$, but it does not hold in the space $l^1$ ([7]). For the properties of the functional $g$ and the q.i.p. spaces, see the recent papers: [3], [4], [5], [6] and [7]. We notice that a q.i.p. space is uniformly smooth and uniformly convex. From this in a q.i.p. space $X$, we have
\[ g(x, y) = ||x|| \lim_{t \to 0} \frac{||x + ty|| - ||x||}{t}, \]
where the functional $g$ is linear in the second argument.

In what follows we assume that $X$ is a complete q.i.p. space. (For example, the space $l^4$ is complete q.i.p. space).

For fixed $x, y \in X \setminus \{0\}$, $x$ and $y$ linearly independent, we consider the real functions
\begin{equation}
f(t) := ||y + tx||, \quad \varphi(t) := ||y + tx||^2 g(y + tx, x) \quad (t \in \mathbb{R}).
\end{equation}
Since $X$ is smooth, the function $F$ is differentiable and we have
\begin{equation}
f'(t) = \frac{g(y + tx, x)}{||y + tx||} \quad (t \in \mathbb{R}).
\end{equation}

Using (1) we get
\begin{equation}
||y + (t + 1)x||^4 - ||y + (t - 1)x||^4
= 8 \left[ ||y + tx||^2 g(y + tx, x) + ||x||^2 g(x, y + tx) \right].
\end{equation}

Besides this we have
\begin{equation}
\varphi(t) = \frac{1}{8} \left[ ||y + (t + 1)x||^4
- ||y + (t - 1)x||^4 - 8t||x||^4 - 8||x||^2 g(x, y) \right] \quad (t \in \mathbb{R}).
\end{equation}

**Lemma 1.** Let $X$ be complete q.i.p. space and $x, y \in X \setminus \{0\}$, $x$ and $y$ be linearly independent. Then there exists a unique $b \in \mathbb{R}$ ($b = b(x, y)$) such that

\[ b = \frac{1}{8} \left[ ||y + (t + 1)x||^4
- ||y + (t - 1)x||^4 - 8t||x||^4 - 8||x||^2 g(x, y) \right] \quad (t \in \mathbb{R}). \]

\[ \text{If } (X, \langle \cdot, \cdot \rangle) \text{ is an i.p. space then } g(x, y) = \langle x, y \rangle \text{ and the equality (1) is the parallelogram equality.} \]
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\[a) \ g(y + bx, x) = 0 \text{ and } \min f(t) = f(b)\; ;
\]
\[b) \ b = 0 \text{ and } \varphi(t) < 0 \iff t < 0.\]

\textbf{Proof.} Since } X \text{ is uniformly convex and complete, the statement } a) \text{ follows from Lemma 4, } [2] \text{ and } (3). \text{ The function } f(x) \text{ is convex. Hence } a) \text{ implies that } b) \text{ is true.} \]

\textbf{Corollary 1.} \textit{Under the conditions of Lemma 1, the following statements are valid:}
\begin{itemize}
  \item \(a) b < 0 \iff g(y, x) > 0 \text{ and } b = 0 \iff g(y, x) = 0;\)
  \item \(b) \text{ The vector } -bx \text{ is the best approximation of vector } y \text{ with vectors from } [x].\)
\end{itemize}

In [5] it is considered the orthogonality \(\perp^9\) defined as
\[x \perp^9 y \iff \|x\|^2 g(x, y) + \|y\|^2 g(y, x) = 0.\]
In an i.p. space \((X, \langle \cdot, \cdot \rangle)\) we have \(x \perp^9 y \iff \langle x, y \rangle = 0.\)
If \(X\) is a q.i.p. space, then the orthogonality \(\perp^9\) is equivalent with James isocceles orthogonality, i.e.
\[x \perp^9 y \iff \|x + y\| = \|x - y\|.\]

\textbf{Lemma 2.} \textit{(Theorem 3.4, [5]) Let } X \text{ be a q.i.p. space and } x, y \in X, \ x \neq 0. \text{ Then there exists a unique } a \in \mathbb{R} \ (a = a(x, y)) \text{ such that } x \perp^9 y + ax, \ i.e.\]
\begin{equation}
\|y + ax\|^2 g(y + ax, x) + \|x\|^2 g(x, y + ax) = 0. \tag{6}
\end{equation}

We say that the vector \(-ax\) is a \(g\)-orthogonal projection of the vector \(y\) on the subspace \([x]\).

\textbf{Theorem 1.} \textit{Let } X \text{ be a complete q.i.p. space, } x \neq 0, \ x \text{ and } y \text{ be linearly independent. Then}
\[b - 1 < a < b + 1.\]
Proof. From (4) we get $f(a + 1) - f(a - 1) = 0$. By the Rolle theorem we obtain $0 = 2f'(c)$, where $c \in (a - 1, a + 1)$. By (3) we have $g(y + cx, x) = 0$ ($c \in (a - 1, a + 1)$). Since $X$ is uniformly convex and $g(y + bx, x) = 0$ we find $c = b$. So, $b \in (a - 1, a + 1)$.

Now we resolve the problem of the relations between the three numbers: $a(x, y), b(x, y)$ and the number $-\frac{g(x, y)}{\|x\|^2}$.

**Theorem 2.** Let $X$ be a complete q.i.p. space and $x \neq 0$, $x$ and $y$ be linearly independent. Then either the number $a$ is between number $b$ and $-\frac{g(x, y)}{\|x\|^2}$, or

$$a = b = -\frac{g(x, y)}{\|x\|^2}.$$ (7)

Proof. Using (4) and (5) we obtain

$$\varphi(a) = -\|x\|^4a - \|x\|^2g(x, y).$$ (8)

According to the property of the function $\varphi$ (Lemma 1 and (6)), we have the following three possibilities:

1. $a < b$. Then $\varphi(a) < 0$, i.e. $a > -\frac{g(x, y)}{\|x\|^2}$. So,

$$-\frac{g(x, y)}{\|x\|^2} < a < b.$$ 1.

2. $a > b$. Then $\varphi(a) > 0$, i.e. $a < -\frac{g(x, y)}{\|x\|^2}$. Hence we have

$$b < a < -\frac{g(x, y)}{\|x\|^2}.$$ 2.

3. $a = b$. Then $a = -\frac{g(x, y)}{\|x\|^2}$, i.e.

$$a = b = -\frac{g(x, y)}{\|x\|^2}.$$
Corollary 2. Under the conditions of Theorem 2, we have
\[ \|y + bx\| \leq \|y + ax\| \leq \|y - \frac{g(x, y)}{\|x\|^2}\|. \]

If \( x \) is orthogonal to \( y \) in the sense of Birkhoff, we denote \( x \perp_B y \).

Lemma 3. (4.4, [1]) A normed space \( X \) is an i.p. space, if and only if the implication
\[ x \perp_B y \Rightarrow \|x + y\| = \|x - y\|, \]
holds.

Lemma 4. (Theorem 2, [4]) A normed space \( X \) is a smooth if and only if the equivalence
\[ g(x, y) = 0 \iff x \perp_B y, \]
holds.

Finally, we give the answer of the question when (7) is valid.

Theorem 3. Let \( X \) be a complete q.i.p. space. \( X \) is an i.p. space if and only if the equality (7) valid.

Proof. If \( X \) is an i.p. space with \((.,.)\), then \( g(x, y) = (x, y) \) and
\[ \left( y - \frac{(x, y)}{\|x\|^2} x, x \right) = 0. \]So, \( a = -\frac{(x, y)}{\|x\|^2} \), i.e. (7) is valid.

Assume that (7) is valid. Let \( g(x, y) = 0 \). Then by Corollary 1 we have \( b = 0 \). In this case, from (7) we obtain \( g(x, y) = 0 \). Then by (1) we get \( \|x + y\| = \|x - y\| \). Since \( X \) is smooth, by Lemma 4, we have \( y \perp_B x \). Hence the implication (9) is valid.

References


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