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On the Orthogonal Projection and the Best Approximation of a Vector in a Quasi-Inner Product Space

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Let X be a quasi-inner product space ([7]), and $x, y \in X\setminus\{0\}$. We resolve the problem of the relations between the three vectors: the so-called g-orthogonal projection of the vector y on the subspace $[x]$ $(-a(x,y)x$, Lemma 2), the best approximation of the vector y with vector from $[x]$ $(-b(x,y)x$, Lemma 1), and the vector $\frac{-g(x,y)}{\|x\|^2}x$. The equality

$$a(x,y) = b(x,y) = \frac{-g(x,y)}{\|x\|^2},$$

is valid if and only if X is an inner product space (i.p. space) [1]

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Let X be a real normed space and

$$g(x,y) := \frac{\|x\|}{2}\left(\lim_{t\to -0}\frac{\|x+ty\|-\|x\|}{t} + \lim_{t\to +0}\frac{\|x+ty\|-\|x\|}{t}\right) \quad (x, y \in X).^2$$

[1]If X is a i.p. space then the vector $\frac{-g(x,y)}{\|x\|^2}x$ is the orthogonal projection of the vector y to the vector x.

[2]The functional g always exists.

We say that X is a quasi-inner product space (q.i.p. space), if the equality

$$\|x+y\|^4 - \|x-y\|^4 = 8\left[\|x\|^2 g(x,y) + \|y\|^2 g(y,x)\right] \quad (x,y\in X)^3 \tag{1}$$

holds ([7]).

The equality (1) holds in the space l^4, but it does not hold in the space l^1 ([7]). For the properties of the functional g and the q.i.p. spaces, see the recent papers: [3], [4], [5], [6] and [7]. We notice that a q.i.p. space is uniformly smooth and uniformly convex. From this in a q.i.p. space X, we have

$$g(x,y) = \|x\| \lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t},$$

where the functional g is linear in the second argument.

In what follows we assume that X is a complete q.i.p. space. (For example, the space l^4 is complete q.i.p. space).

For fixed $x, y \in X\setminus\{0\}$, x and y linearly independent, we consider the real functions

$$f(t) := \|y+tx\|,\quad \varphi(t) := \|y+tx\|^2 g(y+tx,x) \quad (t\in \mathbf{R}). \tag{2}$$

Since X is smooth, the function F is differentiable and we have

$$f'(t) = \frac{g(y+tx,x)}{\|y+tx\|} \quad (t\in\mathbf{R}). \tag{3}$$

Using (1) we get

$$\begin{aligned} &\|y+(t+1)x\|^4 - \|y+(t-1)x\|^4 \\ &= 8\left[\|y+tx\|^2 g(y+tx,x) + \|x\|^2 g(x,y+tx)\right]. \end{aligned} \tag{4}$$

Besides this we have

$$\begin{aligned} \varphi(t) = \frac{1}{8}&\left[\|y+(t+1)x\|^4\right. \\ &\left. -\|y+(t-1)x\|^4 - 8t\|x\|^4 - 8\|x\|^2 g(x,y)\right] \quad (t\in\mathbf{R}). \end{aligned} \tag{5}$$

Lemma 1. *Let X be complete q.i.p. space and $x,y\in X\setminus\{0\}$, x and y be linearly independent. Then there exists a unique $b\in\mathbf{R}$ ($b=b(x,y)$) such that*

[3] If $(X,(.,.))$ is an i.p. space then $g(x,y)=(x,y)$ and the equality (1) is the paralellogram equality.

a) $g(y+bx,x)=0$ *and* $\min f(t)=f(b)$;
b) $b=0$ *and* $\varphi(t)<0 \iff t<0$.

Proof. Since X is uniformly comvex and complete, the statement a) follows from Lemma 4, [2] and (3). The function $f(x)$ is convex. Hence a) implies that b) is true. ■

Corollary 1. *Under the conditions of Lemma 1, the following statements are valid:*

a) $b<0 \iff g(y,x)>0$ *and* $b=0 \iff g(y,x)=0$;
b) The vector $-bx$ *is the best approximation of vector* y *with vectors from* $[x]$.

In [5] it is considered the orthogonality $\perp^g$ defined as

$$x\perp^g y \iff \|x\|^2 g(x,y)+\|y\|^2 g(y,x)=0.$$

In an i.p. space $(X,(.,.))$ we have $x\perp^g y \iff (x,y)=0$.

If X is a q.i.p. space, then the orthogonality $\perp^g$ is equivalent with James isocceles orthogonality, i.e.

$$x\perp^g y \iff \|x+y\|=\|x-y\|.$$

Lemma 2. (Theorem 3.4, [5]) *Let* X *be a q.i.p. space and* $x,y\in X$, $x\neq 0$. *Then there exists a unique* $a\in\mathbf{R}$ $(a=a(x,y))$ *such that* $x\perp^g y+ax$, *i.e.*

$$\|y+ax\|^2 g(y+ax,x)+\|x\|^2 g(x,y+ax)=0. \tag{6}$$

We say that the vector $-ax$ is a g-orthogonal projection of the vector y on the subspace $[x]$.

Theorem 1. *Let* X *be a complete q.i.p. space,* $x\neq 0$, x *and* y *be linearly independent. Then*

$$b-1<a<b+1.$$

Proof. From (4) we get $f(a+1)-f(a-1)=0$. By the Rolle theorem we obtain $0=2f'(c)$, where $c\in(a-1,a+1)$. By (3) we have $g(y+cx,x)=0$ ($c\in(a-1,a+1)$). Since X is uniformly convex and $g(y+bx,x)=0$ we find $c=b$. So, $b\in(a-1,a+1)$. ∎

Now we resolve the problem of the relations between the three numbers: $a(x,y)$, $b(x,y)$ and the number $-\dfrac{g(x,y)}{\|x\|^2}$.

Theorem 2. *Let X be a complete q.i.p. space and $x\neq 0$, x and y be linearly independent. Then either the number a is between number b and $-\dfrac{g(x,y)}{\|x\|^2}$, or*

$$a=b=-\frac{g(x,y)}{\|x\|^2}. \tag{7}$$

Proof. Using (4) and (5) we obtain

$$\varphi(a)=-\|x\|^4a-\|x\|^2g(x,y). \tag{8}$$

According to the property of the function φ (Lemma 1 and (6)), we have the following three possibilities:

1. $a<b$. Then $\varphi(a)<0$, i.e. $a>-\dfrac{g(x,y)}{\|x\|^2}$. So,

$$-\frac{g(x,y)}{\|x\|^2}<a<b.$$

2. $a>b$. Then $\varphi(a)>0$, i.e. $a<-\dfrac{g(x,y)}{\|x\|^2}$. Hence we have

$$b<a<-\frac{g(x,y)}{\|x\|^2}.$$

3. $a=b$. Then $a=-\dfrac{g(x,y)}{\|x\|^2}$, i.e.

$$a=b=-\frac{g(x,y)}{\|x\|^2}.$$

Corollary 2. *Under the conditions of Theorem 2, we have*

$$\|y + bx\| \leq \|y + ax\| \leq \|y - \frac{g(x,y)}{\|x\|^2}\|.$$

If x is orthogonal to y in the sense of Birkhoff, we denote $x \perp_B y$.

Lemma 3. (4.4, [1]) *A normed space X is an i.p. space, if and only if the implication*

(9) $$x \perp_B y \Rightarrow \|x + y\| = \|x - y\|,$$

holds.

Lemma 4. (Theorem 2, [4]) *A normed space X is a smooth if and only if the equivalence*

$$g(x,y) = 0 \iff x \perp_B y,$$

holds.

Finally, we give the answer of the question when (7) is valid.

Theorem 3. *Let X be a complete q.i.p. space. X is a i.p. space if and only if the equality (7) valid.*

Proof. If X is an i.p. space with $(.,.)$, then $g(x,y) = (x,y)$ and $\left(y - \frac{(x,y)}{\|x\|^2}x, x\right) = 0$. So, $a = -\frac{(x,y)}{\|x\|^2}$, i.e. (7) is valid.

Assume that (7) is valid. Let $g(x,y) = 0$. Then by Corollary 1 we have $b = 0$. In this case, from (7) we obtain $g(x,y) = 0$. Then by (1) we get $\|x + y\| = \|x - y\|$. Since X is smooth, by Lemma 4, we have $y \perp_B x$. Hence the implication (9) is valid. ∎

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