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Generalized Hermite Polynomials of Three Variables and Three Indices

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Recently, the multi-variable, multi-index Hermite polynomials have been used in the analysis of charged-beam transport problems in classical mechanics as well as in the formulation of quantum-phase-space mechanics. In this paper we extend Hermite polynomials from two variables and two indices to three variables and three indices. First, we survey some results on the Hermite polynomials of two variables, two indices, including recurrence relations and corresponding orthonormal functions. Then, we deal with the extension to three variables, three indices Hermite polynomials. We derive some recurrence relations and some formulas involving partial derivatives. We analyze the relevant generating functions and integral representations. Along with the 3-variable, 3-index polynomials $H_{j,m,n}(x,p,y)$ and $G_{j,m,n}(x,p,y)$ we introduce the corresponding orthonormal functions. An integral relation and a number of recurrence relations are established for the orthonormal functions.

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1. Introduction

Special functions have played a unique role in applied mathematics and theoretical physics. The complete body of special functions theory and formulae has been reviewed in a number of books ([1] - [7]) and there is a tendency to consider it as a well established field of research, which should not deserve new substantial elements of speculation. The problematics associated with special functions are, however, so rich and wide that it is not surprising when new problems and new classes of non-trivial special functions are proposed [8].

Hermite himself introduced the multivariable, multi-index Hermite polynomials [9] which, unlike the ordinary case, did not find any particular application. More recently their importance has been recognized for the analysis of charged-beam transport problems in classical mechanics [10] and in the more recent formulation of quantum-phase-space mechanics ([11], [12]). Therefore they have been reconsidered and reformulated in a more modern context by introducing the associated orthogonal harmonic oscillator functions and the relevant creation-annihilation operators [13].

The analysis of multivariable, multi-index Hermite polynomials has opened new and fascinating scenarios allowing the possibility of extending the theory of special functions. In this paper we present the theory of multivariable, multi-index Hermite polynomials using, whenever possible, a unified point of view. Subsequently, we extend these polynomials and analyze the relevant generating functions, integral representations and recurrence relations associated with them. Furthermore, we introduce and study the orthonormal functions related to this extension.

2. Two-variable, two-index Hermite polynomials

Ch. Hermite [9] introduced 2-variable, 2-index polynomials through the generating function

$$e^{-\frac{1}{2}(\underline{z}-\underline{u})^T \widehat{M}(\underline{z}-\underline{u})} = e^{-\frac{1}{2}\underline{z}^T \widehat{M}\underline{z}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{t^m}{m!}\frac{h^n}{n!} H_{m,n}(x,p)\,, \tag{1}$$

where $\underline{z}$ and $\underline{u}$ are two-component column vectors defined by

$$\underline{z} = \begin{pmatrix} x \\ p \end{pmatrix}, \quad \underline{u} = \begin{pmatrix} t \\ h \end{pmatrix}, \quad h, t \neq 0\,, \tag{2}$$

and $\widehat{M}$ is the following 2×2 positive definite symmetric matrix

$$\widehat{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad a, c > 0\,, \tag{3}$$

with determinant $\Delta = ac - b^2 > 0$. Along with $H_{m,n}(x,p)$ the associated polynomials $G_{m,n}(\xi,\eta)$ are introduced using the generating function

$$e^{\underline{k}^T \widehat{M}^{-1}\underline{w} - \frac{1}{2}\underline{k}^T \widehat{M}^{-1}\underline{k}} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{r^m}{m!}\frac{s^n}{n!} G_{m,n}(\xi,\eta)\,, \tag{4}$$

with

$$\underline{w} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \widehat{M}\underline{z}, \quad \underline{k} = \begin{pmatrix} r \\ s \end{pmatrix} = \widehat{M}\underline{u}\,.$$

Both $H_{m,n}(x,p)$ and $G_{m,n}(\xi,\eta)$ can be explicitly constructed using a generalized Rodriguez formula, namely

$$H_{m,n}(x,p) = (-1)^{m+n}\, e^{\frac{1}{2}\underline{z}^T \widehat{M} \underline{z}} \frac{\partial^{m+n}}{\partial x^m \partial p^n} e^{-\frac{1}{2}\underline{z}^T \widehat{M} \underline{z}}, \tag{5}$$

$$G_{m,n}(\xi,\eta) = (-1)^{m+n}\, e^{\frac{1}{2}\underline{w}^T \widehat{M}^{-1} \underline{w}} \frac{\partial^{m+n}}{\partial \xi^m \partial \eta^n} e^{-\frac{1}{2}\underline{w}^T \widehat{M}^{-1} \underline{w}}. \tag{6}$$

2.1. Recurrence relations: The 2-variable, 2-index Hermite polynomials satisfy the following recurrence relations [8]:

$$\begin{aligned}
H_{m+1,n}(x,p) &= (ax+bp)H_{m,n}(x,p) - amH_{m-1,n}(x,p) \\
&\quad -bnH_{m,n-1}(x,p), &(7)\\
H_{m,n+1}(x,p) &= (bx+cp)H_{m,n}(x,p) - bmH_{m-1,n}(x,p) \\
&\quad -cnH_{m,n-1}(x,p), &(8)\\
\frac{\partial}{\partial x}H_{m,n}(x,p) &= amH_{m-1,n}(x,p) + bnH_{m,n-1}(x,p), &(9)\\
\frac{\partial}{\partial p}H_{m,n}(x,p) &= bmH_{m-1,n}(x,p) + cnH_{m,n-1}(x,p), &(10)
\end{aligned}$$

and

$$\begin{aligned}
G_{m+1,n}(x,p) &= xG_{m,n}(x,p) - \frac{cm}{\Delta}G_{m-1,n}(x,p) + \frac{bn}{\Delta}G_{m,n-1}(x,p), &(11)\\
G_{m,n+1}(x,p) &= pG_{m,n}(x,p) + \frac{bm}{\Delta}G_{m-1,n}(x,p) - \frac{an}{\Delta}G_{m,n-1}(x,p), &(12)\\
\frac{\partial}{\partial x}G_{m,n}(x,p) &= mG_{m-1,n}(x,p), &(13)\\
\frac{\partial}{\partial p}G_{m,n}(x,p) &= nG_{m,n-1}(x,p). &(14)
\end{aligned}$$

According to the above relations both $H_{m,n}$ and $G_{m,n}$ satisfy the eigenvalue equation

$$\left(\underline{z}^T \underline{\partial}_z - \underline{\partial}_z^T \widehat{M}^{-1} \underline{\partial}_z\right) A_{m,n} = (m+n)A_{m,n}\,, \tag{15}$$

where $\underline{\partial}_z$ denotes the operator

$$\underline{\partial}_z = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial p} \end{pmatrix}. \tag{16}$$

2.2. Orthonormal functions $\mathcal{H}_{m,n}(x,p)$ and $\mathcal{G}_{m,n}(x,p)$: Along with the polynomials $H_{m,n}$ and $G_{m,n}$ it is convenient to introduce the orthonormal functions

$$\mathcal{H}_{m,n}(x,p) = \frac{(\Delta)^{1/4}}{\sqrt{2\pi}} \frac{1}{\sqrt{m!n!}} H_{m,n}(x,p) e^{-\frac{1}{4}\underline{z}^T \hat{M} \underline{z}}, \tag{17}$$

(18) $$\mathcal{G}_{m,n}(x,p) = \frac{(\Delta)^{1/4}}{\sqrt{2\pi}} \frac{1}{\sqrt{m!n!}} G_{m,n}(x,p) e^{-\frac{1}{4}\underline{z}^T \widehat{M} \underline{z}},$$

which satisfy the relation ([6], [9])

(19) $$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [\mathcal{H}_{m,n}(x,p)\mathcal{G}_{r,s}(x,p)]\, dp\, dx = \delta_{m,r}\, \delta_{n,s}\,.$$

The functions (17)-(18) may be viewed as two-variable harmonic oscillator functions and, therefore, operators acting as generalized creation-annihilation operators can be introduced. From the recurrence relations (7)-(10) we define the operators [8]

(20) $$\widehat{a}_{1,+} = \frac{1}{2}(ax + bp) - \frac{\partial}{\partial x}$$

(21) $$\widehat{a}_{1,-} = \frac{1}{\Delta}\left(c\frac{\partial}{\partial x} - b\frac{\partial}{\partial p}\right) + \frac{1}{2}x$$

(22) $$\widehat{a}_{2,+} = \frac{1}{2}(bx + cp) - \frac{\partial}{\partial p}$$

(23) $$\widehat{a}_{2,-} = -\frac{1}{\Delta}\left(b\frac{\partial}{\partial x} - a\frac{\partial}{\partial p}\right) + \frac{1}{2}p,$$

which, upon acting on $\mathcal{H}_{m,n}(x,p)$, yield

(24) $$\widehat{a}_{1,+}\mathcal{H}_{m,n}(x,p) = \sqrt{m+1}\,\mathcal{H}_{m+1,n}(x,p)$$

(25) $$\widehat{a}_{1,-}\mathcal{H}_{m,n}(x,p) = \sqrt{m}\,\mathcal{H}_{m-1,n}(x,p)$$

(26) $$\widehat{a}_{2,+}\mathcal{H}_{m,n}(x,p) = \sqrt{n+1}\,\mathcal{H}_{m,n+1}(x,p)$$

(27) $$\widehat{a}_{2,-}\mathcal{H}_{m,n}(x,p) = \sqrt{n}\,\mathcal{H}_{m,n-1}(x,p)\,.$$

Furthermore, the operators $\widehat{a}_{1,+}\widehat{a}_{1,-}$ and $\widehat{a}_{2,+}\widehat{a}_{2,-}$ act on $\mathcal{H}_{m,n}$ as some kind of number operators, as indicated by the identities

(28) $$\widehat{a}_{1,+}\widehat{a}_{1,-}\mathcal{H}_{m,n} = m\mathcal{H}_{m,n}$$

(29) $$\widehat{a}_{2,+}\widehat{a}_{2,-}\mathcal{H}_{m,n} = n\mathcal{H}_{m,n}\,,$$

which, once summed, allow us to prove that $\mathcal{H}_{m,n}$ satisfies the following equation:

(30) $$\left[\underline{\partial}_z^T \widehat{M}^{-1} \underline{\partial}_z - \frac{1}{4}\underline{z}^T \widehat{M} \underline{z} + (m+n+1)\right] \mathcal{H}_{m,n}(x,p) = 0\,,$$

thus generalizing the eigenvalue equation satisfied by the ordinary harmonic oscillator functions ([3], [7]).

It is worth noting that unlike the one-dimensional case, the creation and annihilation operators are not Hermitian conjugate to each other. Turning to the Hermitian conjugates of (20)-(23), we obtain

$$\hat{a}^+_{1,+} = \frac{\partial}{\partial x} + \frac{1}{2}(ax + bp) \tag{31}$$

$$\hat{a}^+_{1,-} = -\frac{1}{\Delta}\Big(c\frac{\partial}{\partial x} - b\frac{\partial}{\partial p}\Big) + \frac{1}{2}x \tag{32}$$

$$\hat{a}^+_{2,+} = \frac{\partial}{\partial p} + \frac{1}{2}(bx + cp) \tag{33}$$

$$\hat{a}^+_{2,-} = \frac{1}{\Delta}\Big(b\frac{\partial}{\partial x} - a\frac{\partial}{\partial p}\Big) + \frac{1}{2}p\,. \tag{34}$$

The operators $\hat{a}^+$ are the relevant creation-annihilation operators for $\mathcal{G}_{m,n}(x,p)$ functions which can also be shown to satisfy equation (30) as $\mathcal{H}_{m,n}$.

3. Extension to 3-variable, 3-index Hermite polynomials

In this section we consider 3-variable, 3-index Hermite polynomials through the generating function

$$e^{-\frac{1}{2}(\underline{z}-\underline{u})^T\hat{N}(\underline{z}-\underline{u})} = e^{-\frac{1}{2}\underline{z}^T\hat{N}\underline{z}}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^j}{j!}\frac{h^m}{m!}\frac{k^n}{n!}H_{j,m,n}(x,p,y)\,, \tag{35}$$

where $\underline{z}$ and $\underline{u}$ are the three-component column vectors defined by

$$\underline{z} = \begin{pmatrix} x \\ p \\ y \end{pmatrix}, \quad \underline{u} = \begin{pmatrix} t \\ h \\ k \end{pmatrix}, \quad k,h,t \neq 0\,, \tag{36}$$

and $\hat{N}$ is the following 3×3 positive definite symmetric matrix

$$\hat{N} = \begin{pmatrix} a & b & c \\ b & c & d \\ c & d & e \end{pmatrix}, \quad a,c,e > 0\,, \tag{37}$$

with determinant $\Delta = ace + 2bcd - ad^2 - b^2e - c^3 > 0$. Furthermore, we introduce the associated polynomials $G_{j,m,n}(\xi,\eta,\delta)$ using the generating function

$$e^{\underline{w}^T\hat{N}^{-1}\underline{k} - \frac{1}{2}\underline{k}^T\hat{N}^{-1}\underline{k}} = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{r^j}{j!}\frac{s^m}{m!}\frac{v^n}{n!}G_{j,m,n}(\xi,\eta,\delta)\,, \tag{38}$$

where

$$\underline{w} = \begin{pmatrix} \xi \\ \eta \\ \delta \end{pmatrix} = \hat{N}\underline{z}\,, \quad \underline{k} = \begin{pmatrix} r \\ s \\ v \end{pmatrix} = \hat{N}\underline{u}\,. \tag{39}$$

The generalized Rodriguez formulas for $H_{j,m,n}(x,p,y)$ and $G_{j,m,n}(\xi,\eta,\delta)$ are given, respectively, by

$$(40)\ H_{j,m,n}(x,p,y) = (-1)^{j+m+n}\exp(\frac{1}{2}\underline{z}^T\widehat{N}\underline{z})\frac{\partial^{j+m+n}}{\partial x^j\partial p^m\partial y^n}\exp(-\frac{1}{2}\underline{z}^T\widehat{N}\underline{z}),$$

$$(41)\ G_{j,m,n}(\xi,\eta,\delta) = (-1)^{j+m+n}\exp(\frac{1}{2}\underline{w}^T\widehat{N}^{-1}\underline{w})\frac{\partial^{j+m+n}}{\partial\xi^j\partial\eta^m\partial\delta^n}\exp(-\frac{1}{2}\underline{w}^T\widehat{N}^{-1}\underline{w}),$$

with

$$(42)\qquad \widehat{N}^{-1} = \frac{1}{\Delta}\begin{pmatrix} ce-d^2 & cd-be & bd-c^2 \\ cd-be & ae-c^2 & bc-ad \\ bd-c^2 & bc-ad & ac-b^2 \end{pmatrix}.$$

3.1. Recurrence relations: In this section, we derive some recurrence relations involving the functions H and G and their partial derivatives.

Theorem 1. *The Hermite polynomials $H_{j,m,n}(x,p,y)$ satisfy the recurrence relations:*

$$(43)\quad \begin{aligned} H_{j+1,m,n}(x,p,y) &= (ax+bp+cy)\,H_{j,m,n}(x,p,y) - aj\,H_{j-1,m,n}(x,p,y) \\ &\quad -bm\,H_{j,m-1,n}(x,p,y) - cn\,H_{j,m,n-1}(x,p,y), \end{aligned}$$

$$(44)\quad \begin{aligned} H_{j,m+1,n}(x,p,y) &= (bx+cp+dy)\,H_{j,m,n}(x,p,y) - bj\,H_{j-1,m,n}(x,p,y), \\ &\quad -cm\,H_{j,m-1,n}(x,p,y) - dn\,H_{j,m,n-1}(x,p,y), \end{aligned}$$

$$(45)\quad \begin{aligned} H_{j,m,n+1}(x,p,y) &= (cx+dp+ey)\,H_{j,m,n}(x,p,y) - cjH_{j-1,m,n}(x,p,y) \\ &\quad -dm\,H_{j,m-1,n}(x,p,y) - en\,H_{j,m,n-1}(x,p,y). \end{aligned}$$

Proof. It follows from (40) that

$$H_{j+1,m,n}(x,p,y) = -(-1)^{j+m+n}e^{\frac{1}{2}\underline{z}^T\widehat{N}\underline{z}}\frac{\partial^{j+m+n}}{\partial x^j\partial p^m\partial y^n}\frac{\partial}{\partial x}e^{-\frac{1}{2}\underline{z}^T\widehat{N}\underline{z}},$$

and in view of the identity

$$-\frac{1}{2}\underline{z}^T\widehat{N}\underline{z} = -\frac{a}{2}x^2 - bxp - cxy - dpy - \frac{c}{2}p^2 - \frac{e}{2}y^2$$

we have

$$H_{j+1,m,n}(x,p,y) = (-1)^{j+m+n}e^{\frac{1}{2}\underline{z}^T\widehat{N}\underline{z}}\frac{\partial^{j+m+n}}{\partial x^j\partial p^m\partial y^n}(ax+bp+cy)e^{-\frac{1}{2}\underline{z}^T\widehat{N}\underline{z}},$$

which may be written in the form

$$\begin{aligned} H_{j+1,m,n}(x,p,y) &= (-1)^{j+m+n}e^{\frac{1}{2}\underline{z}^T\widehat{N}\underline{z}}\left[\frac{\partial^{m+n}}{\partial p^m\partial y^n}\frac{\partial^j}{\partial x^j}\left(ax\,e^{-\frac{1}{2}\underline{z}^T\widehat{N}\underline{z}}\right)\right. \\ &\quad \left. + \frac{\partial^{j+n}}{\partial x^j\partial y^n}\frac{\partial^m}{\partial p^m}\left(bp\,e^{-\frac{1}{2}\underline{z}^T\widehat{N}\underline{z}}\right) + \frac{\partial^{j+m}}{\partial x^j\partial p^m}\frac{\partial^n}{\partial y^n}\left(cy\,e^{-\frac{1}{2}\underline{z}^T\widehat{N}\underline{z}}\right)\right]. \end{aligned}$$

Using Leibniz's rule for the derivative of the product, we obtain

$$\begin{aligned} H_{j+1,m,n}(x,p,y) &= (-1)^{j+m+n} e^{\frac{1}{2}\underline{z}^T \hat{N}\underline{z}} \left[ax \frac{\partial^{j+m+n}}{\partial x^j \partial p^m \partial y^n} + aj \frac{\partial^{j-1+m+n}}{\partial x^{j-1} \partial p^m \partial y^n} \right. \\ &\quad + bp \frac{\partial^{j+m+n}}{\partial x^j \partial p^m \partial y^n} + bm \frac{\partial^{j+m-1+n}}{\partial x^j \partial p^{m-1} \partial y^n} + \\ &\quad \left. cy \frac{\partial^{j+m+n}}{\partial x^j \partial p^m \partial y^n} + cn \frac{\partial^{j+m+n-1}}{\partial x^j \partial p^m \partial y^{n-1}} \right] e^{-\frac{1}{2}\underline{z}^T \hat{N}\underline{z}}, \end{aligned}$$

which leads to (43) by virtue of (40). The other two relations are obtained in a similar fashion. ∎

Theorem 2. *The first partial derivatives of $H_{j,m,n}(x,p,y)$ satisfy the following recurrence relations:*

$$\frac{\partial}{\partial x} H_{j,m,n} = aj\, H_{j-1,m,n} + bm\, H_{j,m-1,n} + cn\, H_{j,m,n-1}, \tag{46}$$

$$\frac{\partial}{\partial p} H_{j,m,n} = bj\, H_{j-1,m,n} + cm\, H_{j,m-1,n} + dn\, H_{j,m,n-1}, \tag{47}$$

$$\frac{\partial}{\partial y} H_{j,m,n} = cj\, H_{j-1,m,n} + dm\, H_{j,m-1,n} + en\, H_{j,m,n-1}. \tag{48}$$

Proof. From (40) and the identity on $\frac{1}{2}\underline{z}^T \hat{N}\underline{z}$, we have

$$\begin{aligned} \frac{\partial}{\partial x} H_{j,m,n} &= (-1)^{j+m+n} e^{\frac{1}{2}\underline{z}^T \hat{N}\underline{z}} \\ &\quad \times \left[(ax+bp+cy) \frac{\partial^{j+m+n}}{\partial x^j \partial p^m \partial y^n} + \frac{\partial^{j+1+m+n}}{\partial x^{j+1} \partial p^m \partial y^n} \right] e^{-\frac{1}{2}\underline{z}^T \hat{N}\underline{z}} \\ &= (ax+bp+cy)\, H_{j,m,n} - H_{j+1,m,n}, \end{aligned}$$

which, combined with (43), leads to (46). An analogous argument justifies relations (47) and (48). ∎

The corresponding results for $G_{j,m,n}(\xi,\eta,\delta)$ are:

$$\begin{aligned} G_{j+1,m,n} &= \frac{1}{\Delta} \Big[[(ce-d^2)\xi + (cd-be)\eta + (bd-c^2)\delta]\, G_{j,m,n} - \\ &\quad (ce-d^2)\, j\, G_{j-1,m,n} - (cd-be)\, m\, G_{j,m-1,n} - (bd-c^2)\, n\, G_{j,m,n-1} \Big] \end{aligned} \tag{49}$$

$$\begin{aligned} G_{j,m+1,n} &= \frac{1}{\Delta} \Big[[(cd-be)\xi + (ae-c^2)\eta - (ad-bc)\delta]\, G_{j,m,n} - \\ &\quad (cd-be)\, j\, G_{j-1,m,n} - (ae-c^2)\, m\, G_{j,m-1,n} + (ad-bc)\, n\, G_{j,m,n-1} \Big] \end{aligned} \tag{50}$$

$$G_{j,m,n+1} = \frac{1}{\Delta}\Big[[(bd-c^2)\,\xi-(ad-bc)\,\eta+(ac-b^2)\,\delta]\,G_{j,m,n}-$$

$$(bd-c^2)\,j\,G_{j-1,m,n}+(ad-bc)\,m\,G_{j,m-1,n}-(ac-b^2)\,n\,G_{j,m,n-1}\Big], \tag{51}$$

whereas their first partial derivatives satisfy the relations:

$$\frac{\partial}{\partial\xi}G_{j,m,n} = \frac{1}{\Delta}\Big[(ce-d^2)\,j\,G_{j-1,m,n}+(cd-be)\,m\,G_{j,m-1,n}$$

$$+(bd-c^2)\,n\,G_{j,m,n-1}\Big] \tag{52}$$

$$\frac{\partial}{\partial\eta}G_{j,m,n} = \frac{1}{\Delta}\Big[(cd-be)\,j\,G_{j-1,m,n}+(ae-c^2)\,m\,G_{j,m-1,n}$$

$$-(ad-bc)\,n\,G_{j,m,n-1}\Big] \tag{53}$$

$$\frac{\partial}{\partial\delta}G_{j,m,n} = \frac{1}{\Delta}\Big[(bd-c^2)\,j\,G_{j-1,m,n}-(ad-bc)\,m\,G_{j,m-1,n}$$

$$+(ac-b^2)\,n\;G_{j,m,n-1}\Big]. \tag{54}$$

The above relations may also be expressed in terms of the variables (x,p,y). From (36), (37) and (39) we have

$$\xi = ax+bp+cy\,,\quad \eta = bx+cp+dy\,,\quad \delta = cx+dp+ey\,, \tag{55}$$

and relations (49)–(51) are then transformed, respectively, into the following forms:

$$G_{j+1,m,n}(x,p,y) = x\,G_{j,m,n}(x,p,y)-\frac{1}{\Delta}\Big[(ce-d^2)\,j\,G_{j-1,m,n}(x,p,y)+$$

$$(cd-be)\,m\,G_{j,m-1,n}(x,p,y)+(bd-c^2)\,n\,G_{j,m,n-1}(x,p,y)\Big] \tag{56}$$

$$G_{j,m+1,n}(x,p,y) = p\,G_{j,m,n}(x,p,y)-\frac{1}{\Delta}\Big[(cd-be)\,j\,G_{j-1,m,n}(x,p,y)+$$

$$(ae-c^2)\,m\,G_{j,m-1,n}(x,p,y)-(ad-bc)\,n\,G_{j,m,n-1}(x,p,y)\Big] \tag{57}$$

$$G_{j,m,n+1}(x,p,y) = y\,G_{j,m,n}(x,p,y)-\frac{1}{\Delta}\Big[(bd-c^2)\,j\,G_{j-1,m,n}(x,p,y)-$$

$$(ad-bc)\,m\,G_{j,m-1,n}(x,p,y)+(ac-b^2)\,n\,G_{j,m,n-1}(x,p,y)\Big], \tag{58}$$

while (52)–(54) are expressed as:

$$\frac{\partial}{\partial x}G_{j,m,n}(x,p,y) = j\,G_{j-1,m,n}(x,p,y) \tag{59}$$

$$\frac{\partial}{\partial p}G_{j,m,n}(x,p,y) = m\,G_{j,m-1,n}(x,p,y) \tag{60}$$

$$\frac{\partial}{\partial y}G_{j,m,n}(x,p,y) = n\,G_{j,m,n-1}(x,p,y)\,. \tag{61}$$

Combining the former results, other recurrence relations follow. For example, from (43) and (46), we obtain:

$$\left[(ax+bp+cy)-\frac{\partial}{\partial x}\right]H_{j,m,n}(x,p,y)=H_{j+1,m,n}(x,p,y)\,, \tag{62}$$

and relations (44) and (47) lead to:

$$\left[(bx+cp+dy)-\frac{\partial}{\partial p}\right]H_{j,m,n}(x,p,y)=H_{j,m+1,n}(x,p,y)\,, \tag{63}$$

whereas (45) and (48) yield:

$$\left[(cx+dp+ey)-\frac{\partial}{\partial y}\right]H_{j,m,n}(x,p,y)=H_{j,m,n+1}(x,p,y)\,. \tag{64}$$

Furthermore, we have:

$$\frac{1}{\Delta}\left[(ce-d^2)\frac{\partial}{\partial x}+(cd-be)\frac{\partial}{\partial p}+(bd-c^2)\frac{\partial}{\partial y}\right]H_{j,m,n} \;=\; j\,H_{j-1,m,n} \tag{65}$$

$$\frac{1}{\Delta}\left[(cd-be)\frac{\partial}{\partial x}+(ae-c^2)\frac{\partial}{\partial p}+(bc-ad)\frac{\partial}{\partial y}\right]H_{j,m,n} \;=\; m\,H_{j,m-1,n} \tag{66}$$

$$\frac{1}{\Delta}\left[(bd-c^2)\frac{\partial}{\partial x}+(cb-ad)\frac{\partial}{\partial p}+(ac-b^2)\frac{\partial}{\partial y}\right]H_{j,m,n} \;=\; n\,H_{j,m,n-1} \tag{67}$$

To prove these three relations, we use (46), (47) and (48). From (46)–(47) we get

$$\left(c\frac{\partial}{\partial x}-b\frac{\partial}{\partial p}\right)H_{j,m,n} \;=\; (ac-b^2)j\,H_{j-1,m,n}+(c^2-bd)n\,H_{j,m,n-1} \tag{68}$$

$$\left(b\frac{\partial}{\partial x}-a\frac{\partial}{\partial p}\right)H_{j,m,n} \;=\; (b^2-ac)m\,H_{j,m-1,n}+(bc-ad)n\,H_{j,m,n-1}\,, \tag{69}$$

and from (46)–(48)

$$\left(d\frac{\partial}{\partial x}-b\frac{\partial}{\partial y}\right)H_{j,m,n} \;=\; (ad-bc)j\,H_{j-1,m,n}+(cd-be)n\,H_{j,m,n-1} \tag{70}$$

$$\left(c\frac{\partial}{\partial x}-a\frac{\partial}{\partial y}\right)H_{j,m,n} \;=\; (cb-ad)m\,H_{j,m-1,n}+(c^2-ae)n\,H_{j,m,n-1}\,. \tag{71}$$

Eliminating $H_{j,m,n-1}$ from (68) and (70) leads to (65), while eliminating the same from (69) and (71) gives (66); relation (67) follows from (68) and (70) after eliminating $H_{j-1,m,n}$. On the other hand, for $G_{j,m,n}(x,p,y)$ we have:

From (56) and (59)-(61)

$$\left[x-\frac{(ce-d^2)}{\Delta}\frac{\partial}{\partial x}-\frac{(cd-be)}{\Delta}\frac{\partial}{\partial p}-\frac{(bd-c^2)}{\Delta}\frac{\partial}{\partial y}\right]G_{j,m,n}=G_{j+1,m,n}\,. \tag{72}$$

From (57) and (59)-(61):

$$\left[p-\frac{(cd-bc)}{\Delta}\frac{\partial}{\partial x}-\frac{(ae-c^2)}{\Delta}\frac{\partial}{\partial p}+\frac{(ad-bc)}{\Delta}\frac{\partial}{\partial y}\right]G_{j,m,n}=G_{j,m+1,n}\,. \tag{73}$$

From (58) and (59)-(61):

$$\left[y-\frac{(bd-c^2)}{\Delta}\frac{\partial}{\partial x}+\frac{(ad-bc)}{\Delta}\frac{\partial}{\partial p}-\frac{(ac-b^2)}{\Delta}\frac{\partial}{\partial y}\right]G_{j,m,n}=G_{j,m,n+1}\,. \tag{74}$$

The following operators on $H_{j,m,n}(x,p,y)$ are obtained by combining (62) through (67):

$$\frac{1}{\Delta}\left[(ax+bp+cy)-\frac{\partial}{\partial x}\right]\left[(ce-d^2)\frac{\partial}{\partial x}+(cd-be)\frac{\partial}{\partial p}+(bd-c^2)\frac{\partial}{\partial y}\right]=j \tag{75}$$

$$\frac{1}{\Delta}\left[(bx+cp+dy)-\frac{\partial}{\partial p}\right]\left[(cd-be)\frac{\partial}{\partial x}+(ae-c^2)\frac{\partial}{\partial p}+(bc-ad)\frac{\partial}{\partial y}\right]=m \tag{76}$$

$$\frac{1}{\Delta}\left[(cx+dp+ey)-\frac{\partial}{\partial y}\right]\left[(bd-c^2)\frac{\partial}{\partial x}+(cb-ad)\frac{\partial}{\partial p}+(ac-b^2)\frac{\partial}{\partial y}\right]=n. \tag{77}$$

Results (59)-(61) combined with (72)-(74) yield the following operators on $G_{j,m,n}(x,p,y)$:

$$\left[x-\frac{(ce-d^2)}{\Delta}\frac{\partial}{\partial x}-\frac{(cd-be)}{\Delta}\frac{\partial}{\partial p}-\frac{(bd-c^2)}{\Delta}\frac{\partial}{\partial y}\right]\frac{\partial}{\partial x}\;=\;j, \tag{78}$$

$$\left[p-\frac{(cd-be)}{\Delta}\frac{\partial}{\partial x}-\frac{(ae-c^2)}{\Delta}\frac{\partial}{\partial p}+\frac{(ad-bc)}{\Delta}\frac{\partial}{\partial y}\right]\frac{\partial}{\partial p}\;=\;m, \tag{79}$$

$$\left[y-\frac{(bd-c^2)}{\Delta}\frac{\partial}{\partial x}+\frac{(ad-bc)}{\Delta}\frac{\partial}{\partial p}-\frac{(ac-b^2)}{\Delta}\frac{\partial}{\partial y}\right]\frac{\partial}{\partial y}\;=\;n. \tag{80}$$

Summing results (75)-(77), and in view of (42), we observe that $H_{j,m,n}$ satisfies

$$\left(\underline{z}^T\underline{\partial}_z-\underline{\partial}_z^T\widehat{N}^{-1}\underline{\partial}_z\right)H_{j,m,n}(x,p,y)=(j+m+n)H_{j,m,n}(x,p,y)\,,$$

which reduces to the eigenvalue equation:

$$\left(\underline{z}^T\underline{\partial}_z-\underline{\partial}_z^T\widehat{N}^{-1}\underline{\partial}_z\right)H_{j,m,n}(x,p,y)=(j+m+n)H_{j,m,n}(x,p,y)\,, \tag{81}$$

with $\underline{\partial}_z$ denoting the operator

$$\underline{\partial}_z = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial p} \\ \frac{\partial}{\partial y} \end{pmatrix} .$$

Similarly, summing results (78)-(80) we can prove that $G_{j,m,n}(x,p,y)$ satisfies the same eigenvalue equation:

$$(82) \qquad \left(\underline{z}^T \underline{\partial}_z - \underline{\partial}_z^T \widehat{N}^{-1} \underline{\partial}_z\right) G_{j,m,n}(x,p,y) = (j+m+n) G_{j,m,n}(x,p,y) .$$

3.2. Orthonormal functions $\mathcal{H}_{j,m,n}(x,p,y)$ and $\mathcal{G}_{j,m,n}(x,p,y)$: Along with the 3-variable, 3-index polynomials $H_{j,m,n}(x,p,y)$ and $G_{j,m,n}(x,p,y)$ we introduce the orthonormal functions:

$$(83) \qquad \mathcal{H}_{j,m,n}(x,p,y) = \frac{(\Delta)^{1/4}}{(2\pi)^{3/4}} \frac{1}{\sqrt{j!m!n!}} H_{j,m,n}(x,p,y) \exp(-\frac{1}{4} \underline{z}^T \widehat{N} \underline{z}) ,$$

$$(84) \qquad \mathcal{G}_{j,m,n}(x,p,y) = \frac{(\Delta)^{1/4}}{(2\pi)^{3/4}} \frac{1}{\sqrt{j!m!n!}} G_{j,m,n}(x,p,y) \exp(-\frac{1}{4} \underline{z}^T \widehat{N} \underline{z}) .$$

Theorem 3. *The orthonormal functions $\mathcal{H}$ and $\mathcal{G}$ satisfy the integral relation:*

$$(85) \quad \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathcal{H}_{j,m,n}(x,p,y) \, \mathcal{G}_{J,M,N}(x,p,y) \, dx \, dp \, dy = \delta_{j,J} \, \delta_{m,M} \, \delta_{n,N} .$$

P r o o f. The limits $\pm\infty$ on the integrals are omitted and we use the abbreviated form

$$\sum_{j,m,n} \quad \text{for} \quad \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} .$$

It follows from (83) and (84) that

$$(86) \qquad \int\int\int \mathcal{H}_{j,m,n} \mathcal{G}_{J,M,N} dx \, dp \, dy = \frac{\sqrt{\Delta}}{(2\pi)^{3/2}} \frac{1}{\sqrt{j!m!n!}\sqrt{J!M!N!}} I ,$$

where

$$(87) \qquad I = \int\int\int e^{-\frac{1}{2}\underline{z}^T \widehat{N} \underline{z}} H_{j,m,n} G_{J,M,N} dx \, dp \, dy .$$

To evaluate I we multiply both sides of (87) by $(t^j h^m k^n r^J s^M v^N)/(j!m!n!J!M!N!)$ and take the triple sums. Result (35) leads to

$$\sum_{j,m,n} \frac{t^j}{j!}\frac{h^m}{m!}\frac{k^n}{n!} \sum_{J,M,N} \frac{r^J}{J!}\frac{s^M}{M!}\frac{v^N}{N!} I$$

$$= \int\int\int e^{-\frac{1}{2}(\underline{z}-\underline{u})^T \widehat{N}(\underline{z}-\underline{u})} \sum_{J,M,N} \frac{r^J}{J!}\frac{s^M}{M!}\frac{v^N}{N!} G_{J,M,N}\, dx\, dp\, dy \quad . \tag{88}$$

Now, from (38), (39), (42) and the change of variables formula (55) we obtain

$$\sum_{j,m,n} \frac{r^j}{j!}\frac{s^m}{m!}\frac{v^n}{n!} G_{j,m,n}(\xi,\eta,\delta) = \exp\left[xr+ps+yv\right]$$

$$\times \exp\left[-\frac{1}{2\Delta}[(ce-d^2)r^2+2(cd-be)rs+2(bd-c^2)rv+(ae-c^2)s^2-2(ad-bc)sv+(ac-b^2)v^2]\right]. \tag{89}$$

Substituting (36), (37) and (89) in (88), we get

$$\sum_{j,m,n} \frac{t^j}{j!}\frac{h^m}{m!}\frac{k^n}{n!} \sum_{J,M,N} \frac{r^J}{J!}\frac{s^M}{M!}\frac{v^N}{N!} I = \exp\left[-\frac{1}{2\Delta}(\star)\right] \int\int\int \exp[(\star\star)]\, dx\, dp\, dy\,,$$

where

$$(\star) = (ce-d^2)r^2 + 2(cd-be)rs + 2(bd-c^2)rv + (ae-c^2)s^2 - 2(ad-bc)sv + (ac-b^2)v^2$$

$$(\star\star) = -\frac{a}{2}(x-t)^2 - \frac{c}{2}(p-h)^2 - \frac{e}{2}(y-k)^2$$
$$-b(x-t)(p-h) - c(x-t)(y-k) - d(p-h)(y-k) + x\,r + p\,s + y\,v\,.$$

With a changes of variables $u = x-t\,, w = p-h\,, q = y-k$, we get

$$\begin{aligned}(\star) &= (ce-d^2)r^2 + 2(cd-be)rs + 2(bd-c^2)rv + (ae-c^2)s^2 - 2(ad-bc)sv \\ &\quad +(ac-b^2)v^2 - 2\Delta\,(t\,r + h\,s + k\,v) \\ (\star\star) &= -\frac{a}{2}u^2 - buw - cuq - dwq - \frac{c}{2}w^2 - \frac{e}{2}q^2 + ur + ws + qv\,.\end{aligned}$$

Using the well known result [1],

$$\int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi}\,,$$

we evaluate the triple integral and obtain

$$\int\int\int [\ldots]\, du\, dw\, dq = \frac{(2\pi)^{3/2}}{\sqrt{\Delta}} \exp\left[(\star) + \frac{(bc-ad)}{\Delta}(\star\star)\right],$$

where

$$(\star) = \frac{1}{2a}r^2 + \frac{a}{2\tau}s^2 - \frac{b}{\tau}sr + \frac{b^2}{2a\tau}r^2 + \frac{\tau}{2\Delta}v^2 - \frac{c\tau}{a\Delta}vr + \frac{c^2\tau}{2a^2\Delta}r^2$$

$$(\star\star) = -sv - \frac{c}{a}sr - \frac{b}{a}rv + \frac{bc}{a^2}r^2 + \frac{(bc-ad)}{2\tau}s^2 - \frac{b(bc-ad)}{a\tau}sr + \frac{b^2(bc-ad)}{2a^2\tau}r^2\,,$$

and $\tau = ac - b^2 > 0$. Simplifying this expression leads to

$$\sum_{j,m,n} \frac{t^j}{j!}\frac{h^m}{m!}\frac{k^n}{n!} \sum_{J,M,N} \frac{r^J}{J!}\frac{s^M}{M!}\frac{v^N}{N!} I = \frac{(2\pi)^{3/2}}{\sqrt{\Delta}} \exp\left[tr + hs + kv\right]$$

and if we write the power series expansion for the exponential function, we have

$$\sum_{j,m,n} \frac{t^j}{j!}\frac{h^m}{m!}\frac{k^n}{n!} \sum_{J,M,N} \frac{r^J}{J!}\frac{s^M}{M!}\frac{v^N}{N!} I = \frac{(2\pi)^{3/2}}{\sqrt{\Delta}} \sum_{j,m,n=0}^{\infty} \frac{(tr)^j}{j!}\frac{(hs)^m}{m!}\frac{(kv)^n}{n!}\,.$$

Since I is in fact a function of (j, m, n, J, M, N), equating terms in the two series, we find that

$$I = I(j,m,n,J,M,N) = \begin{cases} \frac{(2\pi)^{3/2}}{\sqrt{\Delta}}\, j!\,m!\,n! & \text{if} \quad J = j,\ M = m,\ N = n, \\ 0 & \text{otherwise}. \end{cases}$$

This together with (86)–(87) lead to (85). ■

We now turn to the recurrence relations on $\mathcal{H}$ and its first order derivatives.

Theorem 4. *The orthonormal function $\mathcal{H}_{j,m,n}(x,p,y)$ satisfies the following recurrence relations:*

$$\text{(90)} \quad \left[\frac{1}{2}(ax + bp + cy) - \frac{\partial}{\partial x}\right]\mathcal{H}_{j,m,n}(x,p,y) = \sqrt{j+1}\,\mathcal{H}_{j+1,m,n}(x,p,y),$$

$$\text{(91)} \quad \left[\frac{1}{2}(bx + cp + dy) - \frac{\partial}{\partial p}\right]\mathcal{H}_{j,m,n}(x,p,y) = \sqrt{m+1}\,\mathcal{H}_{j,m+1,n}(x,p,y),$$

$$\text{(92)} \quad \left[\frac{1}{2}(cx + dp + ey) - \frac{\partial}{\partial y}\right]\mathcal{H}_{j,m,n}(x,p,y) = \sqrt{n+1}\,\mathcal{H}_{j,m,n+1}(x,p,y)\,.$$

Proof. We present the proof for relation (90); the other two relations are proved in a similar manner. From (83) we have:

$$\left[(ax + bp + cy) - \frac{\partial}{\partial x}\right]\mathcal{H}_{j,m,n}$$

$$= \frac{(\nabla)^{1/4}}{(2\pi)^{3/4}} \frac{1}{\sqrt{j!m!n!}} \left[(ax + bp + cy) - \frac{\partial}{\partial x} \right] H_{j,m,n}\, e^{-\frac{1}{4}\underline{z}^T \widehat{N} \underline{z}},$$

with

$$\frac{1}{4}\underline{z}^T \widehat{N} \underline{z} = \frac{a}{4}x^2 + \frac{b}{2}xp + \frac{c}{2}xy + \frac{d}{2}py + \frac{c}{4}p^2 + \frac{e}{4}y^2 . \tag{93}$$

It follows from (62) that

$$\left[(ax + bp + cy) - \frac{\partial}{\partial x} \right] \mathcal{H}_{j,m,n}$$

$$= \frac{(\nabla)^{1/4}}{(2\pi)^{3/4}} \frac{1}{\sqrt{j!m!n!}} e^{-\frac{1}{4}\underline{z}^T \widehat{N} \underline{z}} \left[H_{j+1,m,n} + \frac{1}{2}(ax + bp + cy)\, H_{j,m,n} \right]$$

and from the definition of $\mathcal{H}_{j,m,n}(x,p,y)$ we obtain

$$\left[(ax + bp + cy) - \frac{\partial}{\partial x} \right] \mathcal{H}_{j,m,n} = \sqrt{j+1}\, \mathcal{H}_{j+1,m,n} + \frac{1}{2}(ax + bp + cy)\mathcal{H}_{j,m,n} ,$$

which leads to (90). ■

We also derive further recurrence relations involving the three first order partial derivatives of $\mathcal{H}$.

Theorem 5. *The following recurrence relations hold for* $\mathcal{H}_{j,m,n}(x,p,y)$ *:*

$$\left[\frac{1}{\Delta} \left[(ce - d^2)\frac{\partial}{\partial x} + (cd - be)\frac{\partial}{\partial p} + (bd - c^2)\frac{\partial}{\partial y} \right] + \frac{x}{2} \right] \mathcal{H}_{j,m,n} = \sqrt{j}\, \mathcal{H}_{j-1,m,n}, \tag{94}$$

$$\left[\frac{1}{\Delta} \left[(cd - bc)\frac{\partial}{\partial x} + (ac - c^2)\frac{\partial}{\partial p} + (bc - ad)\frac{\partial}{\partial y} \right] + \frac{p}{2} \right] \mathcal{H}_{j,m,n} = \sqrt{m}\, \mathcal{H}_{j,m-1,n}, \tag{95}$$

$$\left[\frac{1}{\Delta} \left[(bd - c^2)\frac{\partial}{\partial x} + (cb - ad)\frac{\partial}{\partial p} + (ac - b^2)\frac{\partial}{\partial y} \right] + \frac{y}{2} \right] \mathcal{H}_{j,m,n} = \sqrt{n}\, \mathcal{H}_{j,m,n-1} . \tag{96}$$

P r o o f. From (83) we have

$$\frac{1}{\Delta} \left[(ce - d^2)\frac{\partial}{\partial x} + (cd - be)\frac{\partial}{\partial p} + (bd - c^2)\frac{\partial}{\partial y} \right] \mathcal{H}_{j,m,n} = \frac{(\Delta)^{1/4}}{(2\pi)^{3/4}} \frac{1}{\sqrt{j!m!n!}} \frac{1}{\Delta}$$

$$\times \left[(ce - d^2)\frac{\partial}{\partial x} + (cd - be)\frac{\partial}{\partial p} + (bd - c^2)\frac{\partial}{\partial y} \right] H_{j,m,n}\, e^{-\frac{1}{4}\underline{z}^T \widehat{N} \underline{z}} .$$

Using (93) and (65) leads, after simplification, to

$$\frac{1}{\Delta}\left[(ce-d^2)\frac{\partial}{\partial x}+(cd-be)\frac{\partial}{\partial p}+(bd-c^2)\frac{\partial}{\partial y}\right]\mathcal{H}_{j,m,n}$$

$$=\frac{(\Delta)^{1/4}}{(2\pi)^{3/4}}\frac{1}{\sqrt{j!m!n!}}j\,H_{j-1,m,n}\,e^{-\frac{1}{4}\underline{z}^T\widehat{N}\underline{z}}-\frac{(\Delta)^{1/4}}{(2\pi)^{3/4}}\frac{1}{\sqrt{j!m!n!}}\frac{x}{2}\,H_{j,m,n}\,e^{-\frac{1}{4}\underline{z}^T\widehat{N}\underline{z}}.$$

Then it follows from (83) that

$$\frac{1}{\Delta}\left[(ce-d^2)\frac{\partial}{\partial x}+(cd-be)\frac{\partial}{\partial p}+(bd-c^2)\frac{\partial}{\partial y}\right]\mathcal{H}_{j,m,n}=\sqrt{j}\,\mathcal{H}_{j-1,m,n}-\frac{x}{2}\mathcal{H}_{j,m,n}\,,$$

which leads to (94). The other two relations are similarly derived. ∎

We now define the following operators:

$$\widehat{a}_{1,+}=\frac{1}{2}(ax+bp+cy)-\frac{\partial}{\partial x}\tag{97}$$

$$\widehat{a}_{1,-}=\frac{1}{\Delta}\left[(ce-d^2)\frac{\partial}{\partial x}+(cd-be)\frac{\partial}{\partial p}+(bd-c^2)\frac{\partial}{\partial y}\right]+\frac{x}{2}\tag{98}$$

$$\widehat{a}_{2,+}=\frac{1}{2}(bx+cp+dy)-\frac{\partial}{\partial p}\tag{99}$$

$$\widehat{a}_{2,-}=\frac{1}{\Delta}\left[(cd-be)\frac{\partial}{\partial x}+(ae-c^2)\frac{\partial}{\partial p}+(bc-ad)\frac{\partial}{\partial y}\right]+\frac{p}{2}\tag{100}$$

$$\widehat{a}_{3,+}=\frac{1}{2}(cx+dp+ey)-\frac{\partial}{\partial y}\tag{101}$$

$$\widehat{a}_{3,-}=\frac{1}{\Delta}\left[(bd-c^2)\frac{\partial}{\partial x}+(cb-ad)\frac{\partial}{\partial p}+(ac-b^2)\frac{\partial}{\partial y}\right]+\frac{y}{2}.\tag{102}$$

From Theorems 4 and 5 above, we deduce the following relations on $\mathcal{H}$:

$$\widehat{a}_{1,+}\mathcal{H}_{j,m,n}(x,p,y)=\sqrt{j+1}\,\mathcal{H}_{j+1,m,n}(x,p,y)\tag{103}$$

$$\widehat{a}_{1,-}\mathcal{H}_{j,m,n}(x,p,y)=\sqrt{j}\,\mathcal{H}_{j-1,m,n}(x,p,y)\tag{104}$$

$$\widehat{a}_{2,+}\mathcal{H}_{j,m,n}(x,p,y)=\sqrt{m+1}\,\mathcal{H}_{j,m+1,n}(x,p,y)\tag{105}$$

$$\widehat{a}_{2,-}\mathcal{H}_{j,m,n}(x,p,y)=\sqrt{m}\,\mathcal{H}_{j,m-1,n}(x,p,y)\tag{106}$$

$$\widehat{a}_{3,+}\mathcal{H}_{j,m,n}(x,p,y)=\sqrt{n+1}\,\mathcal{H}_{j,m,n+1}(x,p,y)\tag{107}$$

$$\widehat{a}_{3,-}\mathcal{H}_{j,m,n}(x,p,y)=\sqrt{n}\,\mathcal{H}_{j,m,n-1}(x,p,y).\tag{108}$$

Taking the combined operators $\widehat{a}_{i,+}\,\widehat{a}_{i,-}$, leads to:

$$\widehat{a}_{1,+}\,\widehat{a}_{1,-}\mathcal{H}_{j,m,n}(x,p,y)=j\,\mathcal{H}_{j,m,n}(x,p,y)\tag{109}$$

$$\widehat{a}_{2,+}\,\widehat{a}_{2,-}\mathcal{H}_{j,m,n}(x,p,y) = m\,\mathcal{H}_{j,m,n}(x,p,y) \tag{110}$$

$$\widehat{a}_{3,+}\,\widehat{a}_{3,-}\mathcal{H}_{j,m,n}(x,p,y) = n\,\mathcal{H}_{j,m,n}(x,p,y). \tag{111}$$

If we add the above identities (109)–(111), we obtain the following eigenvalue equation:

$$\left[\frac{1}{2}\underline{z}^T\widehat{N} - \underline{\partial}_z^T\right]\left[\widehat{N}^{-1}\underline{\partial}_z + \frac{1}{2}\underline{z}\right]\mathcal{H}_{j,m,n}(x,p,y) = (j+m+n)\,\mathcal{H}_{j,m,n}(x,p,y)\,,$$

which is expressed as

$$\left[\frac{1}{2}\underline{z}^T\underline{\partial}_z + \frac{1}{4}\underline{z}^T\widehat{N}\underline{z} - \underline{\partial}_z^T\widehat{N}^{-1}\underline{\partial}_z - \frac{1}{2}\underline{\partial}_z^T\underline{z}\right]\mathcal{H}_{j,m,n}(x,p,y) = (j+m+n)\,\mathcal{H}_{j,m,n}(x,p,y)\,. \tag{112}$$

Finally, we list the corresponding results for the orthonormal functions $\mathcal{G}_{j,m,n}(x,p,y)$. Related to Theorem 4, we have:

$$\left[\frac{\partial}{\partial x} + \frac{1}{2}(ax+bp+cy)\right]\mathcal{G}_{j,m,n}(x,p,y) = \sqrt{j}\,\mathcal{G}_{j-1,m,n}(x,p,y) \tag{113}$$

$$\left[\frac{\partial}{\partial p} + \frac{1}{2}(bx+cp+dy)\right]\mathcal{G}_{j,m,n}(x,p,y) = \sqrt{m}\,\mathcal{G}_{j,m-1,n}(x,p,y) \tag{114}$$

$$\left[\frac{\partial}{\partial y} + \frac{1}{2}(cx+dp+ey)\right]\mathcal{G}_{j,m,n}(x,p,y) = \sqrt{n}\,\mathcal{G}_{j,m,n-1}(x,p,y)\,, \tag{115}$$

while the following relations correspond to Theorem 5:

$$\left[\frac{x}{2} - \frac{(ce-d^2)}{\Delta}\frac{\partial}{\partial x} - \frac{(cd-be)}{\Delta}\frac{\partial}{\partial p} - \frac{(bd-c^2)}{\Delta}\frac{\partial}{\partial y}\right]\mathcal{G}_{j,m,n} = \sqrt{j+1}\,\mathcal{G}_{j+1,m,n} \tag{116}$$

$$\left[\frac{p}{2} - \frac{(cd-be)}{\Delta}\frac{\partial}{\partial x} - \frac{(ae-c^2)}{\Delta}\frac{\partial}{\partial p} + \frac{(ad-bc)}{\Delta}\frac{\partial}{\partial y}\right]\mathcal{G}_{j,m,n} = \sqrt{m+1}\,\mathcal{G}_{j,m+1,n} \tag{117}$$

$$\left[\frac{y}{2} - \frac{(bd-c^2)}{\Delta}\frac{\partial}{\partial x} + \frac{(ad-bc)}{\Delta}\frac{\partial}{\partial p} - \frac{(ac-b^2)}{\Delta}\frac{\partial}{\partial y}\right]\mathcal{G}_{j,m,n} = \sqrt{n+1}\,\mathcal{G}_{j,m,n+1}\,. \tag{118}$$

Similarly, the following operators are introduced:

$$\widehat{a}^{+}_{1,+} = \frac{\partial}{\partial x} + \frac{1}{2}(ax+bp+cy) \tag{119}$$

$$\widehat{a}^{+}_{1,-} = \frac{x}{2} - \frac{1}{\Delta}\left[(ce-d^2)\frac{\partial}{\partial x} + (cd-be)\frac{\partial}{\partial p} + (bd-c^2)\frac{\partial}{\partial y}\right] \tag{120}$$

(121) $$\widehat{a}^{+}_{2,+} = \frac{\partial}{\partial p} + \frac{1}{2}(bx + cp + dy)$$

(122) $$\widehat{a}^{+}_{2,-} = \frac{p}{2} - \frac{1}{\Delta}\left[(cd - bc)\frac{\partial}{\partial x} + (ac - c^2)\frac{\partial}{\partial p} - (ad - bc)\frac{\partial}{\partial y}\right]$$

(123) $$\widehat{a}^{+}_{3,+} = \frac{\partial}{\partial y} + \frac{1}{2}(cx + dp + ey)$$

(124) $$\widehat{a}^{+}_{3,-} = \frac{y}{2} - \frac{1}{\Delta}\left[(bd - c^2)\frac{\partial}{\partial x} - (ad - bc)\frac{\partial}{\partial p} + (ac - b^2)\frac{\partial}{\partial y}\right],$$

which lead to:

(125) $$\widehat{a}^{+}_{1,+}\mathcal{G}_{j,m,n}(x,p,y) = \sqrt{j}\,\mathcal{G}_{j-1,m,n}(x,p,y)$$

(126) $$\widehat{a}^{+}_{1,-}\mathcal{G}_{j,m,n}(x,p,y) = \sqrt{j+1}\,\mathcal{G}_{j+1,m,n}(x,p,y)$$

(127) $$\widehat{a}^{+}_{2,+}\mathcal{G}_{j,m,n}(x,p,y) = \sqrt{m}\,\mathcal{G}_{j,m-1,n}(x,p,y)$$

(128) $$\widehat{a}^{+}_{2,-}\mathcal{G}_{j,m,n}(x,p,y) = \sqrt{m+1}\,\mathcal{G}_{j,m+1,n}(x,p,y)$$

(129) $$\widehat{a}^{+}_{3,+}\mathcal{G}_{j,m,n}(x,p,y) = \sqrt{n}\,\mathcal{G}_{j,m,n-1}(x,p,y)$$

(130) $$\widehat{a}^{+}_{3,-}\mathcal{G}_{j,m,n}(x,p,y) = \sqrt{n+1}\,\mathcal{G}_{j,m,n+1}(x,p,y),$$

and hence to:

(131) $$\widehat{a}^{+}_{1,-}\widehat{a}^{+}_{1,+}\mathcal{G}_{j,m,n}(x,p,y) = j\,\mathcal{G}_{j,m,n}(x,p,y)$$

(132) $$\widehat{a}^{+}_{2,-}\widehat{a}^{+}_{2,+}\mathcal{G}_{j,m,n}(x,p,y) = m\,\mathcal{G}_{j,m,n}(x,p,y)$$

(133) $$\widehat{a}^{+}_{3,-}\widehat{a}^{+}_{3,+}\mathcal{G}_{j,m,n}(x,p,y) = n\,\mathcal{G}_{j,m,n}(x,p,y),$$

which once summed up lead to the following eigenvalue equation for $\mathcal{G}$:

$$\left[\frac{1}{2}\underline{z}^T\underline{\partial}_z + \frac{1}{4}\underline{z}^T\widehat{N}\underline{z} - \underline{\partial}_z^T\widehat{N}^{-1}\underline{\partial}_z - \frac{1}{2}\underline{\partial}_z^T\underline{z}\right]\mathcal{G}_{j,m,n}(x,p,y)$$

(134) $$= (j+m+n)\,\mathcal{G}_{j,m,n}(x,p,y).$$

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