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On the Behavior of Approximations of of the SOR Tanabe's Method

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This article is dedicated to the 70th anniversary of Acad. Bl. Sendov

Convergence properties of the SOR Tanabe's method for the simultaneous determination of a polynomial roots are considered. The choice of the acceleration parameter is discussed.

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1. Introduction

Let

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

be the polynomial with the simple or complex zeros $z_i, i = 1, \ldots, n$. If $z_1^0, z_2^0, \ldots, z_n^0$ are initial approximations to these zeros, then the iteration formula

$$z_i^{k+1} = z_i^k - h_k \sigma_i^k \left(1 - \sum_{j \neq i}^{n} \frac{\sigma_j^k}{z_i^k - z_j^k} \right), \tag{1}$$

$$i = 1, 2, \ldots, n; \;\; k = 0, 1, 2, \ldots,$$

where

$$\sigma_i^k = \frac{f(z_i^k)}{\prod_{j \neq i}^{n} (z_i^k - z_j^k)}, \;\; i = 1, \ldots, n$$

and $h_k \in (0,1]$ is an acceleration parameter, defines the successive overrelaxation (SOR) Tanabe's method [6], [1].

It should be noted that the SOR Tanabe's method (1) has a form of prediction-correction procedure. The method (1) is often used in practice because of its perfect computing properties, and attempts at modification have been made with regard to the rate of convergence [4], [5].

Investigations of divergent starting points for some numerical methods show that for any monic polynomial f of degree n there exists a set $G_f \subset C^n$ such that these methods, starting from $\mathbf{z}^0 = \mathbf{z} \in G_f$, do not converge to the zeros of f, [2].

This set yielding divergent starting points and obtained as the of solutions of nonlinear systems of n equations, will be called a *NS - divergent set*, [3].

The aim of this paper is to prove similar assertion in the case of the SOR method (1).

2. New results. An effective formulation of SOR method (1)

The procedure (1) has several properties, which will be listed below. First, we have the following theorem.

Theorem 1. *Let z_i^{k+1} be determined by (1) for $i = 1,\dots,n$ and $k = 0,1,\dots,$ then the following relations are valid:*

$$
\begin{aligned}
&\frac{1}{h_k}\sum_{i=1}^{n} z_i^{k+1} = \left(\frac{1}{h_k}-1\right)\sum_{i=1}^{n} z_i^k - a_{n-1},\\
&\frac{1}{h_k}\sum_{i=1}^{n} z_i^{k+1}\sum_{j\neq i}^{n} z_j^k = \left(\frac{2}{h_k}-1\right)\sum_{l<s}^{n} z_l^k z_s^k + a_{n-2} - \sum_{l<s}^{n}\sigma_l^k\sigma_s^k,\\
(2)\quad &\frac{1}{h_k}\sum_{i=1}^{n} z_i^{k+1}\sum_{l,s\neq i,\, l<s}^{n} z_l^k z_s^k = \left(\frac{3}{h_k}-1\right)\sum_{l<s<t}^{n} z_l^k z_s^k z_t^k - a_{n-3} - \sum_{l<s}^{n}\sigma_l^k\sigma_s^k\sum_{p\neq l,s}^{n} z_p^k,\\
&\dots\\
&\frac{1}{h_k}\sum_{i=1}^{n} z_i^{k+1}\prod_{j\neq i}^{n} z_j^k = \left(\frac{n}{h_k}-1\right)\prod_{j=1}^{n} z_j^k + (-1)^n a_0 - \sum_{l<s}^{n}\sigma_l^k\sigma_s^k\prod_{p\neq l,s}^{n} z_p^k.
\end{aligned}
$$

Proof. Let $Q(z) = \prod_{j=1}^{n}(z - z_j^k)$. Then $Q'(z_i^k) = \prod_{j\neq i}^{n}(z_i^k - z_j^k)$. Using (1) we obtain

$$\sum_{i=1}^{n} z_i^{k+1} = \sum_{i=1}^{n} z_i^k - h_k \sum_{i=1}^{n} \sigma_i^k + h_k A,$$

where

$$A = \sum_{i=1}^{n} \sigma_i^k \sum_{j \neq i}^{n} \frac{\sigma_j^k}{z_i^k - z_j^k}.$$

Using the Euler formula

$$\sum_{i=1}^{n} \frac{(z_i^k)^t}{Q'(z_i^k)} = \begin{cases} \sum_{i=1}^{n} z_i^k, & t = n \\ 1, & t = n-1 \\ 0, & 0 \le t \le n-2 \end{cases},$$

we have

$$\sum_{i=1}^{n} \sigma_i^k = \sum_{i=1}^{n} \frac{f(z_i^k)}{\prod_{j \neq i}^{n} (z_i^k - z_j^k)} = \sum_{i=1}^{n} \frac{f(z_i^k)}{Q'(z_i^k)}$$

$$= \sum_{i=1}^{n} \frac{(z_i^k)^n + a_{n-1}(z_i^k)^{n-1} + \cdots + a_1 z_i^k + a_0}{Q'(z_i^k)} = \sum_{i=1}^{n} z_i^k + a_{n-1}.$$

Evidently,

$$A = \frac{f(z_1^k)}{\prod_{j \neq 1} (z_1^k - z_j^k)} \left(\frac{f(z_2^k)}{\prod_{j \neq 2} (z_2^k - z_j^k)} \frac{1}{z_1^k - z_2^k} + \cdots + \frac{f(z_n^k)}{\prod_{j \neq n} (z_n^k - z_j^k)} \frac{1}{z_1^k - z_n^k} \right)$$

$$+ \frac{f(z_2^k)}{\prod_{j \neq 2} (z_2^k - z_j^k)} \left(\frac{f(z_1^k)}{\prod_{j \neq 1} (z_1^k - z_j^k)} \frac{1}{z_2^k - z_1^k} + \cdots + \frac{f(z_n^k)}{\prod_{j \neq n} (z_n^k - z_j^k)} \frac{1}{z_2^k - z_n^k} \right) + \cdots$$

$$+ \frac{f(z_n^k)}{\prod_{j \neq n} (z_n^k - z_j^k)} \left(\frac{f(z_1^k)}{\prod_{j \neq 1} (z_1^k - z_j^k)} \frac{1}{z_n^k - z_1^k} + \cdots + \frac{f(z_{n-1}^k)}{\prod_{j \neq n-1} (z_{n-1}^k - z_j^k)} \frac{1}{z_n^k - z_{n-1}^k} \right).$$

$$= \sum_{1\le l,s\le n;\ l<s} \frac{f(z_l^k)f(z_s^k)}{\prod\limits_{j\ne l}^{n}(z_l^k - z_j^k)\prod\limits_{j\ne s}^{n}(z_s^k - z_j^k)} \left(\frac{1}{z_l^k - z_s^k} + \frac{1}{z_s^k - z_l^k}\right) = 0$$

and

$$\sum_{i=1}^{n} z_i^{k+1} = \sum_{i=1}^{n} z_i^k - h_k\left(\sum_{i=1}^{n} z_i^k + a_{n-1}\right),$$

i.e.

$$\frac{1}{h_k}\sum_{i=1}^{n} z_i^{k+1} = \left(\frac{1}{h_k} - 1\right)\sum_{i=1}^{n} z_i^k - a_{n-1}.$$

The other equations (2) are proved similarly. For example, we will prove the last one

$$\sum_{i=1}^{n} z_i^{k+1}\prod_{j\ne i}^{n} z_j^k = B + h_k C,$$

where

$$B = \sum_{i=1}^{n}(z_i^k - h_k\sigma_i^k)\prod_{j\ne i}^{n} z_j^k,$$

$$C = \sum_{i=1}^{n}\sigma_i^k\sum_{j\ne i}^{n}\frac{\sigma_j^k}{z_i^k - z_j^k}\prod_{j\ne i}^{n} z_j^k.$$

Hence

$$B = z_2^k z_3^k \dots z_n^k(z_1^k - h_k\sigma_1^k) + z_1^k z_3^k \dots z_n^k(z_2^k - h_k\sigma_2^k) + \dots + z_1^k z_2^k \dots z_{n-1}^k$$

$$\times(z_n^k - h_k\sigma_n^k) = n\prod_{j=1}^{n} z_j^k - h_k(z_2^k z_3^k \dots z_n^k\sigma_1^k + z_1^k z_3^k \dots z_n^k\sigma_2^k + \dots + z_1^k z_2^k \dots z_{n-1}^k\sigma_n^k)$$

$$= n\prod_{j=1}^{n} z_j^k - h_k\prod_{j=1}^{n} z_j^k\sum_{i=1}^{n}\frac{\sigma_i^k}{z_i^k} = n\prod_{j=1}^{n} z_j^k - h_k\prod_{j=1}^{n} z_j^k\sum_{i=1}^{n}\frac{f(z_i^k)}{z_i^k Q'(z_i^k)}$$

$$= n\prod_{j=1}^{n} z_j^k - h_k\prod_{j=1}^{n} z_j^k\left(\sum_{i=1}^{n}\frac{(z_i^k)^{n-1}}{Q'(z_i^k)} + a_{n-1}\sum_{i=1}^{n}\frac{(z_i^k)^{n-2}}{Q'(z_i^k)} + \dots + a_1\sum_{i=1}^{n}\frac{1}{Q'(z_i^k)}\right.$$

$$\left. + a_0\sum_{i=1}^{n}\frac{1}{z_i^k Q'(z_i^k)}\right) = n\prod_{j=1}^{n} z_j^k - h_k\prod_{j=1}^{n} z_j^k\left(1 + (-1)^{n-1}a_0\frac{1}{\prod\limits_{j=1}^{n} z_j^k}\right)$$

$$= (n - h_k)\prod_{j=1}^{n} z_j^k + (-1)^n a_0 h_k,$$

$$C = \sigma_1^k \left(\frac{\sigma_2^k}{z_1^k - z_2^k} + \cdots + \frac{\sigma_n^k}{z_1^k - z_n^k} \right) z_2^k z_3^k \ldots z_n^k + \sigma_2^k \left(\frac{\sigma_1^k}{z_2^k - z_2^k} + \cdots + \frac{\sigma_n^k}{z_2^k - z_n^k} \right)$$

$$\times z_1^k z_3^k \ldots z_n^k + \cdots = \sigma_{n-1}^k \left(\frac{\sigma_1^k}{z_{n-1}^k - z_1^k} + \cdots + \frac{\sigma_n^k}{z_{n-1}^k - z_n^k} \right) z_1^k \ldots z_{n-2}^k z_n^k$$

$$+ \sigma_n^k \left(\frac{\sigma_1^k}{z_n^k - z_1^k} + \cdots + \frac{\sigma_{n-1}^k}{z_n^k - z_{n-1}^k} \right) z_1^k z_2^k \ldots z_{n-1}^k$$

$$= \frac{\sigma_1^k \sigma_2^k}{z_1^k - z_2^k} z_3^k z_4^k \ldots z_n^k (z_2^k - z_1^k) + \cdots + \frac{\sigma_{n-1}^k \sigma_n^k}{z_n^k - z_{n-1}^k} z_1^k z_2^k \ldots z_{n-2}^k (z_{n-1}^k - z_n^k)$$

$$= -\sum_{l<s}^{n} \sigma_l^k \sigma_s^k \prod_{p \neq l,s}^{n} z_p^k,$$

i.e.

$$\sum_{i=1}^{n} z_i^{k+1} \prod_{j \neq i}^{n} z_i^k = (n - h_k)\prod_{j=1}^{n} z_j^k + (-1)^n a_0 h_k - h_k \sum_{l<s}^{n} \sigma_l^k \sigma_s^k \prod_{p \neq l,s}^{n} z_p^k.$$

Hence

$$\frac{1}{h_k} \sum_{i=1}^{n} z_i^{k+1} \prod_{j \neq i}^{n} z_j^k = \left(\frac{n}{h_k} - 1 \right) \prod_{j=1}^{n} z_j^k + (-1)^n a_0 - \sum_{l<s}^{n} \sigma_l^k \sigma_s^k \prod_{p \neq l,s}^{n} z_p^k.$$

Thus, the theorem is proved. ■

3. The divergent feature

Can we state such initial conditions under which the considered numerical method fails ? We present non-attractive starting point for the algorithm (1) which facilitate the choice of the initial approximations for the user. Actually, we have the following theorem.

Theorem 2. *Let $z_1^0, z_2^0, \ldots, z_n^0$ be the initial approximations and $0 < h_0 < 1$. The SOR Tanabe's method (1) will fail if these approximations satisfy* *the system*

$$
\begin{aligned}
&\left(\tfrac{1}{h_0}-1\right)\sum_{i=1}^{n} z_i^0 - a_{n-1} = 0,\\
(3)\qquad &\left(\tfrac{2}{h_0}-1\right)\sum_{l<s}^{n} z_l^0 z_s^0 + a_{n-2} - \sum_{l<s}^{n}\sigma_l^0\sigma_s^0 = 0,\\
&\dots\\
&\left(\tfrac{n}{h_0}-1\right)\prod_{j=1}^{n} z_j^0 + (-1)^n a_0 - \sum_{l<s}^{n}\sigma_l^0\sigma_s^0\prod_{p\neq l,s}^{n} z_p^0 = 0.
\end{aligned}
$$

P r o o f. for $k=0$, the system of equations (2) in vector form reads:

$$\mathbf{M}\mathbf{z}^1 = \mathbf{r}, \tag{4}$$

where $\mathbf{z}^1 = (z_1^0,\dots,z_n^0)$ is the vector of approximations obtained in the first iterative step by (1), $\mathbf{r}$ is the vector whose components are the terms on the left-hand side of (3) and

$$
M = \begin{pmatrix}
\frac{1}{h_0} & \frac{1}{h_0} & \cdots & \frac{1}{h_0}\\
\frac{1}{h_0}\sum_{j\neq 1} z_j^0 & \frac{1}{h_0}\sum_{j\neq 2} z_j^0 & \cdots & \frac{1}{h_0}\sum_{j\neq n} z_j^0\\
\vdots & \vdots & \cdots & \vdots\\
\frac{1}{h_0}\prod_{j\neq 1} z_j^0 & \frac{1}{h_0}\prod_{j\neq 2} z_j^0 & \cdots & \frac{1}{h_0}\prod_{j\neq n} z_j^0
\end{pmatrix}
$$

is the matrix of system (2). It is easy to find

$$\det\mathbf{M} = \frac{1}{h_0^n}\prod_{i<j}^{n}(z_i^0 - z_j^0) \neq 0.$$

Hence, and from (4), having in mind that $\mathbf{r}=0$ due to (3), we obtain

$$\mathbf{z}^1 = (z_1^0, z_2^0 \dots, z_n^0) = (0,0,\dots,0)$$

and the SOR Tanabe's method is not defined at the 2-nd step. Thus the theorem is proved. ∎

The optimal values of h_i^k (optimal in the sense that convergence is guaranteed) are not known.

4. Numerical method for solving the not-attractive initial approximations

We introduce a mapping $P : C^n \to C^n$,

$$F(Z^0) = \begin{pmatrix} f_1(z_1^0, \ldots, z_n^0) \\ f_2(z_1^0, \ldots, z_n^0) \\ \vdots \\ f_n(z_1^0, \ldots, z_n^0) \end{pmatrix}, \quad Z^0 = \begin{pmatrix} z_1^0 \\ z_2^0 \\ \vdots \\ z_n^0 \end{pmatrix} \in C^n,$$

where

$$f_1(z_1^0, \ldots, z_n^0) = \left(\tfrac{1}{h_0} - 1\right) \sum_{i=1}^{n} z_i^0 - a_{n-1},$$

(5)

$$f_2(z_1^0, \ldots, z_n^0) = \left(\tfrac{2}{h_0} - 1\right) \sum_{l<s}^{n} z_l^0 z_s^0 + a_{n-2} - \sum_{l<s}^{n} \sigma_l^0 \sigma_s^0,$$

$$\ldots$$

$$f_n^0(z_1^0, \ldots, z_n^0) = \left(\tfrac{n}{h_0} - 1\right) \prod_{j=1}^{n} z_j^0 + (-1)^n a_0 - \sum_{l<s}^{n} \sigma_l^0 \sigma_s^0 \prod_{p\neq l,s}^{n} z_p^0.$$

Let $(v_1, \ldots, v_n)^T = (z_1^0, \ldots, z_n^0)^T$ be the vector of solutions obtained by (5).

We apply the modified Newton method for solving nonlinear equation $F(V) = 0$

(6) $$V^{k+1} = V^k - F_1^{-1}(V^k) F(V^k), \quad k = 0, 1, 2, \ldots,$$

where

$$F_1(V^k) = \begin{pmatrix} \frac{\partial f_1(V)}{\partial v_1^k} & & & & \\ & \frac{\partial f_2(V)}{\partial v_2^k} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \frac{\partial f_n(V)}{\partial v_n^k} \end{pmatrix}_{V=V^k}$$

Using Werner's formula

$$\frac{\partial \sigma_i^k}{\partial v_j^k} = \begin{cases} \frac{\sigma_i^k}{v_i^k - v_j^k}, & i \neq j, \\ 1 - \sum\limits_{l \neq i}^{n} \frac{\sigma_l^k}{v_l^k - v_i^k}, & i = j, \end{cases}$$

we find

$$b_{11}^k = \frac{\partial f_1}{\partial v_1^k} = \frac{1}{h_k} - 1,$$

$$\begin{aligned} b_{22}^k &= \frac{\partial f_2}{\partial v_2^k} = \left(\frac{2}{h_k} - 1\right) \sum_{j \neq 2}^{n} v_i^k - \sum_{i=1}^{n} \frac{\partial \sigma_i^k}{\partial v_2^k} \sum_{j \neq i}^{n} \sigma_j^k \\ &= \left(\frac{2}{h_k} - 1\right) \sum_{j \neq 2}^{n} v_i^k - \frac{\partial \sigma_2^k}{\partial v_2^k} \sum_{j \neq i}^{n} \sigma_j^k - \sum_{j \neq 2}^{n} \frac{\partial \sigma_i^k}{\partial v_2^k} \sum_{j \neq i}^{n} \sigma_j^k \\ &= \left(\frac{2}{h_k} - 1\right) \sum_{j \neq 2}^{n} v_i^k - \left(1 - \sum_{l \neq 2}^{n} \frac{\partial \sigma_l^k}{v_l^k - v_2^k}\right) \sum_{j \neq 2}^{n} \sigma_j^k - \sum_{j \neq 2}^{n} \frac{\sigma_i^k}{v_i^k - v_2^k} \sum_{j \neq i}^{n} \sigma_j^k, \end{aligned}$$

$\cdots$

$$\begin{aligned} b_{nn}^k &= \frac{\partial f_n}{\partial v_n^k} = \left(\frac{n}{h_k} - 1\right) \prod_{j=1}^{n-1} v_j^k - \sum_{i=1}^{n} \frac{\partial \sigma_i^k}{\partial v_n^k} \sum_{j \neq i}^{n} \sigma_j^k \prod_{s \neq i,j}^{n} v_s^k - \sum_{l<t}^{n-1} \sigma_l^k \sigma_t^k \prod_{s \neq l,t}^{n-1} v_s^k \\ &= \left(\frac{n}{h_k} - 1\right) \prod_{j=1}^{n-1} v_j^k - \frac{\partial \sigma_n^k}{\partial v_n^k} \sum_{j \neq i} \sigma_j^k \prod_{s \neq n,j} v_s^k - \sum_{i=1}^{n-1} \frac{\partial \sigma_i^k}{\partial v_n^k} \sum_{j \neq i}^{n} \sigma_j^k \prod_{s \neq i,j} v_s^k \\ &\quad - \sum_{l<t}^{n-1} \sigma_l^k \sigma_t^k \prod_{s \neq l,t}^{n-1} v_s^k \\ &= \left(\frac{n}{h_k} - 1\right) \prod_{j=1}^{n-1} v_j^k - \left(1 - \sum_{l \neq n} \frac{\partial \sigma_l^k}{v_l^k - v_n^k}\right) \sum_{j \neq n} \sigma_j^k \prod_{s \neq n,j} v_s^k \\ &\quad - \sum_{j=1}^{n-1} \frac{\sigma_i^k}{v_i^k - v_n^k} \sum_{j \neq i} \sigma_j^k \prod_{s \neq i,j} v_s^k - \sum_{l<t}^{n-1} \sigma_l^k \sigma_t^k \prod_{s \neq l,t}^{n-1} v_s^k. \end{aligned}$$

Evidently, the matrix F_1 is non-singular. Then the matrix

$$F_1^{-1} = \begin{pmatrix} (b_{11}^k)^{-1} & & & & & \\ & (b_{22}^k)^{-1} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & (b_{nn}^k)^{-1} \end{pmatrix}$$

exists and (6) converges for $h_k < 1$.

5. Convergence properties of the SOR Tanabe's method

We will consider method

$$x_i^{k+1} = x_i^k - h_k I_i(x_i^k) \qquad i = 1, 2, \dots, n; \quad k = 0, 1, \dots, \tag{7}$$

where

$$I_i(x_i^k) = W_i(x_i^k)\Big(1 - \sum_{j \neq i}^{n} \frac{W_j(x_j^k)}{x_i^k - x_j^k}\Big) \qquad i = 1, 2, \dots, n; \quad k = 0, 1, \dots .$$

Let $W^k = \max_{1 \le i \le n} |W_i(x_i^k)|$, $d_k = \min_{i \neq j} |x_i^k - x_j^k|$. In practice, the initial conditions for the convergence of an iteration method for the simultaneous determination of polynomial roots is often given in the form of inequality $W^0 < \omega d_0 / n$, where $\omega > 0$ is a constant. In the case of SOR Tanabe's method this condition is replaced by

$$hW < \frac{\omega d}{2n}. \tag{8}$$

(For simplicity, we will omit iteration index and write h, d, W, instead h_k, d_k, W^k.)

In further consideration we will find the upper bound for ω.

Lemma 1. *If the inequality (8) holds, then*

(*i*) $|1 - \sigma_i^k| > \dfrac{2nh - n\omega + \omega}{2nh}$, *where* $\sigma_i^k = \sum_{j \neq i}^{n} \dfrac{W_j(x_j^k)}{x_i^k - x_j^k}$,

(*ii*) $|1 - \sigma_i^k| < \dfrac{2nh + n\omega - \omega}{2nh}$,

(*iii*) $|x_i^{k+1} - x_i^k| < \dfrac{\omega(2nh + n\omega - \omega)}{4n^2 h} d$,

(*iv*) $|x_i^{k+1} - x_j^k| > \dfrac{4n^2 h - (2nh + n\omega - \omega)\omega}{4n^2 h} d$,

$$(v)\quad |x_i^{k+1} - x_j^{k+1}| > \frac{4n^2h - 2(2nh + n\omega - \omega)\omega}{4n^2h} d.$$

Proof. By the definition of d and (8), we find

$$\begin{aligned} |1 - \sigma_i^k| &> 1 - \left| \sum_{j \neq i}^{n} \frac{W_j(x_j^k)}{x_i^k - x_j^k} \right| > 1 - (n-1)W/d \\ &> 1 - (n-1)\omega/2nh = \frac{2nh - n\omega + \omega}{2nh} \end{aligned}$$

which proves (i). Hence, we estimate

$$\begin{aligned} |1 - \sigma_i^k| &< 1 + \left| \sum_{j \neq i}^{n} \frac{W_j(x_j^k)}{x_i^k - x_j^k} \right| < 1 + (n-1)W/d \\ &< 1 + (n-1)\omega/2nh = \frac{2nh + n\omega - \omega}{2nh} \end{aligned}$$

$$|x_i^{k+1} - x_i^k| = hI_i(x_i^k) < hW|1 - \sigma_i^k| < \frac{(2nh + n\omega - \omega)\omega}{4n^2h} d$$

$$\begin{aligned} |x_i^{k+1} - x_j^k| &= |x_i^{k+1} - x_i^k + x_i^k - x_j^k| > |x_i^k - x_j^k| - |x_i^{k+1} - x_i^k| \\ &> d - \frac{(2nh + n\omega - \omega)\omega}{4n^2h} d = \frac{4n^2h - (2nh + n\omega - \omega)\omega}{4n^2h} d \\ |x_i^{k+1} - x_j^{k+1}| &= |x_i^{k+1} - x_i^k + x_i^k - x_j^k + x_j^k - x_j^{k+1}| \\ &> |x_i^k - x_j^k| - |x_i^{k+1} - x_i^k| - |x_i^{k+1} - x_i^k| \\ &> d - 2\frac{(2nh + n\omega - \omega)\omega}{4n^2h} d = \frac{4n^2h - 2(2nh + n\omega - \omega)\omega}{4n^2h} d. \end{aligned}$$

Here we assume that $\dfrac{4n^2h - 2(2nh + n\omega - \omega)\omega}{4n^2h} d > 0$ for every $n \geq 2$ which request $\omega/h < \sqrt{5} - 1$. ■

Lemma 2. *If the inequality (8) holds, then*

$$\prod_{j \neq i}^{n} \left| \frac{x_i^{k+1} - x_j^k}{x_i^{k+1} - x_j^{k+1}} \right| < e^{\frac{\omega(2h+\omega)}{4h}} \left(1 + \frac{\omega(2h + \omega)}{4h(1 + R(\omega, h))} \right)^{-R(\omega,h)},$$

where $R(\omega, h) = 1 - \omega - \omega^2/2h$.

Proof.

$$\prod_{j\neq i}^{n}\left|\frac{x_i^{k+1}-x_j^k}{x_i^{k+1}-x_j^{k+1}}\right| \leq \prod_{j\neq i}^{n}\left(1+\left|\frac{x_i^{k+1}-x_i^k}{x_i^{k+1}-x_j^{k+1}}\right|\right)$$

$$< \left(1+\frac{(2nh+n\omega-\omega)\omega}{4n^2h-2\omega(2nh+n\omega-\omega)}\right)^{n-1}$$

$$= \left(1+\frac{n\omega(2h+\omega)-\omega^2}{4nh(n-1)+4nh(1-\omega-\omega^2/2h)+2\omega^2}\right)^{n-1}$$

$$< \left(1+\frac{\omega(2h+\omega)}{4h(n-1)+4hR(\omega,h)}\right)^{n-1}$$

$$= \left(1+\frac{\omega(2h+\omega)}{4h(n-1)+4hR(\omega,h)}\right)^{n-1+R(\omega,h)}\left(1+\frac{\omega(2h+\omega)}{4h(n-1)+R(\omega,h))}\right)^{-R(\omega,h)}$$

$$< e^{\frac{\omega(2h+\omega)}{4h}}\left(1+\frac{\omega(2h+\omega)}{4h(1+R(\omega,h))}\right)^{-R(\omega,h)}.$$

By the Lagrangean interpolation formula we have

$$f(x)=\prod_{i=1}^{n}(x-x_i^k)\left(1+\sum_{j=1}^{n}\frac{W_i(x_i^k)}{x-x_j^k}\right). \tag{9}$$

■

Lemma 3. *If the inequality (8) holds, then*

$$|W_i(x_i^{k+1})| < e^{\frac{\omega(2h+\omega)}{4h}}\left(1+\frac{\omega(2h+\omega)}{4h(1+R(\omega,h))}\right)^{-R(\omega,h)}$$

$$\times\left(1-h+\frac{\omega^2(1+\omega/4)}{4h}\right)|W_i(x_i^k)|.$$

Proof.

$$|W_i(x_i^{k+1})| = |x_i^{k+1}-x_i^k|\prod_{j\neq i}^{n}\left|\frac{x_i^{k+1}-x_j^k}{x_i^{k+1}-x_j^{k+1}}\right|\left|1+\sum_{j=1} n\frac{W_j(x_j^k)}{x_i^{k+1}-x_j^k}\right|$$

$$< h|W_i(x_i^k)||1-\sigma_i^k|\left|1+\sum_{j=1}^{n}\frac{W_i(x_j^k)}{x_i^{k+1}-x_j^k}\right| e^{\frac{\omega(2h+\omega)}{4h}}\left(1+\frac{\omega(2h+\omega)}{4h(1+R(\omega,h))}\right)^{-R(\omega,h)}$$

$$\left|1+\sum_{j=1}^{n}\frac{W_i(x_j^k)}{x_i^{k+1}-x_j^k}\right| = \left|1+\frac{W_i(x_j^k)}{x_i^{k+1}-x_j^k}+\sum_{j\neq i}^{n}\frac{W_i(x_j^k)}{x_i^{k+1}-x_j^k}\right|.$$

From (7) we obtain

$$\frac{W_i(x_i^k)}{x_i^{k+1}-x_i^k} = -\frac{1}{h(1-\sigma_i^k)}$$

$$\left|1+\sum_{j=1}^{n}\frac{W_i(x_j^k)}{x_i^{k+1}-x_j^k}\right| = \left|1-\frac{1}{h(1-\sigma_i^k)}+\sum_{j\neq i}^{n}\frac{W_i(x_j^k)}{x_i^{k+1}-x_j^k}\right|$$

$$= \frac{1}{|1-\sigma_i^k|}\left|1-\frac{1}{h}+\sum_{j\neq i}^{n}\frac{W_i(x_j^k)}{x_i^{k+1}-x_j^k}-\sigma_i^k+\sigma_i^k\sum_{j\neq i}^{n}\frac{W_i(x_j^k)}{x_i^{k+1}-x_j^k}\right|$$

$$= \frac{1}{|1-\sigma_i^k|}\left|\frac{h-1}{h}+\sum_{j\neq i}^{n}\frac{W_i(x_j^k)(x_i^k-x_i^{k+1})}{(x_i^{k+1}-x_j^k)(x_i^k-x_j^k)}+\sigma_i^k\sum_{j\neq i}^{n}\frac{W_i(x_j^k)}{x_i^{k+1}-x_j^k}\right|$$

$$< \frac{1}{|1-\sigma_i^k|}\left(\frac{1-h}{h}+\frac{(n-1)W|x_i^k-x_i^{k+1}|}{|x_i^{k+1}-x_j^k|d}+|\sigma_i^k|\frac{(n-1)W}{|x_i^{k+1}-x_j^k|}\right)$$

$$= \frac{1}{|1-\sigma_i^k|}\left(\frac{1-h}{h}+\frac{(n-1)W}{|x_i^{k+1}-x_j^k|d}\left(|x_i^{k+1}-x_i^k|+|\sigma_i^k|d\right)\right)$$

$$< \frac{1}{|1-\sigma_i^k|}\left(\frac{1-h}{h}+\frac{(n-1)\omega 4n^2h}{2nh(4n^2h-\omega(2nh+n\omega-\omega))d}\left(\frac{\omega d|1-\sigma_i^k|}{2n}+|\sigma_i^k|d\right)\right)$$

$$< \frac{1}{|1-\sigma_i^k|}\left(\frac{1-h}{h}+\frac{(n-1)\omega 2n(\omega+|\sigma_i^k|\omega+2n|\sigma_i^k|)}{(4hn(n-1)+4nh-2nh\omega-n\omega^2+\omega^2)2n}\right)$$

$$= \frac{1}{|1-\sigma_i^k|}\left(\frac{1-h}{h}+\frac{(n-1)\omega(\omega+(2n+\omega)|\sigma_i^k|)}{4hn(n-1)+n(4hR(\omega,h)+2h\omega+\omega^2)+\omega^2}\right)$$

$$< \frac{1}{|1-\sigma_i^k|}\left(\frac{1-h}{h}+\frac{(n-1)\omega(2nh\omega+(2n+\omega)(n-1)\omega)}{2n^2h(4h(n-1)+4hR(\omega,h)+2h\omega+\omega^2)}\right)$$

$$< \frac{1}{|1-\sigma_i^k|}\left(\frac{1-h}{h}+\frac{(n-1)\omega^2(2n+\omega-2(1-h))}{2nh(4h(n-1)+4hR(\omega,h)+2h\omega+\omega^2)}\right)$$

$$< \frac{1}{|1-\sigma_i^k|}\left(\frac{1-h}{h}+\frac{\omega^2(1+\omega/2n)}{4h^2+\dfrac{4hR(\omega,h)+2h\omega+\omega^2}{n-1}}\right).$$

Having in mind that $4hR(\omega,h)+2h\omega+\omega^2>0$ for $\omega/h<\sqrt{5}-1$ we find

$$\left|1+\sum_{j=1}^{n}\frac{W_i(x_j^k)}{x_i^{k+1}-x_j^k}\right|<\frac{1}{|1-\sigma_i^k|}\left(\frac{1-h}{h}+\frac{\omega^2(1+\omega/2n)}{4h^2}\right)$$

$$<\frac{1}{|1-\sigma_i^k|}\left(\frac{1-h}{h}+\frac{\omega^2(1+\omega/4)}{4h^2}\right),$$

$$|W_i(x_i^{k+1})|<e^{\dfrac{\omega(2h+\omega)}{4h}}\left(1+\frac{\omega(2h+\omega)}{4h(1+R(\omega,h))}\right)^{-R(\omega,h)}$$

$$\times\left(1-h+\frac{\omega^2(1+\omega/4)}{4h}\right)|W_i(x_i^k)|.$$

Let

$$x(\omega,h)=e^{\dfrac{\omega(2h+\omega)}{4h}}\left(1+\frac{\omega(2h+\omega)}{4h(1+R(\omega,h))}\right)^{-R(\omega,h)}\left(1-h+\frac{\omega^2(1+\omega/4)}{4h}\right),$$

$$\alpha=\frac{h_{k+1}}{h}=\frac{h^*}{h},\quad y(\omega,h)=\frac{x(\omega,h)}{1-\dfrac{(2+\omega/h)\omega}{4}},\quad W^*=W^{k+1},\quad d^*=d_{k+1}.$$

■

Lemma 4. *If* $\alpha y(\omega,h)<1$, $y(\omega,h)<1$, *then*

(*i*) $h^*W^*<\dfrac{\omega d^*}{2n}$,

(*ii*) $|I_i(x_i^{k+1}|<\dfrac{2h+\omega}{2h-\omega}x(\omega,h)|I_i(x_i^k)|$.

P r o o f. Evidently,

$$h^*W^*<\frac{h^*}{h}x(\omega,h)hW\frac{d^*}{d^*}<\frac{h^*x(\omega,h)d^*d\omega}{hd^*2n}=\frac{h^*x(\omega,h)d}{hd^*}\frac{d^*\omega}{2n}$$

$$<\frac{h^*x(\omega,h)d}{h\dfrac{4n^2h-2(2nh+n\omega-\omega)\omega}{4n^2h}d}\frac{d^*\omega}{2n}<\frac{h^*x(\omega,h)}{h-\dfrac{(2h+\omega)\omega}{2n}}\frac{d^*\omega}{2n}$$

$$<\frac{h^*}{h}\frac{x(\omega,h)}{1-\dfrac{(2+\omega/h)\omega}{4}}\frac{d^*\omega}{2n}=\alpha y(\omega,h)\frac{d^*\omega}{2n}<\frac{d^*\omega}{2n},$$

$$|I_i(x_i^{k+1})|=|1-\sigma_i^{k+1}||W_i(x_i^{k+1})|<\left(1+(n-1)\frac{W^*}{d^*}\right)x(\omega,h)W_i(x_i^k)\frac{|1-\sigma_i^k|}{|1-\sigma_i^k|}$$

$$< \left(1 + (n-1)y(\omega, h)\frac{W}{d}\right)x(\omega, h)\frac{|I_i(x_i^k)|}{|1-\sigma_i^k|}$$

$$< \left(1 + \frac{(n-1)\omega}{2nh}\right)x(\omega, h)\frac{|I_i(x_i^k)|}{1 - \dfrac{(n-1)\omega}{2nh}} < \frac{2h+\omega}{2h-\omega}x(\omega, h)|I_i(x_i^k)|.$$

■

Define $q(\omega, h) = \dfrac{2h+\omega}{2h-\omega}x(\omega, h)$.

Lemma 5. *If $\omega/h < 0.828$, then $q(\omega, h) > y(\omega, h)$.*

Proof.

$$q(\omega, h) > y(\omega, h) \iff \frac{2h+\omega}{2h-\omega} > \frac{1}{1 - \dfrac{(2+\omega/h)\omega}{4}} \iff$$

$$\iff (2h+\omega)(4-\omega(2+\omega/h)) > 4(2h-\omega) \iff -\omega^2/h - 4\omega - 4h + 8 > 0 \iff$$

$$\iff h(2+\omega/h)^2 < 8,$$

and from $\omega/h < 0.828 \Longrightarrow h(2+\omega/h)^2 < 8$. ■

Theorem 3. *Let $h_k \leq h_{k+1}$ and $\alpha = \max\limits_k \dfrac{h_{k+1}}{h_k}$. If the inequalities*

$$\left|\begin{array}{l} \alpha q(\omega, h_0) < 1 \\ \dfrac{4h_0}{\omega(2h_0-\omega)} > g(\alpha q(\omega, h_0)) \end{array}\right.$$

hold and $\omega/h_0 < 0.828$, then the iterative process (7) is convergent.

Proof. For fixed $\omega \leq 0.82$ the function $q(\omega, h)$ is monotonically decreasing and from $q(\omega, h_0) < 1 \Rightarrow q(\omega, h_k) < 1$ for every k.

From $\alpha q(\omega, h_0) < 1 \Rightarrow q(\omega, h_0) < 1$ and if $\omega/h_0 < 0.828$, then from Lemma 4 and Lemma 5, we obtain

$$(i)\quad h_k W^k < \frac{\omega}{2n} \quad k = 0, 1, \ldots$$

$$(ii)\quad h_{k+1}|I_i(x_i^{k+1})| < \alpha q(\omega, h_0)h_k|I_i(x_i^k)| \quad k = 0, 1, \ldots,$$

which means $x_i^k \longrightarrow x_i^*$ and x_i^* are zeros of $f(x)$. The next condition guarantees that $x_i^* \neq x_j^*$ for $i \neq j$. Evidently,

$$d_0 > g(\alpha q(\omega, h_0))(h_0|I_i(x_i^0)| + h_0|I_j(x_j^0)|),$$

and from $h_0 W^0 < \dfrac{\omega}{2n}$, we have

$$d_0 > \frac{n}{\omega}(h_0|W_i(x_i^0)| + h_0|W_j(x_j^0)|) > \frac{n2h_0}{\omega(2h_0-\omega)}(h_0|I_i(x_i^0)| + h_0|I_j(x_j^0)|) \Longrightarrow$$

$$\Longrightarrow \frac{n2h_0}{\omega(2h_0-\omega)} > g(\alpha q(\omega, h_0)).$$

This must be right for $n \geq 2$, but:

$$\frac{n2h_0}{\omega(2h_0-\omega)} > \frac{4h_0}{\omega(2h_0-\omega)} > g(\alpha q(\omega, h_0)),$$

which completes the proof of the theorem. ■

We represent in Figure 1 and Figure 2 the quantities $q(\omega, h)$ and $\frac{4h}{\omega(2h-\omega)} - g(q(\omega, h))$ as functions of ω taking h as a parameter. From the equation $q(\omega, 1) = 1$ we find the upper bound for ω. From the above consideration it is clear that the choice of a small h is not advisable.

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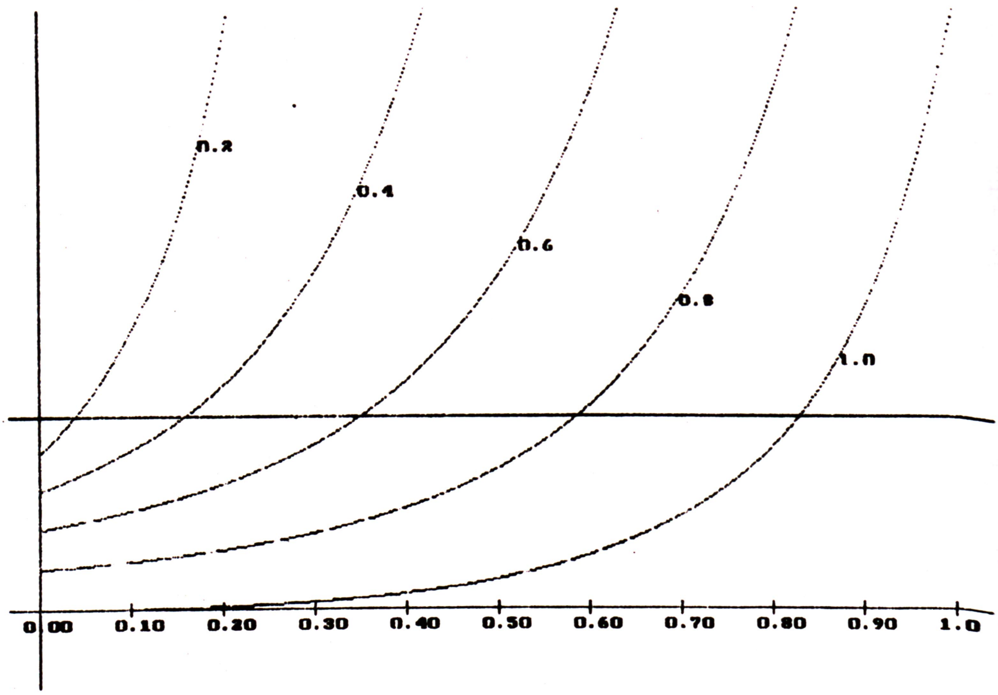

Fig. 1

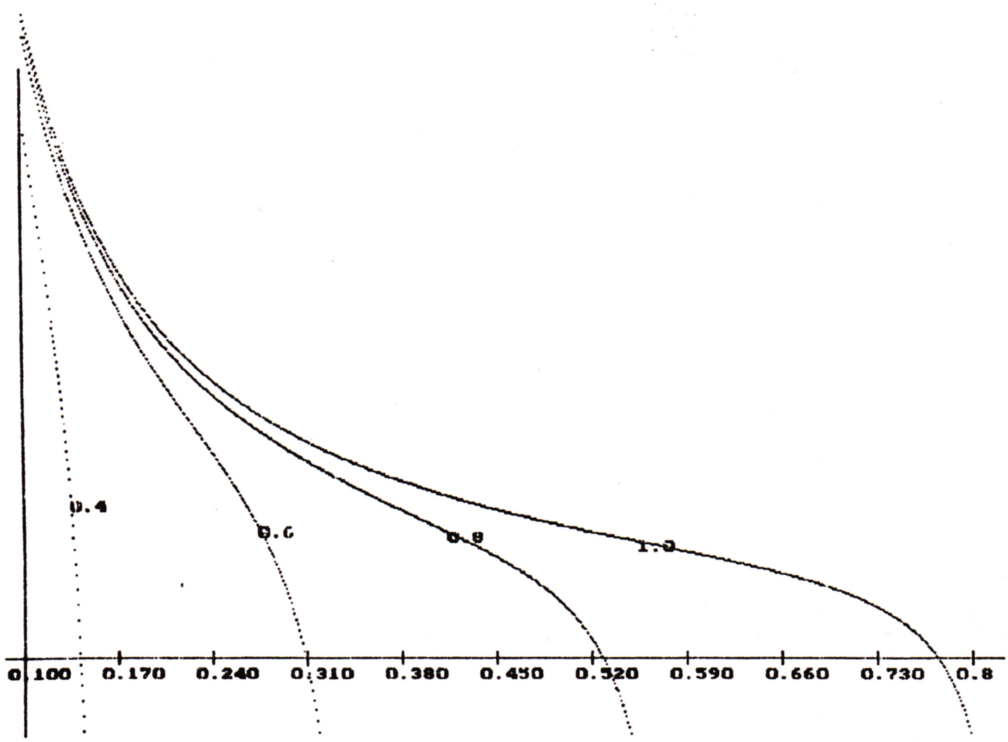
0.4
0.6
0.8
1.0
0.100
0.170
0.240
0.310
0.380
0.450
0.520
0.590
0.660
0.730
0.8

Fig. 2