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Pure Multigrades

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This article is dedicated to the 70th anniversary of Acad. Bl. Sendov

Two sets of integers A and B form a pure multigrade of type (D, n) if $A \cap B$ is empty,

$$\sum_{x_k \in A} x_k^d = \sum_{y_k \in B} y_k^d \qquad d = 0, 1, 2, \ldots, D, \tag{1}$$

and $|A| = |B| = n$. A pure multigrade is perfect, if $A \cup B = \{1, 2, \ldots, 2n\}$. We prove some new theorems about pure multigrades and state some open problems.

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1. Introduction

A multigrade of type (D, n) is a pair of multisets A and B of integers such that $|A| = |B| = n$, and

$$\sum_{x_k \in A} x_k^d = \sum_{y_k \in B} y_k^d \tag{2}$$

for $d = 1, 2, \ldots, D$. Here, D is the *degree* of the multigrade (A, B) and n is the *size*.

The word multiset implies that some integers may occur more than once in A or B. Equation (2) and its various generalizations and specializations have been investigated for many years; see Gloden [1] and Borwein and Ingalls [2].

Clearly, if A and B contain a common element q it can be dropped from both sets without harm to the condition (2). In this paper, A and B will be true sets with no repetitions permitted.

We say that A and B form a *pure multigrade* (A, B) of type (D, n) if A and B are sets that satisfy equation (2) and $A \cap B = \emptyset$. Equation (2) can be extended to include more than two sets and in this situation the existence of a solution was widely known as the Tarry-Escott problem. It was a surprise when Wright [3] pointed out that Prouhet [4] had announced that the enlarged system ((2) for more than two sets) always has a solution and indicated his method for computing such asolution. Further, Wright explained in some detail the solution that Prouhet had merely hinted at. Prouhet's work was published 70 years before the Tarry and Escott papers appeared and 70 years before the problem was formulated. However, there are still many open questions related to equation (2) and its variations.

We will need two well-known theorems that are basic to the subject. We say that the set $S = \{u_1, u_2, \ldots, u_n\}$ is subjected to the affine transformation $T : v = au + b$, where a and b are integers if $T(S)$ is the set $\{v_k = au_k + b \mid k = 1, 2, \ldots, n\}$. Then we have the following theorem.

Theorem 1. (Frolov [5]) *If the sets A and B satisfy the system of equations* (2)*, then for any T the sets $T(A)$ and $T(B)$ satisfy the same system of equations.*

Thus, the transformation T takes a multigrade of type (D, n) into another multigrade of the sametype.

Theorem 1 shows that for a multigrade (A, B) the sets A or B may contain negativeintegers, so some normalization should be made if we wish to avoid this.

We will call a multigrade, (A, B), normalized if $1 \in A$ and all other integers in $A \cup B$ are greater than one. It is clear that any pure multigrade of type (D, n) can be transformed into a normalized pure multigrade of the same type by using a suitable T and Theorem 1. Gloden [1, p. 21] has a different definition for a normalized multigrade.

Two multigrades, (A, B) and (C, D), are said to be equivalent if either can be obtained from the other by a suitable affine transformation $T : v = au + b$, where a and b are integers. Notice that this set of transformations does not form a group, but only a semigroup, because the inverse of atransformation may not have the proper form.

Theorem 2. (Bastien [6]) *If (A, B) is a pure multigrade of type (D, n), then*

$$n \geq D + 1. \tag{3}$$

It is convenient to have the standard notation

$$x_1, x_2, \ldots, x_n \stackrel{D}{=} y_1, y_2, \ldots, y_n$$

for a solution of (2). With this notation we have the following examples of pure normalized multigrades of type $(2, n)$:

(4) $\quad n = 3 \quad 1, 5, 6 \stackrel{2}{=} 2, 3, 7$

(5) $\quad n = 4 \quad 1, 4, 6, 7 \stackrel{2}{=} 2, 3, 5, 8$

(6) $\quad n = 5 \quad 1, 5, 8, 10, 13 \stackrel{2}{=} 2, 3, 9, 11, 12$

2. The union of two multigrades

If (A_1, B_1) and (A_2, B_2) are two multigrades of type (D_1, n_1) and (D_2, n_2), respectively, we can form a new multigrade (A^*, B^*) by taking their union

(7) $$A^* = A_1 \cup A_2 \quad \text{and} \quad B^* = B_1 \cup B_2.$$

We indicate this union by writing

$$(A^*, B^*) = (A_1, B_1) \cup (A_2, B_2).$$

Here (A^*, B^*) has type (D^*, n^*), where $n^* = n_1 + n_2$ and $D^* = \min\{D_1, D_2\}$. Henceforth, we will assume that $D_1 = D_2 = D$.

If (A_1, B_1) and (A_2, B_2) are pure multigrades, it does not follow that (A^*, B^*) will be a pure multigrade. But the union will be a pure multigrade of type (D, n^*) if $\max\{A_1 \cup B_1\} < \min\{A_2 \cup B_2\}$. For example, if we transform the multigrade (5) by $T : v = u+7$, then we obtain the multigrade $8, 11, 13, 14 \stackrel{2}{=} 9, 10, 12, 15$. Then the union of this multigrade and the one given by (4) is:

(8) $$1, 5, 6, 8, 11, 13, 14 \stackrel{2}{=} 2, 3, 7, 9, 10, 12, 15,$$

a normalized multigrade of type $(2, 7)$.

Definition 1. A pure multigrade is said to be *primitive*, if it cannot be obtained as the union of two pure multigrades with no elements in common.

At present, we do not have any efficient general method for determining whether a pure multigrade is primitive or not. However, we can prove the following.

Theorem 3. *If $D \geq 2$ and $n \leq 5$, then a pure multigrade of type (D, n) is primitive.*

P r o o f. If the components in the union have type (D, n_1) and (D, n_2), then $n_1 + n_2 \leq 5$. Hence $n_k \leq 2$ for some k, and by Bastien's theorem this is impossible, if $D \geq 2$. ■

Theorem 4. *The decomposition of a pure multigrade into the union of primitive pure multigrades is not always unique.*

P r o o f. We consider the example

$$1, 4, 6, 7, 9, 12, 15, 19, 24 \stackrel{2}{=} 2, 3, 5, 8, 10, 11, 14, 21, 23\,, \tag{9}$$

a pure multigrade of type $(2, 9)$. A reasonable amount of computation will show that

$$(A, B) = (C_1, D_1) \cup (E_1, F_1) \quad \text{and} \quad (A, B) = (C_2, D_2) \cup (E_2, F_2)\,, \tag{10}$$

where

$$\begin{aligned} C_1 &= \{1, 4, 6, 7\} & D_1 &= \{2, 3, 5, 8\} \\ E_1 &= \{9, 12, 15, 19, 24\} & F_1 &= \{10, 11, 14, 21, 23\} \end{aligned} \tag{11}$$

and

$$\begin{aligned} C_2 &= \{5, 8, 10, 11\} & D_2 &= \{6, 7, 9, 12\} \\ E_2 &= \{2, 3, 14, 21, 23\} & F_2 &= \{1, 4, 15, 19, 24\}\,. \end{aligned} \tag{12}$$

Now each of the sets in (11) and (12) has $n = 4$ or 5, so each of the multigrades in the unions (10) is a primitive multigrade. The multigrade (C_1, D_1) isequivalent to (C_2, D_2) under $T : v = u + 4$ but (E_1, F_1) is not equivalent to either (E_2, F_2) or (F_2, E_2). Hence, (10) gives a decomposition of (A, B) into a union of primitive multigrades in two different ways. ■

Counting the numbers of primes less than a given constant has long been a fascinating problem. Thus, it may be of interest to count the number of primitive normalized pure multigrades for which $M = \max\{A \cup B\} \leq M_0$. Here, of course, if a number of primitive multigrades are equivalent we count only the one with the smallest M.

3. Perfect multigrades

It will be very convenient to have a short symbol for certain sums that occur frequently. Thus we will let

$$S_d(C) = \sum_{c_k \in C} c_k^d. \tag{13}$$

We will call a pure multigrade of type (D, n) *perfect*, if

$$A \cup B = \{1, 2, \ldots, 2n\},$$

the first $2n$ integers.

Theorem 5. *If $n = 4k$, with $k \geq 1$ there is a perfect multigrade of type $(2, n)$.*

Proof. If $k = 1$ we have the perfect multigrade given in equation (5), where $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

For larger values of k we first observe that for any integer q, we have $q + (q + 3) = (q + 1) + (q + 2)$ so we can place one pair in A and one pair in B without disturbing the equality $\Sigma_A x = \Sigma_B y$. For the squares we have $q^2 + (q + 3)^2 - (q + 1)^2 - (q + 2)^2 = 4$, so if one pair is in A and the other pair is in B, the equality $\Sigma_A x^2 = \Sigma_B y^2$ is unbalanced by 4. But if $|A| = |B| = 4k$, we can alternate the placement of these pairs so that $\Sigma_A x^2 = \Sigma_B y^2$.

For example, if $k = 3$ so that $4k = 12$ and $|A \cup B| = 24$, we can set

$$A = \{1, 4, \quad 6, 7, \quad 9, 12, \quad 14, 15, \quad 17, 20, \quad 22, 23\}$$

and

$$B = \{2, 3, \quad 5, 8, \quad 10, 11, \quad 13, 16 \quad 18, 19, \quad 21, 24\}.$$

Then (A, B) is a perfect normalized multigrade of type $(2, 12)$.

In general, if $n = 4k$, this method will give a perfect normalized multigrade of type $(2, 4k)$ in which $M = \max\{A \cup B\} = 8k = 2n$. ■

Theorem 6. *If $n = 4k + 2$ with $k \geq 1$, there is a perfect multigrade of type $(2, n)$.*

Proof. For $k = 1$ we have $n = 6$. Set $A = \{1, 3, 7, 8, 9, 11\}$ and $B = \{2, 4, 5, 6, 10, 12\}$. Then $S_1(A) = S_1(B) = 39$ and $S_2(A) = S_2(B) = 325$, so (A, B) form a perfect multigrade of type $(2, 6)$. For larger values of k we add integers q, $q + 3$, $q + 1$, $q + 2$ in alternating pairs as in the proof of Theorem 5. ■

For example if $k = 2$, so $n = 10$ we have

$$A = \{1, 3, 7, 8, 9, 11, 13, 16, 18, 19\}$$

and

$$B = \{2, 4, 5, 6, 10, 12, 14, 15, 17, 20\}.$$

Here $S_1(A) = S_1(B) = 105$ and $S_2(A) = S_2(B) = 1435$.

If n is odd, then $\sum_1^{2n} k = n(2n+1)$, an odd number. Thus a partition with $S_1(A) = S_1(B)$ is impossible. Consequently, there are no perfect multigrades with n odd. But if n is odd, there are multigrades that are as close to perfect as possible, namely $A \cup B \subset \{1, 2, 3, \ldots, 2n+1\}$.

Theorem 7. *If $n = 4k+1$ and $k \geq 2$, then there is a normalized pure multigrade of type $(2, n)$ with $M = \max(A \cup B) = 2n+1$.*

Proof. For $k = 2$ we have $n = 9$ and we set

$$A = \{1, 3, 7, 8, 10, 11, 13, 17, 19\},$$

and

$$B = \{2, 4, 5, 6, 9, 14, 15, 16, 18\}.$$

Here $S_1(A) = S_1(B) = 89$ and $S_2(A) = S_2(B) = 1163$.

For larger values of k add integers q, $q+3$, $q+1$, $q+2$ in alternating pairs as in the proof of Theorem 5. ■

If $k = 1$, there is NO normalized pure multigrade of type $(2, 5)$ with $M = 11$.

Theorem 8. *If $n = 4k+3$ and $k \geq 0$, then there is a normalized pure multigrade of type $(2, n)$ with $M = 2n+1$.*

Proof. For $k = 0$, use the multigrade given in equation (4), where $M = 7 = 2(3) + 1$.

For larger values of k add integers q, $q+3$, $q+1$, $q+2$ in alternating pairs as in the proof of Theorem 5. ■

For example if $k = 2$, so $n = 11$ we have

$$A = \{1, 5, 6, 8, 11, 13, 14, 16, 19, 21, 22\}$$

and

$$B = \{2, 3, 7, 9, 10, 12, 15, 17, 18, 20, 23\}.$$

Here $S_1(A) = S_1(B) = 136$ and $S_2(A) = S_2(B) = 2154$.

4. Multigrades as vectors

In this section, we regard any set $C = \{c_1, c_2, \ldots, c_n\}$ as a vector with components c_i. As usual, addition of two such sets will be termwise and we will be using the standard dot product $A \cdot B$ of two sets.

Theorem 9. *Let (A, B) and (C, D) be two multigrades each of type $(2, n)$. Then the vector sums $(A + C, B + D)$ will be a multigrade of the same type iff as a dot product $A \cdot C = B \cdot D$.*

Proof. The condition $\sum_1^n a_i + c_i = \sum_1^n b_i + d_i$ is trivial. For the sum of the squares we need

$$\sum_{i=1}^{n} (a_i + c_i)^2 = \sum_{i=1}^{n} (b_i + d_i)^2 . \tag{14}$$

Since $S_2(A) = S_2(B)$ and $S_2(C) = S_2(D)$, then (14) holds iff

$$\sum_{i=1}^{n} a_i c_i = \sum_{i=1}^{n} b_i d_i, \tag{15}$$

or $A \cdot C = B \cdot D$. ■

Of course the new multigrade will not always be perfect, or even pure, but $(A+C, B+D)$ will always have only integers and can always be normalized. The following two examples show that the theorem is not empty.

Set $A = \{1, 5, 6\}$, $B = \{2, 3, 7\}$, $C = \{1, 20, 12\}$, and $D = \{10, 2, 21\}$. Then (A, B) and (C, D) are multigrades of type $(2, 3)$. Further, $A \cdot C = B \cdot D = 173$. Consequently, $(A + C, B + D)$ is a multigrade of type $(2, 3)$. Here, $(A+C) = \{2, 25, 18\}$ and $(B+D) = \{12, 5, 28\}$ with $S_1(A+C) = S_1(B+D) = 45$ and $S_2(A + C) = S_2(B + D) = 953$.

For a more general example, we can use the formula given by Gloden [1, p.33]. For any 8 integers a, b, c, d, r, s, t, and u let

$$A = \{ab, cd, ac + bd\}, \quad B = \{ac, bd, ab + cd\},$$

and

$$C = \{rs, tu, rt + su\}, \quad D = \{rt, su, rs + tu\}.$$

Then (A, B) and (C, D) are multigrades of type $(2, 3)$. A brief computation will show that $A \cdot C = B \cdot D$ iff $au = dr$ or $cs = bt$. It is clear that one can select the variables in many ways so that all of the multigrades involved are pure.

5. Open problems

Let $\mathcal{G}$ be a collection of pure normalized multigrades (A, B) of type $(2, 3)$ with the following properties:

(1) no two multigrades in $\mathcal{G}$ are equivalent,

(2) every pure multigrade of type $(2, 3)$ is equivalent to one of the multigrades in $\mathcal{G}$.

It is clear that such a set $\mathcal{G}$ exists. What is lacking, is a formula that will give $\mathcal{G}$. The beautiful formula given by Gloden [1, p.33] gives all multigrades of type $(2, 3)$, but it gives far too many multigrades. It gives some multigrades that are NOT pure, it violates both the conditions for $\mathcal{G}$, and there is no hint of the restrictions on the parameters that will give a formula for $\mathcal{G}$.

The sets (A, B) defined by

$$A = \{1, 2 + x + 2s + kx + sx, 2 + 2x + x^2 + s - k + 2sx\}$$

and

$$B = \{2 + x + s - k, 1 + kx + sx, 2 + 2x + x^2 + 2s + 2sx\}$$

satisfy the equations $S_1(A) = S_1(B)$ and $S_2(A) = S_2(B)$ for all values of the three parameters k, s, and x. Hence, these formulas generate a multigrade of type $(2, 3)$ for all integervalues of the three parameters. What restrictions on these parameters will turn this collection of pairs (A, B) into a set $\mathcal{G}$?

We can introduce a measure $L(D, n)$, called the length. As usual let $M = \max\{A \cup B\}$ and for fixed (D, n), let $L(D, n) = \min\{M\}$ for the set of all normalized pure multigrades of type (D, n). For $D = 2$, the value of $L(D, n)$ was obtained in Theorems 5, 6, 7, and 8. For $D \geq 3$, the value of $L(D, n)$ is unknown, but a computer search will give these values for small D and n.

During the preparation of this work, Prof. Gerald Myerson made important contributions which I am happy to acknowledge. Prof. Myerson proved Theorems 6 and 7. He also found the nice conditions $au = dr$ or $cs = bt$ for $A \cdot C = B \cdot D$ in Theorem 9. Further, he found the two interesting examples given at the end of that theorem.

In a letter, [8] Myerson proved that for every $D \geq 2$, there is a perfect multigrade of type $(D, 2^D)$.

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