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# On Walsh Functions with Respect to Weights [1]

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*Dedicated to the 70th birthday of Professor Bl. Sendov*

The orthogonal polynomials are constructed from the power functions by means of the Schmidt orthogonalization process in which the scalar product is generated by weight functions. There is another general construction with probabilistic method in the background that can be applied to obtain orthogonal and biorthogonal systems. Namely, an important class of orthogonal systems can be received by starting from normed martingale differences, or more generally, from conditionally orthonormal systems, and taking their product system. Among others, the Walsh system and its generalizations can be originated in this way. These systems are effectively applied in image and data processing. Moreover, the Fourier coefficients with respect to these systems can be calculated by FFT algorithms.

In this paper we consider the dyadic stochastic basis, but instead of the Lebesgue measure we take a probability measure, generated by a positive weight function $\rho \in L^1$. For every such weight function there exists a sequence of martingale differences, the product system of which is orthogonal with respect to the weight function. In particular, with proper choice of the weight function we receive the so called Walsh similar functions constructed by Sendov [12], [13].

In this paper we construct Walsh-type systems based on a product decomposition of $\rho$ but under more general condition than in [7]. We show that under fairly general conditions also the reciprocal system is the product system of a normed dyadic martingale system if a proper weight function is used. The Fourier partial sums with respect to product systems can be expressed by martingale transform operators. This can be applied for the study of convergence problems of these series.

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## 1. Introduction

In this paper we fix a weight function $\rho \in L^1([0,1))$, $\rho > 0$, with $\int_0^1 \rho(x)\,dx = 1$ and investigate dyadic martingale with respect to the probability measure spaces $(\mathbb{I}, \mathcal{A}, \mu)$, where $\mathcal{A}$ is the collection of Lebesgue-measurable sets in $\mathbb{I} := [0,1)$ and $\mu(A) = \int_A \rho(x)\,dx$ $(A \in \mathcal{A})$. Denote by

$$\mathcal{A}_n := \{[k2^{-n}, (k+1)2^{-n}) : k = 0, 1, \cdots, 2^n - 1\} \quad (n \in \mathbb{N})$$

the set of dyadic intervals with the length $2^{-n}$ and let $\mathcal{A}_n$ the $\sigma$-algebra, generated by $\mathcal{A}_n$. The set of real $\mathcal{A}_n$ measurable functions, defined on $\mathbb{I}$ is denoted by $L(\mathcal{A}_n)$. Obviously

$$L(\mathcal{A}_n) = \operatorname{span}\{\chi_I : I \in \mathcal{J}_n\} \quad (n \in \mathbb{N}),$$

where $\chi_I$ is the characteristic function of the set $I$. The conditional expectation $E_n^\rho$ with respect to $\mathcal{A}_n$ is of the form

$$(1.1) \qquad (E_n^\rho f)(x) = \frac{\int_I f(s)\rho(s)\,ds}{\int_I \rho(s)\,ds} \quad (x \in I \in \mathcal{A}_n, f \in L_\rho^1(\mathbb{I})).$$

It is known (see [2], [3], [8]), that

$$(1.2) \qquad E_n^\rho(\lambda f) = \lambda E_n^\rho f, \quad E_n^\rho(E_m^\rho f) = E_{\min\{m,n\}}^\rho f$$

$$(\lambda \in L(\mathcal{A}_n), f \in L_\rho^1(\mathbb{I}), m, n \in \mathbb{N}).$$

In the case $\rho = 1$ we shall use the notation $E_n := E_n^1$ $(n \in \mathbb{N})$. Obviously,

$$(1.3) \qquad E_n^\rho f = \frac{E_n(\rho f)}{E_n \rho} \quad (f \in L_\rho^1(\mathbb{I}), n \in \mathbb{N}).$$

The sequence $\Phi = (\phi_n, n \in \mathbb{N})$ is called a *normalized dyadic martingale difference sequence* in the probability space $(\mathbb{I}, \mathcal{A}, \mu)$, if

$$(1.4) \quad i)\ \phi_n \in L(\mathcal{A}_{n+1}), \qquad ii)\ E_n^\rho(\phi_n) = 0, \qquad iii)\ E_n^\rho(|\phi_n|^2) = 1 \quad (n \in \mathbb{N}).$$

It is clear that in the case $\rho = 1$ the Rademacher system $R = (r_n, n \in \mathbb{N})$ satisfies (1.4). Recall (see [8]) that

$$r_n(x) := \begin{cases} 1, & (x \in [k2^{-n}, (k+1/2)2^{-n})) \\ -1, & (x \in [(k+1/2)2^{-n}, (k+1)2^{-n})) \end{cases} \qquad (k = 0, 1, \cdots, 2^n - 1, n \in \mathbb{N}).$$

In the general case, taking the standardization of $r_n$ in the space $(\mathbb{I}, \mathcal{A}, \mu)$ we get the system

$$(1.5)\quad \phi_n := \frac{r_n - E_n^\rho r_n}{\sqrt{E_n^\rho(|r_n - E_n^\rho r_n|^2)}} = \frac{r_n - b_n}{\sqrt{1-b_n^2}}, \quad b_n := E_n^\rho r_n = \frac{E_n(\rho r_n)}{E_n(\rho)} \quad (n \in \mathbb{N}),$$

satisfying (1.4).

We remark that if for the sequence $a_n \in L(\mathcal{A}_n)$ $(n \in \mathbb{N})$ the condition $|a_n| = 1$ $(n \in \mathbb{N})$ holds, then $\varphi_n := a_n\phi_n$ $(n \in \mathbb{N})$ satisfies (1.4) too. Moreover, every sequence $\varphi_n$ $(n \in \mathbb{N})$ satisfying (1.4) can be written in this form.

The product system of the system $\Phi = (\phi_n, n \in \mathbb{N})$ (see [1], [5], [8]) is defined by

$$(1.6)\qquad \psi_m := \prod_{k=0}^{\infty} \phi_k^{m_k} \quad (m \in \mathbb{N}),$$

where the numbers $m_k \in \{0, 1\}$ are the digits in the dyadic representation

$$(1.7)\qquad m = \sum_{k=0}^{\infty} m_k 2^k.$$

Especially, the product system of the Rademacher system is the Walsh system in Paley's enumeration, which is orthonormed in $L^2(\mathbb{I})$ (see [8]).

In the papers [4], [5], [9] and [10] product systems of normalized martingale difference systems have been investigated in a more general form. Applying these results for this special case (see Lemma in [5]) we get the following result.

**Theorem A.**
*The product system $\Psi^\rho = (\psi_n, n \in \mathbb{N})$ is orthonormal in $L^2_\rho$, i.e.*

$$\langle \psi_n, \psi_m \rangle := \int_{\mathbb{I}} \psi_n(x)\psi_m(x)\rho(x)\, dx = \delta_{mn} \quad (m, n \in \mathbb{N}),$$

*where $\delta_{mn}$ is the Kronecker symbol (see Lemma in [5]).*

The weight function $\rho$ can be written in the product form

$$\rho = \prod_{j=0}^{\infty}(1 + b_j r_j), \quad b_j = \frac{E_j(\rho r_j)}{E_j(\rho)} \quad (j \in \mathbb{N}).$$

Indeed, it is easy to see that for the partial product we have

$$\rho_n := \prod_{j=0}^{n-1}(1 + b_j r_j) = E_n\rho \quad (n \in \mathbb{N})$$

and consequently, $\rho_n \to \rho$ a.e. and in $L^1(\mathbb{I})$ norm as $n \to \infty$ (see e.g. [8]).

Indeed, $\rho_0 := 1 = E_0(\rho)$. Since (see [8]) $E_{n+1}\rho = E_n\rho + r_n E_n(r_n\rho)$, by induction and by definition we get

$$\rho_{n+1} = \rho_n + r_n\rho_n b_n = E_n\rho + r_n E_n\rho \frac{E_n(r_n\rho)}{E_n\rho} = E_{n+1}\rho.$$

We remark that the construction of Walsh similar functions by Sendov (see [11], [12], [7]) was based on a starting function $\psi > 0$ given in the form

$$\psi^2 = \prod_{j=0}^{\infty}(1 + b_j r_j),$$

where $b_j \in \mathbb{R}$ are numbers and

$$\sum_{j=0}^{\infty} |b_j| < \infty, \quad |b_j| < 1 \ (j \in \mathbb{N}).$$

It turns auto that in this case the Walsh-similar functions introduced in [11] can be expressed by the product system generated by the weight $\rho = \psi^2$, namely $(\sqrt{\rho}\psi_n, n \in \mathbb{N})$ coincides with the Walsh-similar system of Sendov.

By (1.5) the reciprocal function of $\phi_n$ is of the form

$$\phi_n^{-1} := 1/\phi_n = \frac{r_n + b_n}{\sqrt{1 - b_n^2}} = \frac{r_n(1 + b_n r_n)}{\sqrt{1 - b_n^2}} \quad (n \in \mathbb{N}). \tag{1.8}$$

We show that under certain conditions the product system of this system, i.e. $\Psi^{-1} := (\psi_n^{-1}, n \in \mathbb{N})$ is orthonormal with respect to the weight function

$$\rho^- := \prod_{j=0}^{\infty}(1 - b_j r_j). \tag{1.9}$$

We remark that the partial product

$$\rho_0^- := 1, \quad \rho_n^- := \prod_{j=0}^{n-1}(1 - b_j r_j) \quad (n \in \mathbb{N}^*)$$

forms a dyadic martingale with respect to the Lebesgue-space, i.e.

$$\rho_n^- \in L(\mathcal{A}_n), \quad E_n(\rho_{n+1}^-) = \rho_n^- \quad (n \in \mathbb{N}).$$

Furthermore,

$$\rho_n^-(x) > 0, \quad \int_{\mathbb{I}} \rho_n^-(t)\,dt = 1 \quad (n \in \mathbb{N}, x \in \mathbb{I}),$$

i.e. the martingale $(\rho_n^-, n \in \mathbb{N})$ is $L^1$-bounded. This implies (see [2],[3]) that the infinite product in (1.9) converges a.e. to the limit $\rho^- \geq 0$ and $\rho^- \in L^1(\mathbb{I})$ (see [2], [3]).

We will prove (see next section) the following theorem.

**Theorem 1.**
*Suppose that the maximal function of the martingale* $(\rho_n^-, n \in \mathbb{N})$ *satisfies*

$$\sup_{n\in\mathbb{N}} \rho_n^- \in L^1(\mathbb{I}). \tag{1.10}$$

*Then* $\Psi^{-1}$ *is orthonormal with respect to the weight function* $\rho^-$.

The partial sums

$$S_n^\rho f := 0, \quad S_n^\rho f := \sum_{k=0}^{n-1} \langle f, \psi_k \rangle \psi_k \quad (n \in \mathbb{N}^*) \tag{1.11}$$

of the function $f \in L_\rho^1(\mathbb{I})$ can be expressed by the conditional expectations $E_n^\rho$ $(n \in \mathbb{N})$ (see [5]).

**Theorem B.** *For every* $f \in L_\rho^1(\mathbb{I})$ *and for any* $m, n \in \mathbb{N}$ *we have*

$$S_{2^n}^\rho f = E_n^\rho f, \quad S_m^\rho f = \sum_{k=0}^{\infty} m_k \psi_{m^{k+1}} E_k^\rho (f \psi_{m^{k+1}}), \tag{1.12}$$

*where*

$$m^k := \sum_{j=k}^{\infty} m_j 2^j \quad (k \in \mathbb{N}).$$

In the case when $\rho = 1$ we have $\psi_n^2 = 1$ and the partial sums can be expressed by the martingale transform operator

$$T_m f := \sum_{j=0}^{\infty} m_j r_j E_j(f r_j) = \sum_{j=0}^{\infty} m_j (E_{j+1} - E_j) f \quad (f \in L^1(\mathbb{I})).$$

Namely,

$$S_m f = \psi_m T_m (f \psi_m) \quad (m \in \mathbb{N}). \tag{1.13}$$

This formula is the background of the convergence theorems with respect to the Walsh system.

In order to get the analogue of this formula in the general case we introduce the martingale transform operators

$$T_m^\rho f := \sum_{k=0}^{\infty} a_m^k E_k^{\rho^-}(\phi_k^{-1} f)\phi_k^{-1} \quad (f \in L^1_{\rho^-}, m \in \mathbb{N}), \tag{1.14}$$

where the $\mathcal{A}_k$ measurable coefficients are defined by

$$a_m^k := m_k \prod_{j=0}^{k-1} \frac{(1 - b_j r_j)^{1-m_j}}{(1 + b_j r_j)^{1-m_j}} \quad (m, k \in \mathbb{N}). \tag{1.15}$$

We remark that in the case $\rho = 1$ the martingale transform operator $T_m^\rho$ coincides with $T_m$.

We will prove (see next section) the following generalization of (1.12).

**Theorem 2.** *For any function $f \in L^1_\rho(\mathbb{I})$ we have*

$$S_m^\rho f = \psi_m T_m^\rho (f \widehat{\rho} \psi_m) \quad (\widehat{\rho} := \rho/\rho^-, m \in \mathbb{N}). \tag{1.16}$$

## 2. Proofs

Proof of Theorem 1. We show that the system $(\phi_n^{-1}, n \in \mathbb{N})$ is a normalized dyadic martingale difference with respect to the weight $\rho^-$. Since $b_j \in L(\mathcal{A}_j)$ $(j \in \mathbb{N})$ we have

$$E_j(1 - b_j r_j) = E_j 1 = 1.$$

Thus for the products

$$\rho_{nm} := \prod_{j=n}^{m} (1 - b_j r_j), \quad \rho^n := \prod_{j=n}^{\infty} (1 - b_j r_j)$$

we get by (1.2) that

$$E_n \rho_{nm} = E_n(\rho_{n(m-1)} E_m(1 - b_m r_m)) = E_n(\rho_{n(m-1)}) = \cdots = E_n(\rho_{nn}) = 1.$$

Taking the limit as $m \to \infty$ condition (1.10) implies

$$E_n \rho^n = E_n(\lim_{m\to\infty} \rho_{nm}) = \lim_{m\to\infty} E_n \rho_{nm} = 1 \quad (n \in \mathbb{N}). \tag{2.1}$$

Consequently by (1.2) we have

$$E_n(\rho^-) = \rho_n^- E_n(\rho^n) = \rho_n^- \quad (n \in \mathbb{N}). \tag{2.2}$$

In order to show (1.4) for the system $(\phi_n^{-1}, n \in \mathbb{N})$ we apply (1.2), (1.8) and (2.1) to get

$$E_n(\phi_n^{-1}\rho^-) = \frac{E_n(r_n(1+b_nr_n)\rho^-)}{\sqrt{1-b_n^2}} = \frac{\rho_n^- E_n(r_n(1+b_nr_n)(1-b_nr_n)E_{n+1}\rho^{n+1})}{\sqrt{1-b_n^2}}$$
$$= \frac{\rho_n^-(1-b_n^2)E_nr_n}{\sqrt{1-b_n^2}} = 0.$$

Consequently,

$$E_n^{\rho^-}(\phi_n^{-1}) = \frac{E_n(\phi_n^{-1}\rho^-)}{E_n(\rho^-)} = 0.$$

In a similar way, by (1.2),(1.8) and (2.1) we get

$$E_n(|\phi_n^{-1}|^2\rho^-) = \frac{E_n((1+b_nr_n)^2\rho^-)}{1-b_n^2} = \frac{\rho_n^- E_n((1+b_nr_n)^2(1-b_nr_n)E_{n+1}\rho^{n+1})}{1-b_n^2}$$
$$= \frac{\rho_n^-(1-b_n^2)E_n(1+b_nr_n)}{1-b_n^2} = \rho_n^-.$$

Then by (2.2) we have

$$E_n^{\rho^-}(|\phi_n^{-1}|^2) = \frac{E_n(|\phi_n^{-1}|^2\rho^-)}{E_n(\rho^-)} = 1.$$

Applying Theorem A we can finish the proof of Theorem 1. ■

P r o o f  o f  T h e o r e m 2. Applying Theorem B we get

$$S_m^\rho f := \sum_{k=0}^{\infty} m_k E_k^\rho(f\psi_{m^{k+1}})\psi_{m^{k+1}}.$$

Set

$$g = f\psi_m\rho/\rho^-.$$

If $m_k = 1$ then for the $k$-th term of the last sum we get by (1.2), (1.6) and (1.7) that

$$E_k^\rho(f\psi_{m^{k+1}})\psi_{m^{k+1}} = \psi_m \prod_{j=0}^{k-1} \phi_j^{-2m_j} \frac{1}{\rho_k} E_k(f\psi_m\rho\phi_k^{-1})\phi_k^{-1}$$

$$= \psi_m \prod_{j=0}^{k-1} \phi_j^{-2m_j} \frac{\rho_k^-}{\rho_k} E_k^{\rho^-}(g\phi_k^{-1})\phi_k^{-1}.$$

Thus, in the case $m_k = 1$ we have

$$\frac{\rho_k^-}{\rho_k} \prod_{j=0}^{k-1} \phi_j^{-2m_j} = \prod_{j=0}^{k-1} \frac{(1-b_jr_j)^{1-m_j}}{(1+b_jr_j)^{1-m_j}} \prod_{j=0,m_j=1}^{k-1} \frac{(1-b_jr_j)}{(1+b_jr_j)} \frac{(1+b_jr_j)^2}{1-b_j^2}$$

$$= \prod_{j=0}^{k-1} \frac{(1-b_jr_j)^{1-m_j}}{(1+b_jr_j)^{1-m_j}} = a_m^k,$$

therefore

$$m_k E_k^{\rho}(f\psi_{m^{k+1}})\psi_{m^{k+1}} = m_k a_m^k E_k^{\rho^-}(g\phi_k^{-1})\phi_k^{-1}.$$

Applying Theorem B, we get Theorem 2. ■

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