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On the Location of the Zeros and Critical Points of a Polynomial

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1. Introduction and statement of results

If all the zeros of a polynomial $P(z)$ of degree n lie in the disk $D = \{z : |z - c| \leq R\}$, then according to the Gauss-Lucas theorem, every critical point of $P(z)$ also lies in D. This theorem has been rather thoroughly investigated [4] and sharpened in several ways. Recently, Brown [3] conjectured that if $P(z) = z\prod_{k=1}^{n-1}(z - z_k)$ is a polynomial of degree n with $|z_k| \geq 1$, $k = 1, 2, \ldots, n-1$, then $P'(z) \neq 0$ for $|z| < 1/n$. This conjectured result was verified by Aziz and Zargar [2] by proving the following more general result.

Theorem A. *If all the zeros of the polynomial* $R(z) = \prod_{j=1}^{n-k}(z - z_j)$ *lie in* $|z| \geq 1$ *and* $P(z) = z^k R(z)$, *then* $P'(z)$ *has* $(k-1)$ *fold zeros at the origin and the remaining* $(n-k)$ *zeros of* $P'(z)$ *lie in* $|z| \geq k/n$.

The aim of this paper is to add three more interesting results to the location of zeros and the critical points of polynomials. The first two results are extensions of Theorem A and include Brown's conjecture as a special case. The third theorem among other interesting things yields a generalization of the Gauss-Lucas theorem. We first prove the following theorem.

Theorem 1. *If all zeros of a polynomial* $P(z)$ *of degree* n *lie in* $|z| \geq 1$, *then for every real or complex number* λ *with* $\Re(\lambda) > 0$, *the polynomial* $\lambda P(z) + zP'(z)$ *does not vanish in the disk* $|z| < \dfrac{\Re(z)}{n + \Re(z)}$.

R e m a r k 1. If all the zeros of a polynomial $R(z) = \prod_{j=1}^{n-k}(z - z_j)$ lie in $|z| \geq 1$ and $P(z) = z^k R(z)$, then clearly

$$P'(z) = z^{k-1}(kR(z) + zR'(z)).$$

Applying Theorem 1 with $\lambda = k$ to the polynomial $R(z)$, which is of degree $n - k$, we immediately obtain Theorem A.

Next we prove

Theorem 2. *If all the zeros of a polynomial $P(z)$ of degree n lie in $|z| \geq 1$, then for every real or complex number λ with $\Re(z) \geq -n/2$, the polynomial $\lambda P(z) + zP'(z)$ does not vanish in the disk $|z| < \dfrac{|\lambda|}{|n+\lambda|}$. In case $\Re(\lambda) < -n/2$, then the polynomial $\lambda P(z) + zP'(z)$ does not vanish in $|z| \leq 1$.*

R e m a r k 2. Theorem 2 includes the validity of the Brown conjecture as a special case when $\lambda = 1$ and $P(z)$ is a polynomial of degree $n - 1$.

Finally, we present the following result, which among other things yields an interesting generalization of the Gauss-Lucas theorem as a special case.

Theorem 3. *If all the zeros of a polynomial $P(z)$ of degree n lie in the disk $|z| \leq R$, then for every real or complex number λ with $\Re(\lambda) > -m/2$, $m \leq n$, the polynomial $zP^{(n-m+1)} + \lambda P^{(n-m)}(z)$ has all its zeros in $|z| \leq R$.*

The following corollary which is an interesting generalization of the Gauss-Lucas theorem is obtained by taking $m = n$ in Theorem 3.

Corollary 1. *If all the zeros of a polynomial $P(z)$ of degree n lie in the disk $|z| \leq R$, then for every real or complex number λ with $\Re(\lambda) + n/2 > 0$, the polynomial $\lambda P(z) + zP'(z)$ has also all zeros in $|z| \leq R$.*

Applying Corollary 1 to the polynomial $P(z + c)$, the following result is immediate.

Corollary 2. *If all the zeros of a polynomial $P(z)$ lie in the disk $|z - c| \leq R$, then for every real or complex number λ with $\Re(\lambda) + n/2 > 0$, the polynomial $\lambda P(z) + (z - c)P'(z)$ has also all its zeros $|z - c| \leq R$.*

R e m a r k 3. For $\lambda = 0$, Corollary 2 reduces to the Gauss-Lucas theorem.

Taking in particular $\lambda = -n/2$ in Corollary 2, we get the following interesting result:

Corollary 3. *If all zeros of a polynomial $P(z)$ of degree n lie in the disk $|z-c| \leq R$, then the polynomial $nP(z) + 2(c-z)P(z)$ has its zeros in $|z-c| \leq R$.*

2. Lemmas

For the proof of these theorems, we need the following lemmas. The first lemma is the Coincidence theorem of Walsh [4, p.62], see also [1].

Lemma 1. *Let $G(z_1, z_2, \ldots, z_n)$ be a linear symmetric form of the total degree n in $z_1, z_2, \ldots, z_n$ and let C be a circular region containing the n points $w_1, w_2, \ldots, w_n$, then there exists at least one point α belonging to C such that*

$$G(\alpha, \alpha, \ldots, \alpha) = G(w_1, w_2, \ldots, w_n).$$

The next lemma which we need is due to Aziz [1].

Lemma 2. *If all the zeros of $P(z) = \sum_{j=0}^{n} C(n,j) a_j z^j$ of degree n lie in $|z| \leq r$ and all the zeros of $Q(z) = \sum_{j=0}^{m} C(m,j) b_j z^j$ of degree m lie in $|z| < s$, $m \leq n$, then all the zeros of the polynomial $R(z) = \sum_{j=0}^{m} C(m,j) a_{n-m+j} b_j z^j$ of degree m lie in $|z| < rs$.*

We also need

Lemma 3. *If $P(z) = \sum_{k=0}^{n} C(n,k) a_k z^k$, then $\lambda P^{(n-m)}(z) + zP^{(n-m+1)}(z)$*

$$= \sum_{k=n-m}^{n} C(n,k) \frac{k!}{(k-(n-m))!} (k-(n-m)+\lambda) a_k z^{k-(n-m)}.$$

The proof of this lemma is simple and we leave it to the reader.

3. Proof of the theorems

Proof øf Theorem 1. Let $z_1, z_2, \ldots, z_n$ be the zeros of $P(z)$, so that by hypothesis $|z_j| \geq 1$ for $j = 1, 2, \ldots, n$. If w is any zero of $\lambda P(z) + zP'(z)$, then

$$\lambda P(w) + wP'(w) = 0. \tag{1}$$

If $w = z_j$, for some $j = 1, 2, \ldots, n$, then clearly

$$|w| = |z_j| \geq 1 \geq \frac{\Re(\lambda)}{n + \Re(\lambda)}.$$

Henceforth, we suppose that $w \neq z_j$ for any $j = 1, 2, \ldots, n$, so that $P(w) \neq 0$ and from (1), we have

$$\frac{wP'(w)}{P(w)} = -\lambda.$$

This implies

$$\sum_{j=1}^{n} \frac{w}{w - z_j} = -\lambda,$$

which gives

$$n + \sum_{j=1}^{n} \left(\frac{2w}{w - z_j} - 1 \right) = -2\lambda.$$

Or equivalently

$$\sum_{j=1}^{n} \left(\frac{z_j + w}{z_j - w} \right) = n + 2\lambda. \tag{2}$$

Taking real part on both sides of (2), we get

$$\sum_{j=1}^{n} \Re \left(\frac{z_j + w}{z_j - w} \right) = n + 2\Re(\lambda),$$

which implies

$$\sum_{j=1}^{n} \frac{|z_j|^2 - |w|^2}{|z_j - w|^2} = n + 2\Re(\lambda). \tag{3}$$

If $|w| \geq 1$, then the result follows immediately, so we assume that $|w| < 1 \leq |z_j|$ for all $j = 1, 2, \ldots, n$ and from (3), we have

$$n + 2\Re(\lambda) \leq \sum_{j=1}^{n} \frac{|z_j|^2 - |w|^2}{(|z_j| - |w|)^2} = \sum_{j=1}^{n} \left(\frac{|z_j| + |w|}{|z_j| - |w|} \right)$$

$$\leq \sum \left(\frac{1 + |w|}{1 - |w|} \right) = n \left(\frac{1 + |w|}{1 - |w|} \right),$$

which after a short simplification leads to

$$|w| \geq \frac{\Re(\lambda)}{n+\Re(\lambda)}.$$

This shows that all the zeros of $P(z)$ lie in

$$|z| \geq \frac{\Re(\lambda)}{n+\Re(\lambda)},$$

and this proves Theorem 1 completely. ■

Proof of Theorem 2. If $z_1, z_2, \ldots, z_n$ are the zeros of $P(z)$, then $|z_j| \geq 1$ for all $j = 1, 2, \ldots, n$ and we have

$$P(z) = c\prod_{j=1}^{n}(z - z_j).$$

We first assume that $\Re(\lambda) \geq -n/2$, then it can be easily seen $|n+\lambda| \geq |\lambda|$. Let w be any zero of $F(z) = \lambda P(z) + wP'(z)$, then

$$F(w) = \lambda P(w) + wP'(w) = 0. \tag{4}$$

This is an equation which is linear and symmetric in the zeros $z_1, z_2, \ldots, z_n$ of $P(z)$. Hence an application of Lemma 1 with circular region $C = \{z : |z| \geq 1\}$ shows that (4) is also satisfied when we subtitute $P(z) = (z-\alpha)^n$, where α is a suitably chosen point in the given region $|z| \geq 1$. That is, w also satisfies the equation

$$\lambda(w-\alpha)^n + w_n(w-\alpha)^{n-1} = 0.$$

Equivalently,

$$\lambda(w-\alpha)^{n-1}\{w(\lambda+n) - \lambda\alpha\} = 0.$$

Thus w has the values $w = \alpha$ and $w = \dfrac{\lambda\alpha}{n+\alpha}$. If $w = \alpha$, then clearly $|w| = |\alpha| \geq 1 \geq \dfrac{|\lambda|}{|n+\alpha|}$ and if $w = \dfrac{\lambda\alpha}{n+\alpha}$, then $|w| = \dfrac{|\lambda||\alpha|}{|n+\lambda|} \geq \dfrac{|\lambda|}{|n+\lambda|}$. Hence in any case $|w| \geq \dfrac{|\lambda|}{|n+\lambda|}$. Since w is any zero of the polynomial $F(z) = \lambda P(z) + zP'(z)$, it follows that $F(z)$ does not vanish in the disk $|z| < \dfrac{|\lambda|}{|n+\lambda|}$, if $\Re(\lambda) \geq -n/2$.

In case $\Re(\lambda) < -n/2$, then it can easily verified that $\dfrac{|\lambda|}{|n+\lambda|} > 1$, so that if $w = \dfrac{\lambda\alpha}{n+\alpha}$, then $|w| = \dfrac{|\lambda||\alpha|}{|n+\lambda|} > 1$ and if $w = \alpha$, then $|w| = |\alpha| \geq 1$. Thus in this case the polynomial $F(z) = \lambda P(z) + zP'(z)$ does not vanish in $|z| < 1$. This completes the proof of Theorem 2. ■

Proof of Theorem 3. We have $P(z) = \sum_{k=0}^{n} C(n,k) a_k z_k$.
If $R(z) = \lambda P^{(n-m)}(z) + zP^{(n-m+1)}(z)$, then by Lemma 3, we can write

$$R(z) = \sum_{k=n-m}^{n} C(n,k) \frac{k!}{(k+m-n)!}(k+m-n+\lambda) a_k z^{k+m-n}.$$

Now, if the polynomial

$$Q(z) = \sum_{k=n-m}^{n} C(n,k) \frac{k!}{(k+m-n)!}(k+m-n+\lambda) z^{k+m-n}$$

$$= \lambda \sum_{k=n-m}^{n} C(n,k) \frac{k!}{(k+m-n)!} z^{k+m-n}$$

$$+ z \sum_{k=n-m}^{n} C(n,k) \frac{k!}{(k+m-n-1)} z^{k+m-n-1}$$

$$= n(n-1)\dots(m+1)(1+z)^{m-1}(\lambda + \lambda z + mz),$$

then, the zeros of $Q(z)$ are -1 and $-\dfrac{\lambda}{\lambda+m}$. It can be easily verified that $\left| -\dfrac{\lambda}{\lambda+m} \right| \leq 1$ if and only if $\Re(\lambda) \geq -m/2$. Using Lemma 2 with $r = R$ and $s = 1$, it follows that all the zeros of the polynomial $\lambda P^{(n-m)}(z) + zP^{(n-m+1)}(z)$ lie in $|z| \leq R$, if $\Re(\lambda) \geq -m/2$. This completes the proof of Theorem 3. ■

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