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On H-Fuzzy Differentiation

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Dedicated to the 70th anniversary of the famous Bulgarian mathematician, Bl. Sendov

The concept of H-fuzzy differentiation is discussed thoroughly in the univariate and multivariate cases. Basic H-derivatives are calculated and then important theorems are established on the topic, such as, the H-mean value theorem, the univariate and multivariate H-chain rules, and the interchange of the order of H-fuzzy differentiation. And finally is given the multivariate H-fuzzy Taylor formula.

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0. Introduction

Fuzzyness was first introduced in the celebrated paper [12]. For the notion of H-fuzzy derivative, see [7] and [9]. First we give some background from Fuzzyness, motivation and justification, necessary for the results to follow. In Propositions 1–4 we calculate basic H-fuzzy derivatives. In Lemmas 1 and 2 we give results on fuzzy continuity, and in Propositions 5 and 6 we give basic properties of H-fuzzy differentiation. Then come the main results.

Theorem 1 is on H-Fuzzy Mean Value Theorem, Lemmas 3, 4 and 5 are auxiliary on fuzzy convergence and fuzzy continuity, Theorem 2 is on univariate H-fuzzy chain rule, and Theorem 3 is on multivariate H-fuzzy chain rule.

We conclude with Theorem 4 on the interchange of the order of H-fuzzy differentiation, and the development of the multivariate H-fuzzy Taylor formula with integral remainder, see Theorem 5 and Corollary 1.

1. Background

We start with the following

Definition A (see [9]). Let $\mu: \mathbb{R} \to [0,1]$ with the following properties:

(i) is *normal*, i.e., $\exists x_0 \in \mathbb{R}$: $\mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$, $\forall x, y \in \mathbb{R}$, $\forall \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is *upper semicontinuous* on $\mathbb{R}$, i.e., $\forall x_0 \in \mathbb{R}$ and $\forall \varepsilon > 0$, $\exists$ neighborhood $V(x_0)$: $\mu(x) \leq \mu(x_0) + \varepsilon$, $\forall x \in V(x_0)$.

(iv) The set $\overline{\operatorname{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\operatorname{supp}(\mu) := \{x \in \mathbb{R};\ \mu(x) > 0\}$).

We call μ a *fuzzy real number*. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\mathcal{X}_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\mathcal{X}_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R}\colon\ \mu(x) \geq r\}$ and

$$[\mu]^0 := \overline{\{x \in \mathbb{R}\colon \mu(x) > 0\}}\,.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [9]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = [u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} < u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall r \in [0,1]$.

Define

$$D\colon \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+ \cup \{0\}$$

by

$$D(u,v) := \sup_{r \in [0,1]} \max\{|u_-^{(r)} - v_-^{(r)}|, |u_+^{(r)} - v_+^{(r)}|\},$$

where $[v]^r = [v_-^{(r)}, v_+^{(r)}]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [9], with the properties

$$D(u \oplus w, v \oplus w) = D(u,v), \quad \forall u, v, w \in \mathbb{R}_{\mathcal{F}},$$
$$D(k \odot u, k \odot v) = |k| D(u,v), \quad \forall u, v \in \mathbb{R}_{\mathcal{F}},\ \forall k \in \mathbb{R},$$
$$D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e), \quad \forall u, v, w, e \in \mathbb{R}_{\mathcal{F}}.$$

Let $f, g\colon \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be *fuzzy real number valued functions.* The distance between f, g is defined by

$$D^*(f,g) := \sup_{x\in\mathbb{R}} D(f(x), g(x)).$$

On $\mathbb{R}_{\mathcal{F}}$ we define a *partial order by* "$\leq$": $u, v \in \mathbb{R}_{\mathcal{F}}$, $u \leq v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, $\forall r \in [0,1]$.

We need

Lemma 2.2 ([3]). *For any $a, b \in \mathbb{R}$: $a, b \geq 0$ and any $u \in \mathbb{R}_{\mathcal{F}}$ we have*

$$D(a \odot u, b \odot u) \leq |a - b| \cdot D(u, \tilde{o}),$$

where $\tilde{o} \in \mathbb{R}_{\mathcal{F}}$ is defined by $\tilde{o} := \mathcal{X}_{\{0\}}$.

Lemma 4.1 ([3]).

(i) *If we denote $\tilde{o} := \mathcal{X}_{\{0\}}$, then $\tilde{o} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e., $u \oplus \tilde{o} = \tilde{o} \oplus u = u$, $\forall u \in \mathbb{R}_{\mathcal{F}}$.*

(ii) *With respect to $\tilde{o}$, none of $u \in \mathbb{R}_{\mathcal{F}}$, $u \neq \tilde{o}$ has opposite in $\mathbb{R}_{\mathcal{F}}$.*

(iii) *Let $a, b \in \mathbb{R}$: $a \cdot b \geq 0$, and any $u \in \mathbb{R}_{\mathcal{F}}$, we have $(a+b) \odot u = a \odot u \oplus b \odot u$. For general $a, b \in \mathbb{R}$, the above property is false.*

(iv) *For any $\lambda \in \mathbb{R}$ and any $u, v \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.*

(v) *For any $\lambda, \mu \in \mathbb{R}$ and $u \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda \odot (\mu \odot u) = (\lambda \cdot \mu) \odot u$.*

(vi) *If we denote $\|u\|_{\mathcal{F}} := D(u, \tilde{o})$, $\forall u \in \mathbb{R}_{\mathcal{F}}$, then $\|\cdot\|_{\mathcal{F}}$ has the properties of a usual norm on $\mathbb{R}_{\mathcal{F}}$, i.e.,*

$$\|u\|_{\mathcal{F}} = 0 \;\; \textit{iff}\; u = \tilde{o}, \;\; \|\lambda \odot u\|_{\mathcal{F}} = |\lambda| \cdot \|u\|_{\mathcal{F}},$$

$$\|u \oplus v\|_{\mathcal{F}} \leq \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}, \;\; \|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}} \leq D(u, v).$$

Notice that $(\mathbb{R}_{\mathcal{F}}, \oplus, \odot)$ is *not* a linear space over $\mathbb{R}$, and consequently $(\mathbb{R}_{\mathcal{F}}, \|\cdot\|_{\mathcal{F}})$ is *not* a normed space.

We need

Definition B (see [9]). Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists a $z \in \mathbb{R}_{\mathcal{F}}$ such that $x = y + z$, then we call z the *H-difference* of x and y, denoted by $z := x - y$.

Definition 3.3 (see [9]). Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f\colon T \to \mathbb{R}_{\mathcal{F}}$ is *H-differentiable* at $x \in T$ if there exists a $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to metric D)

$$\lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h\to 0^+} \frac{f(x) - f(x-h)}{h}$$

exist and are equal to $f'(x)$. We call f' the *derivative* or *H-derivative* of f at x. If f is H-differentiable at any $x \in T$, we call f *differentiable* or *H-differentiable* and it has *H-derivative over* T the function f'.

The last definition was given first by M. Puri and D. Ralescu [7].

Example. Let $f: \mathbb{R}_+ \to \mathbb{R}_{\mathcal{F}}$ be such that for any $\lambda, \mu \geq 0$ it holds

$$f(\lambda x + \mu y) = \lambda \odot f(x) \oplus \mu \odot f(y), \quad \forall x, y \in \mathbb{R}_+.$$

Then the H-derivative $f'(x) = f(1)$, $\forall x \in \mathbb{R}_+$.

Proof. By $f(x+h) = f(x) \oplus f(h)$, that is the H-difference

$$f(x+h) - f(x) = f(h) \in \mathbb{R}_{\mathcal{F}}.$$

Thus

$$\frac{f(x+h) - f(x)}{h} = f(1), \quad h > 0.$$

Similarly, $f(x) = f(x-h) \oplus f(h)$, for $h > 0$ small, that is the H-difference $f(x) - f(x-h) = f(h) \in \mathbb{R}_{\mathcal{F}}$. Hence

$$\frac{f(x) - f(x-h)}{h} = f(1).$$

But

$$\lim_{h \to 0^+} D(f(1), f(1)) = 0.$$

Clearly for $f'(0)$ we take the right-hand side H-derivative. ■

We need also a particular case of the *Fuzzy Henstock integral* ($\delta(x) = \frac{\delta}{2}$) introduced in [9], Definition 2.1.

That is,

Definition 13.14 (see [5], p. 644). Let $f: [a,b] \to \mathbb{R}_{\mathcal{F}}$. We say that f is *Fuzzy-Riemann integrable* to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any division $P = \{[u,v]; \xi\}$ of $[a,b]$ with the norms $\Delta(P) < \delta$, we have

$$D\left(\sum_P{}^{*} (v-u) \odot f(\xi), I\right) < \varepsilon,$$

where $\sum^*$ denotes the fuzzy summation. We choose to write

$$I := (FR) \int_a^b f(x) dx.$$

We also call an f as above (FR)-*integrable.*

We are based on the following fundamental theorem of Fuzzy Calculus:

Corollary A ([1]). *If $f\colon [a,b] \to \mathbb{R}_{\mathcal{F}}$ has a fuzzy continuous H-derivative f' on $[a,b]$, then $f'(x)$ is (FR)-integrable over $[a,b]$ and*

$$f(s) = f(t) \oplus (FR)\int_t^s f'(x)dx, \quad \text{for any } s \ge t,\ s,t \in [a,b].$$

Note. In Corollary A when $s < t$ the formula is invalid! since fuzzy real numbers correspond to closed intervals etc.

We need also

Lemma 1 ([1]). *If $f,g\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ are fuzzy continuous (with respect to metric D), then the function $F\colon [a,b] \to \mathbb{R}_+ \cup \{0\}$ defined by $F(x) := D(f(x),g(x))$ is continuous on $[a,b]$, and*

$$D\left((FR)\int_a^b f(u)du, (FR)\int_a^b g(u)du\right) \le \int_a^b D(f(x),g(x))dx.$$

Lemma 2 ([1]). *Let $f\colon [a,b] \to \mathbb{R}_{\mathcal{F}}$ fuzzy continuous (with respect to metric D), then $D(f(x),\tilde{o}) \le M$, $\forall x \in [a,b]$, $M > 0$, that is f is fuzzy bounded. Equivalently we get $\chi_{-M} \le f(x) \le \chi_M$, $\forall x \in [a,b]$.*

Lemma 3 ([1]). *Let $f\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then*

$$(FR)\int_a^x f(t)dt \quad \textit{is a fuzzy continuous function in } x \in [a,b].$$

Lemma 4 ([1]). *Let $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ fuzzy continuous, $r \in \mathbb{N}$. Then the following integrals*

$$(FR)\int_a^{s_{r-1}} f(s_r)ds_r, (FR)\int_a^{s_{r-2}}\left(\int_a^{s_{r-1}} f(s_r)ds_r\right)ds_{r-1},$$
$$\dots, (FR)\int_a^s\left(\int_a^{s_1}\cdots\left(\int_a^{s_{r-2}}\left(\int_a^{s_{r-1}} f(s_r)ds_r\right)ds_{r-1}\right)\cdots\right)ds_1,$$

are fuzzy continuous functions in $s_{r-1}, s_{r-2}, \dots, s$, respectively. Here $a \le s_{r-1} \le s_{r-2} \le \cdots \le s \le b$.

Additionally we mention

Lemma 5 ([2]). *Let $f\colon [a,b] \to \mathbb{R}_{\mathcal{F}}$ have an existing H-fuzzy derivative f' at $c \in [a,b]$. Then f is fuzzy continuous at c.*

We need the Fuzzy Taylor formula

Theorem 1 ([1]). *Let* $T := [x_0, x_0+\beta] \subset \mathbb{R}$, *with* $\beta > 0$. *We assume that* $f^{(i)}: T \to \mathbb{R}_{\mathcal{F}}$ *are* H*-differentiable for all* $i = 0, 1, \dots, n-1$, *for any* $x \in T$. *(I.e., there exist in* $\mathbb{R}_{\mathcal{F}}$ *the* H*-differences* $f^{(i)}(x+h) - f^{(i)}(x)$, $f^{(i)}(x) - f^{(i)}(x-h)$, $i = 0, 1, \dots, n-1$ *for all small* $h: 0 < h < \beta$. *Furthermore there exist* $f^{(i+1)}(x) \in \mathbb{R}_{\mathcal{F}}$ *such that the limits in* D*-distance exist and*

$$f^{(i+1)}(x) = \lim_{h \to 0^+} \frac{f^{(i)}(x+h) - f^{(i)}(x)}{h} = \lim_{h \to 0^+} \frac{f^{(i)}(x) - f^{(i)}(x-h)}{h},$$

for all $i = 0, 1, \dots, n-1$.*) Also we assume that* $f^{(n)}$, *is fuzzy continuous on* T. *Then for* $s \geq a$; $s, a \in T$ *we obtain*

$$f(s) = f(a) \oplus f'(a) \odot (s-a) \oplus f''(a) \odot \frac{(s-a)^2}{2!} \oplus \cdots \oplus f^{(n-1)}(a) \odot \frac{(s-a)^{n-1}}{(n-1)!} \oplus R_n(a,s),$$

where

$$R_n(a,s) := (FR) \int_a^s \left(\int_a^{s_1} \cdots \left(\int_a^{s_{n-1}} f^{(n)}(s_n) ds_n \right) ds_{n-1} \right) \dots \Big) ds_1.$$

Here $R_n(a,s)$ *is fuzzy continuous on* T *as a function of* s.

Note. This formula is invalid when $s < a$, as it is totally based on Corollary A.

For the interest of the reader we given the following

Theorem 5.2 ([6]). *Let* $f: [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ *be* H*-fuzzy differentiable. Let* $t \in [a,b]$, $0 \leq r \leq 1$. *(Clearly*

$$[f(t)]^r = [(f(t))_-^{(r)}, (f(t))_+^{(r)}] \subseteq \mathbb{R}.)$$

Then $(f(t))_{\pm}^{(r)}$ *are differentiable and*

$$[f'(t)]^r = [((f(t))_-^{(r)})', ((f(t))_+^{(r)})'].$$

The last can be used to find f'.

Next $\overline{C}[0,1]$ stands for the class of all real-valued bounded functions f on $[0,1]$ such that f is left continuous for any $x \in (0,1]$ and f has a right limit for any $x \in [0,1)$, especially f is right continuous at 0. With the norm $\|f\| = \sup_{x \in [0,1]} |f(x)|$, $\overline{C}[0,1]$ is a Banach space [10].

We mention

Theorem (*) (Wu and Ma [10]). *For $u \in \mathbb{R}_{\mathcal{F}}$, denote $j\colon j(u) := (u_-, u_+)$, where $u_\pm = u_\pm(r) := u_\pm^{(r)}$, $\quad 0 \le r \le 1$. Then $j(\mathbb{R}_{\mathcal{F}})$ is a closed convex cone with vertex 0 in $\overline{C}[0,1] \times \overline{C}[0,1]$ (here $\overline{C}[0,1] \times \overline{C}[0,1]$ is a Banach space with the norm defined by $\|(f,g)\| := \max(\|f\|, \|g\|)$), and $j\colon \mathbb{R}_{\mathcal{F}} \to \overline{C}[0,1] \times \overline{C}[0,1]$ satisfies*

(1) *for all $u, v \in \mathbb{R}_{\mathcal{F}}$, $s \ge 0$, $t \ge 0$, $j(su + tv) = sj(u) + tj(v)$,*

(2) *$D(u,v) = \|j(u) - j(v)\|$, i.e., j embeds $\mathbb{R}_{\mathcal{F}}$ into $\overline{C}[0,1] \times \overline{C}[0,1]$ isometrically and isomorphically.*

We finally mention the important connections of the H-fuzzy derivative to the Fréchet derivative.

Lemma (*) (Wu and Ma [11]). *If $f\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ satisfies the condition H: for any $x \in [a,b]$, there exists $\beta > 0$ such that the H-differences of $f(x+h) - f(x)$, $f(x) - f(x-h)$ exist for all $0 < h < \beta$, then the H-differentiability of $f(x)$ implies the differentiability of $(j \circ f)(x)$ and $(j \circ f)'(x) \in j(\mathbb{R}_{\mathcal{F}})$, where the differentiability of $(j \circ f)(x)$ on $\overline{C}[0,1] \times \overline{C}[0,1]$ is in the Fréchet's sense.*

Lemma (**) (Wu and Ma [11]). *If $(j \circ f)(x)$ is Fréchet differentiable and $(j \circ f)'(x) \in j(\mathbb{R}_{\mathcal{F}})$, then $f(x)$ is H-differentiable, and $f'(x) = j^{-1}((j \circ f)'(x))$. Here $f\colon [a,b] \to \mathbb{R}_{\mathcal{F}}$, $j\colon \mathbb{R}_{\mathcal{F}} \to (\overline{C}[0,1])^2$, and $(j \circ f)\colon [a,b] \to (\overline{C}[0,1])^2$.*

2. Results

We present

Proposition 1. *Let $F(t) := t^n \odot u$, $t \ge 0$, $n \in \mathbb{N}$, and $u \in \mathbb{R}_{\mathcal{F}}$ be fixed. Then (the H-derivative)*

$$F'(t) = nt^{n-1} \odot u. \tag{1}$$

In particular when $n = 1$ then $F'(t) = u$.

Proof. We need to establish that

$$F'(t) = F'_+(t) = F'_-(t),$$

where

$$F'_+(t) := \lim_{h \to 0^+} \frac{(t+h)^n \odot u - t^n \odot u}{h},$$

and

$$F'_-(t) := \lim_{h \to 0^+} \frac{t^n \odot u - (t-h)^n \odot u}{h},$$

the limits are taken with respect to the D-metric.

First we take care of the case $t > 0$, $n \geq 2$. Here h is a small positive quantity approaching zero. By Lemma 4.1 (iii) of [3] we notice that

$$(t+h)^n \odot u = t^n \odot u \oplus \left(\sum_{k=1}^{n} \binom{n}{k} t^{n-k} h^k \right) \odot u,$$

where

$$t^n, \sum_{k=1}^{n} \binom{n}{k} t^{n-k} h^k > 0.$$

That is the H-difference

$$(t+h)^n \odot u - t^n \odot u = \left(\sum_{k=1}^{n} \binom{n}{k} t^{n-k} h^k \right) \odot u$$

exists, and

$$\frac{(t+h)^n \odot u - t^n \odot u}{h} = \left(\sum_{k=1}^{n} \binom{n}{k} t^{n-k} h^{k-1} \right) \odot u.$$

Then we observe that

$$\begin{aligned}
&\lim_{h \to 0^+} D\left(\frac{(t+h)^n \odot u - t^n \odot u}{h},\ n t^{n-1} \odot u \right) \\
&= \lim_{h \to 0^+} D\left(\left(\sum_{k=1}^{n} \binom{n}{k} t^{n-k} h^{k-1} \right) \odot u, n t^{n-1} \odot u \right) \\
&\leq \text{ (by Lemma 2.2 of [3])} \\
&\qquad \lim_{h \to 0^+} \left| \left(\sum_{k=1}^{n} \binom{n}{k} t^{n-k} h^{k-1} \right) - n t^{n-1} \right| D(u, \tilde{o}) \\
&= \lim_{h \to 0^+} \left(\sum_{k=2}^{n} \binom{n}{k} t^{n-k} h^{k-1} \right) D(u, \tilde{o}) = 0 D(u, \tilde{o}) = 0.
\end{aligned}$$

That is

$$F'_+(t) = n t^{n-1} \odot u, \quad t > 0, \quad n \geq 2.$$

Furthermore we notice that

$$F'_-(t) = \lim_{h \to 0^+} \frac{((t-h)+h)^n \odot u - (t-h)^n \odot u}{h}.$$

We set $\beta := t-h$, which for sufficiently small $h>0$ is positive, i.e., $\beta>0$. Thus

$$F'_{-}(t)=\lim_{h\to 0^+}\frac{(\beta+h)^n\odot u-\beta^n\odot u}{h}.$$

Again we have

$$(\beta+h)^n\odot u=\beta^n\odot u\oplus\left(\sum_{k=1}^{n}\binom{n}{k}\beta^{n-k}h^k\right)\odot u,$$

where

$$\beta^n,\ \sum_{k=1}^{n}\binom{n}{k}\beta^{n-k}h^k>0.$$

That is the H-difference

$$(\beta+h)^n\odot u-\beta^n\odot u=\left(\sum_{k=1}^{n}\binom{n}{k}\beta^{n-k}h^k\right)\odot u$$

exists, and

$$\frac{(\beta+h)^n\odot u-\beta^n\odot u}{h}=\left(\sum_{k=1}^{n}\binom{n}{k}\beta^{n-k}h^{k-1}\right)\odot u.$$

Then we observe that

$$\begin{aligned}
&\lim_{h\to 0^+}D\left(\frac{t^n\odot u-(t-h)^n\odot u}{h},nt^{n-1}\odot u\right)\\
&= \lim_{h\to 0^+}D\left(\left(\sum_{k=1}^{n}\binom{n}{k}\beta^{n-k}h^{k-1}\right)\odot u,nt^{n-1}\odot u\right)\\
&\le \lim_{h\to 0^+}\left|\sum_{k=1}^{n}\binom{n}{k}\beta^{n-k}h^{k-1}-nt^{n-1}\right|D(u,\tilde{o})\\
&= \lim_{h\to 0^+}\left|n(t-h)^{n-1}+\sum_{k=2}^{n}\binom{n}{k}(t-h)^{n-k}h^{k-1}-nt^{n-1}\right|D(u,\tilde{o})\\
&= 0D(u,\tilde{o})=0.
\end{aligned}$$

Hence $F'_{-}(t)=nt^{n-1}\odot u$, $t>0$, $n\ge 2$. That is,

$$F'(t)=nt^{n-1}\odot u,\quad t>0,\quad n\ge 2.$$

Next we treat separately the case of $n = 1$, $t > 0$ for the sake of clarity. Here

$$\lim_{h\to 0^+} D\left(\frac{(t+h)\odot u - t\odot u}{h}, u\right) = \lim_{h\to 0^+} D\left(\frac{h\odot u}{h}, u\right) = \lim_{h\to 0^+} D(u,u) = 0.$$

I.e., $F'_+(t) = u$, $t > 0$, $n = 1$. And we see that

$$\begin{aligned}
&\lim_{h\to 0^+} D\left(\frac{t\odot u - (t-h)\odot u}{h}, u\right)\\
&= \lim_{h\to 0^+} D\left(\frac{((t-h)+h)\odot u - (t-h)\odot u}{h}, u\right)\\
&= \lim_{h\to 0^+} D\left(\frac{(\beta+h)\odot u - \beta\odot u}{h}, u\right)\\
&= \lim_{h\to 0^+} D\left(\frac{h\odot u}{h}, u\right) = \lim_{h\to 0^+} D(u,u) = 0,
\end{aligned}$$

where $\beta := t - h > 0$, for sufficiently small $h > 0$. I.e., $F'_-(t) = u$, $t > 0$, $n = 1$. That is

$$F'(t) = u, \quad t > 0, \quad n = 1.$$

At last we do the case of $t = 0$. Here we need to find

$$F'_+(0) = \lim_{h\to 0^+} \frac{h^n \odot u}{h} = \lim_{h\to 0^+} h^{n-1}\odot u.$$

For $n = 1$, we see that

$$\lim_{h\to 0^+} D(h^{n-1}\odot u, u) = \lim_{h\to 0^+} D(u,u) = 0.$$

Thus

$$F'(0) = F'_+(0) = u, \quad \text{for } n = 1.$$

For $n \ge 2$ we see that

$$\lim_{h\to 0^+} D(h^{n-1}\odot u, \tilde{o}) = D(\tilde{o}, \tilde{o}) = 0.$$

Therefore

$$F'(0) = F'_+(0) = \tilde{o}, \quad \text{for } n \ge 2.$$

That is

$$F'(t) = nt^{n-1}\odot u \quad \text{is true for } t = 0.$$

■

Remark 1. Let a_i, $i = 1, \ldots,$ be a sequence of real numbers all of the same sign such that $|\sum_{i=1}^{\infty} a_i| < +\infty$. Then

$$\left(\sum_{i=1}^{n} \alpha_i\right) \odot u = \sum_{i=1}^{n}{}^{*} (a_i \odot u), \quad u \in \mathbb{R}_{\mathcal{F}}, \quad \forall n \in \mathbb{N},$$

by Lemma 4.1 (iii) of [3]. Since

$$D\left(\left(\sum_{i=1}^{n} a_i\right) \odot u, \sum_{i=1}^{n}{}^{*} (\alpha_i \odot u)\right) = 0,$$

one obtains

$$\lim_{n\to+\infty} D\left(\left(\sum_{i=1}^{n} a_i\right) \odot u,\ \sum_{i=1}^{n}{}^{*} (a_i \odot u)\right) = 0.$$

That is

$$\left(\sum_{i=1}^{\infty} a_i\right) \odot u = \sum_{i=1}^{\infty}{}^{*} (a_i \odot u) \in \mathbb{R}_{\mathcal{F}}.$$

Next we give

Proposition 2. *Let $F(x) = x^p \odot u$, $x \geq 0$, $u \in \mathbb{R}_{\mathcal{F}}$, and $p > 0$ not an integer. Then*

(2) $$F'(x) = px^{p-1} \odot u, \quad p > 0, \quad x > 0,$$

and

(3) $$F'(o) = \tilde{o}, \quad \text{for } p > 1.$$

Proof. When $p > 0$ and $-1 \leq x \leq 1$ from [8], p. 232 we obtain the Binomial series, which converges absolutely

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}x^n + \cdots.$$

In the last we plug in instead of x, $\frac{h}{x}$ for $h, x > 0$ and $h \leq x$. Clearly $-1 \leq \frac{h}{x} \leq 1$ is automatically fulfilled and $x + h > 0$. That is

$$\left(1 + \frac{h}{x}\right)^p = 1 + p\frac{h}{x} + \frac{p(p-1)}{2!}\,\frac{h^2}{x^2} + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}\,\frac{h^n}{x^n} + \cdots.$$

And

$$(x+h)^p = x^p + phx^{p-1} + \frac{p(p-1)}{2!}h^2x^{p-2} + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}h^n x^{p-n} + \cdots.$$

By $x+h > x$ we have $(x+h)^p > x^p > 0$ and $(x+h)^p - x^p > 0$. Consequently it holds

$$\Delta := phx^{p-1} + \frac{p(p-1)}{2!}h^2x^{p-2} + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}h^n x^{p-n} + \cdots > 0.$$

Therefore

$$(x+h)^p \odot u - x^p \odot u = \Delta \odot u \ \text{ exists in } \mathbb{R}_{\mathcal{F}}.$$

Hence

$$\begin{aligned}
&\lim_{h\to 0^+} D\left(\frac{(x+h)^p \odot u - x^p \odot u}{h}, px^{p-1} \odot u\right) \\
&= \lim_{h\to 0^+} D\left(\frac{\Delta}{h} \odot u, px^{p-1} \odot u\right) \le \lim_{h\to 0^+} \left|\frac{\Delta}{h} - px^{p-1}\right| D(u,\tilde{o}) \\
&= \lim_{h\to 0^+} \left| px^{p-1} + \frac{p(p-1)}{2!}hx^{p-2} + \cdots \right. \\
&\quad \left. + \frac{p(p-1)\cdots(p-n+1)}{n!}h^{n-1}x^{p-n} + \cdots - px^{p-1}\right| D(u,\tilde{o}) \\
&= 0D(u,\tilde{o}) = 0.
\end{aligned}$$

I.e.,

$$F'_+(x) = (x^p \odot u)'_+ = px^{p-1} \odot u, \quad p > 0, \quad x > 0.$$

Next we evaluate in D-metric

$$\begin{aligned}
F'_-(t) &= \lim_{h\to 0^+} \frac{x^p \odot u - (x-h)^p \odot u}{h} \\
&= \lim_{h\to 0^+} \frac{((x-h)+h)^p \odot u - (x-h)^p \odot u}{h} \\
&= \lim_{h\to 0^+} \frac{(\beta+h)^p \odot u - \beta^p \odot u}{h},
\end{aligned}$$

where $\beta := x - h > 0$, for $h > 0$ small enough. In fact we choose h such that $2h < x$, that is, $h < x - h = \beta$. I.e., $0 < h < \beta$. Next we apply the Binomial series for $\frac{h}{\beta}$. Thus

$$(\beta+h)^p = \beta^p + ph\beta^{p-1} + \frac{p(p-1)}{2!}h^2\beta^{p-2} + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}h^n\beta^{p-n} + \cdots.$$

Clearly $\beta + h > \beta$ and $(\beta+h)^p > \beta^p > 0$, by $p > 0$. And $(\beta+h)^p - \beta^p > 0$. Hence

$$\Delta^* := ph\beta^{p-1} + \frac{p(p-1)}{2!}h^2\beta^{p-2} + \cdots + \frac{p(p-1)\cdots(p-n+1)}{n!}h^n\beta^{p-n} + \cdots > 0.$$

Therefore

$$(\beta+h)^p \odot u - \beta^p \odot u = \Delta^* \odot u \quad \text{exists in } \mathbb{R}_{\mathcal{F}}.$$

Furthermore we have

$$\begin{aligned}
&\lim_{h\to 0^+} D\left(\frac{(\beta+h)^p \odot u - \beta^p \odot u}{h}, px^{p-1} \odot u\right) \\
&= \lim_{h\to 0^+} D\left(\frac{\Delta^*}{h} \odot u, px^{p-1} \odot u\right) \\
&= \lim_{h\to 0^+} D\Bigg(\bigg(p\beta^{p-1} + \frac{p(p-1)}{2!}h\beta^{p-2} + \cdots \\
&\quad + \frac{p(p-1)\cdots(p-n+1)}{n!}h^{n-1}\beta^{p-n}\bigg) \odot u, px^{p-1} \odot u\Bigg) \\
&= D(px^{p-1} \odot u, px^{p-1} \odot u) = 0.
\end{aligned}$$

I.e., $F'_-(x) = (x^p \odot u)'_- = px^{p-1} \odot u$, $p > 0$, $x > 0$. That is

$$F'(x) = (x^p \odot u)' = px^{p-1} \odot u, \quad p > 0, \quad x > 0.$$

Finally at $x = 0$ we get

$$F'(0) = F'_+(0) = \lim_{h\to 0^+} \frac{(o+h)^p \odot u}{h} = \lim_{h\to 0^+} h^{p-1} \odot u.$$

Hence

$$\lim_{h\to 0^+} D(h^{p-1} \odot u, \tilde{o}) = D(\tilde{o}, \tilde{o}) = 0, \quad p > 1.$$

I.e., $F'(0) = (x^p \odot u)'|_{x=0} = \tilde{o}$, $p > 1$. ∎

It follows

Proposition 3. *Let $u \in \mathbb{R}_{\mathcal{F}}$ be fixed. Then*

$$(e^x \odot u)' = e^x \odot u, \quad \text{any } x \in \mathbb{R}. \tag{4}$$

P r o o f. We have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots, \quad -\infty < x < +\infty.$$

Then

$$e^{x+h} = 1 + (x+h) + \frac{(x+h)^2}{2!} + \frac{(x+h)^3}{3!} + \cdots + \frac{(x+h)^n}{n!} + \cdots, \quad h > 0.$$

Consequently we get

$$e^{x+h} - e^x = h + \left(\frac{2xh + h^2}{2!}\right) + \left(\frac{3x^2h + 3xh^2 + h^3}{3!}\right) + \cdots + \left(\frac{\sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^k}{n!}\right) + \cdots =: \Delta.$$

Here $x \in \mathbb{R}$ and $x + h > x$. Since e^x is increasing then $e^{x+h} > e^x > 0$ and $e^{x+h} - e^x > 0$. I.e., $\Delta > 0$.

Therefore the next H-difference and quotient makes sense in $\mathbb{R}_{\mathcal{F}}$,

$$\begin{aligned} \frac{e^{x+h} \odot u - e^x \odot u}{h} &= \frac{\Delta}{h} \odot u \\ &= \left\{1 + \left(\frac{2x + h}{2!}\right) + \left(\frac{3x^2 + 3xh + h^2}{3!}\right) + \cdots \right. \\ &\quad \left. + \left(\frac{\sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^{k-1}}{n!}\right) + \cdots \right\} \odot u =: K \odot u, \quad K > 0. \end{aligned}$$

Thus

$$\begin{aligned} \lim_{h \to 0^+} D(K \odot u, e^x \odot u) &\le \lim_{h \to 0^+} |K - e^x| D(u, \tilde{o}) \\ &= \left| 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots - e^x \right|. \end{aligned}$$

$$D(u, \tilde{o}) = |e^x - e^x| D(u, \tilde{o}) = 0.$$

We prove that $(e^x \odot u)'_+ = e^x \odot u$.

Next we evaluate

$$(e^x \odot u)'_- = \lim_{h \to 0^+} \frac{e^x \odot u - e^{x-h} \odot u}{h}, \quad x \in \mathbb{R},\ u \in \mathbb{R}_{\mathcal{F}}.$$

By setting $\beta := x - h$ we get

$$(e^x \odot u)'_- = \lim_{h \to 0^+} \frac{e^{\beta+h} \odot u - e^{\beta} \odot u}{h}.$$

Again we have $\beta + h > \beta$ and $e^{\beta+h} > e^{\beta} > 0$, and $e^{\beta+h} - e^{\beta} > 0$. Furthermore it holds

$$e^{\beta+h} - e^{\beta} = h + \left(\frac{2\beta h + h^2}{2!}\right) + \left(\frac{3\beta^2 h + 3\beta h^2 + h^3}{3!}\right) + \cdots + \left(\frac{\sum_{k=1}^{n} \binom{n}{k} \beta^{n-k} h^k}{n!}\right) + \cdots =: \Delta^*.$$

Clearly $0 < \Delta^* < +\infty$.

The next make sense in $\mathbb{R}_{\mathcal{F}}$

$$\begin{aligned} \frac{e^{\beta+h} \odot u - e^{\beta} \odot u}{h} &= \frac{\Delta^*}{h} \odot u \\ &= \left\{1 + \left(\frac{2\beta + h}{2!}\right) + \left(\frac{3\beta^2 + 3\beta h + h^2}{3!}\right) + \cdots \right. \\ &\quad \left. + \left(\frac{\sum_{k=1}^{n} \binom{n}{k} \beta^{n-k} h^{k-1}}{n!}\right) + \cdots \right\} \odot u \\ &=: K^* \odot u, \quad K^* > 0. \end{aligned}$$

Thus

$$\begin{aligned} \lim_{h \to 0^+} D(K^* \odot u, e^x \odot u) &\le \lim_{h \to 0^+} |K^* - e^x| D(u, \tilde{o}) \\ &= \left|1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots - e^x\right| D(u, \tilde{o}) = 0. \end{aligned}$$

We have established

$$(e^x \odot u)'_- = e^x \odot u,$$

and finally proved (4). ■

N o t e. Clearly $(e^x \odot u)^{(\ell)} = e^x \odot u$, $\ell \in \mathbb{N}$, $u \in \mathbb{R}_{\mathcal{F}}$ is fixed, $x \in \mathbb{R}$.

Next we need

Bernstein's Theorem 13-31 (see [4], p. 418). *Assume that $f \in C^\infty$ on an open interval of the form $(a-\delta, b)$, where $\delta > 0$, and suppose that f and all its derivatives are non-negative in the half-open interval $[a, b)$. Then, for every x_0 in $[a, b)$, we have*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \quad \text{if } x_0 \le x < b.$$

We present

Proposition 4. *Let $u \in \mathbb{R}_{\mathcal{F}}$ be fixed, and $f \in C^\infty(-\varepsilon, r)$, $\varepsilon > 0$, $r > 0$ and assume that $f, f', f'', \ldots \ge 0$ on $[0, r)$, with $f(0) = 0$. Then*

$$(f(x) \odot u)' = f'(x) \odot u, \quad \text{for } 0 \le x < r. \tag{5}$$

Clearly

$$(f(x) \odot u)^{(\ell)} = f^{(\ell)}(x) \odot u, \quad \text{for } 0 \le x < r, \quad \ell \in \mathbb{N}. \tag{6}$$

E.g., $f(x) = \sinh x$.

P r o o f. By Bernstein's Theorem we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

and

$$f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x+h)^n, \quad x \in [0, r)$$

and $h > 0$ such that $x + h \in [0, r)$. Since f is non-decreasing we have $f(x+h) \ge f(x) \ge 0$, and $f(x+h) - f(x) \ge 0$. Consequently we see that

$$\begin{aligned} f(x+h) - f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}((x+h)^n - x^n) \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \left(\sum_{k=1}^{n} \binom{n}{k} x^{n-k} h^k \right) \ge 0. \end{aligned}$$

Thus

$$\frac{f(x+h)-f(x)}{h}=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left(\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^{k-1}\right)\geq 0.$$

Therefore the next makes sense in $\mathbb{R}_{\mathcal{F}}$

$$\frac{f(x+h)\odot u-f(x)\odot u}{h}=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left(\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^{k-1})\right)\odot u.$$

Then

$$\begin{aligned}
&\lim_{h\to 0^+} D\left(\left(\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left(\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^{k-1}\right)\right)\odot u, f'(x)\odot u\right)\\
&\leq \lim_{h\to 0^+}\left|\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left(\sum_{k=1}^{n}\binom{n}{k}x^{n-k}h^{k-1})\right)-f'(x)\right| D(u,\tilde{o})\\
&=\left|\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}(nx^{n-1})-f'(x)\right| D(u,\tilde{o})\\
&=|f'(x)-f'(x)|D(u,\tilde{o})=0.
\end{aligned}$$

I.e.,

$$(f(x)\odot u)'_+=f'(x)\odot u,\quad 0\leq x<r.$$

Call $\beta:=x-h$, $x>0$, $x>h$ as $h\to 0^+$. Clearly $\beta>0$. Here

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}(\beta+h)^n,$$

and

$$f(x-h)=f(\beta)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\beta^n.$$

Also $f(x)$, $f(x-h)\geq 0$ and $f(x)\geq f(x-h)$. Thus

$$\begin{aligned}
f(x)-f(x-h)&=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}((\beta+h)^n-\beta^n)\\
&=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left(\sum_{k=1}^{n}\binom{n}{k}\beta^{n-k}h^k\right)\geq 0.
\end{aligned}$$

Furthermore

$$\frac{f(x)-f(x-h)}{h}=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left(\sum_{k=1}^{n}\binom{n}{k}\beta^{n-k}h^{k-1}\right)\geq 0.$$

Consequently

$$\begin{aligned}
&\lim_{h\to 0^+} D\left(\frac{f(x)\odot u-f(x-h)\odot u}{h}, f'(x)\odot u\right)\\
&\quad=\lim_{h\to 0^+} D\left(\left(\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left(\sum_{k=1}^{n}\binom{n}{k}\beta^{n-k}h^{k-1}\right)\right)\odot u, f'(x)\odot u\right)\\
&\quad\leq \lim_{h\to 0^+}\left|\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left(\sum_{k=1}^{n}\binom{n}{k}\beta^{n-k}h^{k-1}\right)-f'(x)\right|.
\end{aligned}$$

$$\begin{aligned}
D(u,\tilde{o}) &= \left|\sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}(nx^{n-1})-f'(x)\right| D(u,\tilde{o})\\
&= |f'(x)-f'(x)|D(u,\tilde{o})=0.
\end{aligned}$$

I.e.,

$$(f(x)\odot u)'_- = f'(x)\odot u,\quad 0<x<r.$$

We have established (5). ∎

Note. One can do other examples of calculation of H-derivatives of basic fuzzy functions, working as above with power series over appropriate intervals.

We mention

Lemma 1. *Let $f,g\colon (a,b)\subseteq \mathbb{R}\to\mathbb{R}_{\mathcal{F}}$ be fuzzy continuous functions. Assume that the H-difference function $f-g$ exists on (a,b). Then $f-g$ is a fuzzy continuous function on (a,b).*

Proof. Let $x_n, x\in(a,b)$ such that $x_n\to x$, as $n\to+\infty$. We observe that

$$\begin{aligned}
&D(f(x_n)-g(x_n), f(x)-g(x))\\
&\qquad = D(f(x_n)-g(x_n)\oplus g(x_n), g(x_n)\oplus f(x)-g(x))\\
&\qquad = D(f(x_n), g(x_n)\oplus f(x)-g(x))\\
&\qquad = D(f(x_n)\oplus g(x), g(x_n)\oplus f(x)-g(x)\oplus g(x))\\
&\qquad = D(f(x_n)\oplus g(x), g(x_n)\oplus f(x))\\
&\qquad \leq D(f(x_n), f(x))+D(g(x_n), g(x))\to 0.
\end{aligned}$$

■

Lemma 2. *Let U be an open subset of $\mathbb{R}^2$ and let $f, g: U \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous (jointly) in $(x, y) \in U$. Then $D(f(x, y), g(x, y))$ is continuous (jointly) in (x, y).*

Proof. It is similar to [5], p. 644, Lemma 13.2 (ii). It goes as follows: Let $U \ni z_n := (x_n, y_n) \to z := (x, y)$, as $n \to +\infty$. We have

$$D(f(z_n), g(z_n)) \leq D(f(z_n), f(z)) + D(f(z), g(z)) + D(g(z), g(z_n)),$$

and

$$D(f(z), g(z)) \leq D(f(z), f(z_n)) + D(f(z_n), g(z_n)) + D(g(z_n), g(z)).$$

Passing to the limit as $n \to +\infty$, from the continuity of f and g we obtain

$$\lim_{n \to +\infty} D(f(z_n), g(z_n)) = D(f(z), g(z)).$$

■

We give

Proposition 5. *Let I be an open interval of $\mathbb{R}$ and let $f, g: I \to \mathbb{R}_{\mathcal{F}}$ be fuzzy differentiable functions with H-derivatives f', g'. Then $(f \oplus g)'$ exists and*

$$(f \oplus g)' = f' \oplus g'. \tag{7}$$

Proof. Let $h \to 0^+$, then by assumption

$$\alpha := f(x+h) - f(x), \quad \beta := g(x+h) - g(x) \in \mathbb{R}_{\mathcal{F}}.$$

Hence $f(x+h) = \alpha \oplus f(x)$, $g(x+h) = \beta \oplus g(x)$. Thus

$$(f \oplus g)(x+h) = \alpha \oplus \beta \oplus (f \oplus g)(x),$$

i.e.,

$$(f \oplus g)(x+h) - (f \oplus g)(x) = \alpha \oplus \beta.$$

Therefore

$$\begin{aligned} &D\left(\frac{(f \oplus g)(x+h) - (f \oplus g)(x)}{h}, f'(x) \oplus g'(x)\right) \\ &\quad = D\left(\frac{\alpha}{h} \oplus \frac{\beta}{h}, f'(x) \oplus g'(x)\right) \\ &\quad \leq D\left(\frac{\alpha}{h}, f'(x)\right) + D\left(\frac{\beta}{h}, g'(x)\right) \to 0, \quad \text{as } h \to 0^+. \end{aligned}$$

Next we set

$$\gamma := f(x) - f(x-h), \quad \delta := g(x) - g(x-h).$$

Clearly $\gamma, \delta \in \mathbb{R}_{\mathcal{F}}$. Then $f(x) = \gamma \oplus f(x-h)$, $g(x) = \delta \oplus g(x-h)$. Hence

$$(f \oplus g)(x) = (\gamma \oplus \delta) \oplus (f \oplus g)(x-h),$$

i.e.,

$$(f \oplus g)(x) - (f \oplus g)(x-h) = \gamma \oplus \delta.$$

Therefore

$$\begin{aligned} & D\left(\frac{(f \oplus g)(x) - (f \oplus g)(x-h)}{h}, f'(x) \oplus g'(x)\right) \\ & \quad = D\left(\frac{\gamma \oplus \delta}{h}, f'(x) \oplus g'(x)\right) \\ & \quad \leq D\left(\frac{\gamma}{h}, f'(x)\right) + D\left(\frac{\delta}{h}, g'(x)\right) \to 0, \quad \text{as } h \to 0^+. \end{aligned}$$

That is, proving the claim. ∎

The counterpart of the above follows.

Proposition 6. *Let I be an open interval of $\mathbb{R}$ and let $f\colon \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable, $c \in \mathbb{R}$. Then*

(8) $$(c \odot f)' \ \text{ exists and } \ (c \odot f)' = c \odot f'(x).$$

P r o o f. We see

$$\begin{aligned} & D\left(\frac{(c \odot f)(x+h) - (c \odot f)(x)}{h}, c \odot f'(x)\right) \\ & \quad = D\left(\frac{c \odot f(x+h) - c \odot f(x)}{h}, c \odot f'(x)\right) =: (*). \end{aligned}$$

Here $\alpha := f(x+h) - f(x) \in \mathbb{R}_{\mathcal{F}}$, so that $f(x+h) = \alpha \oplus f(x)$. Then

$$c \odot f(x+h) = c \odot \alpha \oplus c \odot f(x).$$

I.e., $c \odot f(x+h) - c \odot f(x) = c \odot a$. Therefore

$$\begin{aligned} (*) &= D\left(\frac{c \odot a}{h}, c \odot f'(x)\right) \\ &= |c| D\left(\frac{a}{h}, f'(x)\right) \to 0, \quad \text{as } h \to 0^+. \end{aligned}$$

Next let $\beta := f(x) - f(x-h) \in \mathbb{R}_{\mathcal{F}}$, so that $f(x) = \beta \oplus f(x-h)$. Hence

$$c \odot f(x) = c \odot \beta \oplus c \odot f(x-h),$$

i.e.,

$$c \odot f(x) - c \odot f(x-h) = c \odot \beta.$$

Therefore

$$\begin{aligned} D & \left(\frac{(c \odot f)(x) - (c \odot f)(x-h)}{h}, c \odot f'(x) \right) \\ &= D \left(\frac{c \odot f(x) - c \odot f(x-h)}{h}, c \odot f'(x) \right) \\ &= D \left(\frac{c \odot \beta}{h}, c \odot f'(x) \right) = |c| D \left(\frac{\beta}{h}, f'(x) \right) \to 0, \quad \text{as } h \to 0^+. \end{aligned}$$

That is establishing the claim. ■

Note. Linearity is true in H-fuzzy differentiation, that is

$$(\lambda \odot f \oplus \mu \odot g)' = \lambda \odot f' \oplus \mu \odot g',$$

when $\lambda, \mu \in \mathbb{R}$ and f, g are H-fuzzy differentiable.

3. Main Results

We present the "Fuzzy Mean Value Theorem".

Theorem 1. *Let $f \colon [a,b] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy differentiable function on $[a,b]$ with H-fuzzy derivative f' which is assumed to be fuzzy continuous. Then*

$$D(f(d), f(c)) \le (d-c) \sup_{t \in [c,d]} D(f'(t), \tilde{o}), \tag{9}$$

for any $c, d \in [a,b]$ with $d \ge c$.

Proof. By Corollary A of [1] it holds that

$$f(c) = f(a) \oplus (FR) \int_a^c f'(t) dt,$$

and

$$f(d) = f(a) \oplus (FR) \int_a^d f'(t) dt.$$

Then

$$\begin{aligned} D(f(d), f(c)) &= D\left(f(a) \oplus (FR)\int_a^d f'(t)dt, f(a) \oplus (FR)\int_a^c f'(t)dt\right) \\ &= D\left((FR)\int_a^d f'(t)dt, (FR)\int_a^c f'(t)dt\right) \\ &= D\left((FR)\int_a^c f'(t)dt \oplus (FR)\int_c^d f'(t)dt, (FR)\int_a^c f'(t)dt\right) \\ &= D\left((FR)\int_c^d f'(t)dt, \tilde{o}\right) =: (*). \end{aligned}$$

Clearly $k \odot \tilde{o} = \tilde{o}$ for $k \in \mathbb{R}$. And

$$\tilde{o} = \tilde{o} \odot (d-c) = \tilde{o} \odot \int_c^d 1\, dt = (FR)\int_c^d (\tilde{o} \odot 1)dt = (FR)\int_c^d \tilde{o}\, dt.$$

Hence

$$\begin{aligned} (*) &= D\left((FR)\int_c^d f'(t)dt, (FR)\int_c^d \tilde{o}\, dt\right) \\ &\overset{\text{(by Lemma 1, [1])}}{\leq} \int_c^d D(f'(t), \tilde{o})dt \leq (d-c) \sup_{t \in [c,d]} D(f'(t), \tilde{o}) < +\infty, \end{aligned}$$

by Lemma 2 of [1]. ∎

We need

Lemma 3. *Let $u_n, v_n, u, v \in \mathbb{R}_{\mathcal{F}}$, $n \in \mathbb{N}$. Let $u_n \to u$, $v_n \to v$, as $n \to +\infty$. Then $D(u_n, v_n) \to D(u, v)$, as $n \to +\infty$ (i.e., $D(u, v)$ is continuous in (u, v)). In particular $D(u_n, v) \to D(u, v)$, as $n \to +\infty$. We write*

$$\lim_{n\to+\infty} D(u_n, v_n) = D\left(\lim_{n\to+\infty} u_n, \lim_{n\to+\infty} v_n\right) = D(u, v).$$

Lemma 4. *Let $u_n, u \in \mathbb{R}_{\mathcal{F}}$; $c_n, c \in \mathbb{R}_+$, such that $u_n \to u$ and $c_n \to c$, as $n \to +\infty$. Then in D-metric*

$$u_n \odot c_n \to u \odot c, \quad \text{as } n \to +\infty,$$

i.e.,

$$\lim_{n\to+\infty} (u_n \odot c_n) = \left(\lim_{n\to+\infty} u_n\right) \odot \left(\lim_{n\to+\infty} c_n\right) = u \odot c.$$

P r o o f. We notice that

$$\begin{aligned} D(u_n \odot c_n, u \odot c) &\le D(u_n \odot c_n, u_n \odot c) + D(u_n \odot c, u \odot c) \\ &\overset{\text{(by Lemma 2.2, [3])}}{\le} |c_n - c| D(u_n, \tilde{o}) + c D(u_n, u) \\ &\overset{\text{(by Lemma 3)}}{\to} 0 D(u, \tilde{o}) + c0 = 0. \end{aligned}$$

That is

$$\lim_{n \to +\infty} D(u_n \odot c_n, u \odot c) = 0.$$

■

We present the "Univariate Fuzzy Chain Rule".

Theorem 2. *Let I be a closed interval in $\mathbb{R}$. Here $g: I \to \zeta := g(I) \subseteq \mathbb{R}$ is differentiable, and $f: \zeta \to \mathbb{R}_{\mathcal{F}}$ is H-fuzzy differentiable. Assume that g is strictly increasing. Then $(f \circ g)'(x)$ exists and*

$$(f \circ g)'(x) = f'(g(x)) \odot g'(x), \quad \forall x \in I. \tag{10}$$

P r o o f. Call $u := g(x)$. Let $\Delta x > 0$, such that $\Delta x \to 0^+$.

i) Let $\Delta u := g(x + \Delta x) - g(x)$. Then $\Delta u > 0$, and as $\Delta x \to 0^+$ we get $\Delta u \to 0^+$ by continuity of g. See that $g(x + \Delta x) = u + \Delta u$. We observe that

$$\begin{aligned} &\lim_{\Delta x \to 0^+} D\left(\frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x}, f'(g(x)) \odot g'(x)\right) \\ &= \lim_{\Delta x \to 0^+} D\left(\left(\frac{f(g(x + \Delta x)) - f(g(x))}{g(x + \Delta x) - g(x)}\right)\right. \\ &\quad \left. \odot \left(\frac{g(x + \Delta x) - g(x)}{\Delta x}\right), f'(g(x)) \odot g'(x)\right) \\ &= \lim_{\Delta x \to 0^+} D\left(\left(\frac{f(u + \Delta u) - f(u)}{\Delta u}\right)\right. \\ &\quad \left. \odot \left(\frac{g(x + \Delta x) - g(x)}{\Delta x}\right), f'(g(x)) \odot g'(x)\right) \\ &= D(f'(u) \odot g'(x), f'(g(x)) \odot g'(x)) = 0, \end{aligned}$$

by Lemmas 3 and 4. I.e.,

$$(f \odot g)'_+ = f'(g(x)) \odot g'(x).$$

ii) Let $\Delta u := g(x) - g(x - \Delta x)$. Then $\Delta u > 0$, and as $\Delta x \to 0^+$ we get $\Delta u \to 0^+$ by continuity of g. Notice that $g(x - \Delta x) = u - \Delta u$. We observe that

$$\begin{aligned}
&\lim_{\Delta x \to 0^+} D\left(\frac{f(g(x)) - f(g(x - \Delta x))}{\Delta x}, f'(g(x)) \odot g'(x)\right) \\
&= \lim_{\Delta x \to 0^+} D\left(\left(\frac{f(g(x)) - f(g(x - \Delta x))}{g(x) - g(x - \Delta x)}\right)\right. \\
&\quad \left.\odot \left(\frac{g(x) - g(x - \Delta x}{\Delta x}\right), f'(g(x)) \odot g'(x)\right) \\
&= \lim_{\Delta x \to 0^+} D\left(\left(\frac{f(u) - f(u - \Delta u)}{\Delta u}\right)\right. \\
&\quad \left.\odot \left(\frac{g(x) - g(x - \Delta x)}{\Delta x}\right), f'(g(x)) \odot g'(x)\right) \\
&= D(f'(u) \odot g'(x), f'(g(x)) \odot g'(x)) = 0,
\end{aligned}$$

by Lemmas 3 and 4. I.e.,

$$(f \circ g)'_- = f'(g(x)) \odot g'(x).$$

At the endpoints of I we take one-sided derivatives. ∎

Next follows the multivariate fuzzy chain rule.

Theorem 3. *Let $\phi_i\colon [a, b] \subseteq \mathbb{R} \to \phi_i([a, b]) := I_i \subseteq \mathbb{R}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, are strictly increasing and differentiable functions. Denote $x_i := x_i(t) := \phi_i(t)$, $t \in [a, b]$, $i = 1, \ldots, n$. Consider U an open subset of $\mathbb{R}^n$ such that $\times_{i=1}^n I_i \subseteq U$. Consider $f\colon U \to \mathbb{R}_{\mathcal{F}}$ a fuzzy continuous function. Assume that $f_{x_i}\colon U \to \mathbb{R}_{\mathcal{F}}$, $i = 1, \ldots, n$, the H-fuzzy partial derivatives of f, exist and are fuzzy continuous. Call $z := z(t) := f(x_1, \ldots, x_n)$. Then $\frac{dz}{dt}$ exists and*

$$\frac{dz}{dt} = \sum_{i=1}^{n}{}^{*} \frac{dz}{dx_i} \odot \frac{dx_i}{dt}, \quad \forall t \in [a, b] \tag{11}$$

where $\frac{dz}{dt}$, $\frac{dz}{dx_i}$, $i = 1, \ldots, n$ are the H-fuzzy derivatives of f with respect to t, x_i, respectively.

Proof. Let first $t \in (a, b)$. Let a general $(x_1, x_2, \ldots, x_n) \in U$ be fixed and let $\Delta x_i > 0$, $i = 1, \ldots, n$, be small.

I) Call

$$\begin{aligned}
\alpha_1 := {} & f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n) \\
& - f(x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n) \in \mathbb{R}_{\mathcal{F}}.
\end{aligned}$$

That is

$$f(x_1+\Delta x_1, x_2+\Delta x_2, \ldots, x_n+\Delta x_n) = \alpha_1 \oplus f(x_1, x_2+\Delta x_2, \ldots, x_n+\Delta x_n).$$

Call

$$\begin{aligned}\alpha_2 := f(&x_1, x_2+\Delta x_2, \ldots, x_n+\Delta x_n)\\ &- f(x_1, x_2, x_3+\Delta x_3, \ldots, x_n+\Delta x_n) \in \mathbb{R}_{\mathcal{F}}.\end{aligned}$$

That is

$$f(x_1, x_2+\Delta x_2, \ldots, x_n+\Delta x_n) = \alpha_2 \oplus f(x_1, x_2, x_3+\Delta x_3, \ldots, x_n+\Delta x_n).$$

Call

$$\begin{aligned}\alpha_3 := f(&x_1, x_2, x_3+\Delta x_3, \ldots, x_n+\Delta x_n)\\ &- f(x_1, x_2, x_3, x_4+\Delta x_4, \ldots, x_n+\Delta x_n) \in \mathbb{R}_{\mathcal{F}}.\end{aligned}$$

That is

$$f(x_1, x_2, x_3+\Delta x_3, \ldots, x_n+\Delta x_n) = \alpha_3 \oplus f(x_1, x_2, x_3, x_4+\Delta x_4, \ldots, x_n+\Delta x_n),$$

etc. Call

$$a_n := f(x_1, x_2, \ldots, x_{n-1}, x_n+\Delta x_n) - f(x_1, x_2, \ldots, x_n) \in \mathbb{R}_{\mathcal{F}}.$$

That is

$$f(x_1, x_2, \ldots, x_{n-1}, x_n+\Delta x_n) = \alpha_n \oplus f(x_1, x_2, \ldots, x_n).$$

I.e., it holds

$$\mathbb{R}_{\mathcal{F}} \in f(x_1+\Delta x_1, x_2+\Delta x_2, \ldots, x_n+\Delta x_n) - f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}{}^{*} \alpha_i.$$

Since the partial derivatives f_{x_i} exist, the above H-differences α_i, $i = 1, \ldots, n$ exist in $\mathbb{R}_{\mathcal{F}}$ for small $\Delta x_i > 0$. In particular we define

$$\Delta x_i := \phi_i(t+\Delta t) - \phi_i(t), \quad \Delta t > 0, \quad i = 1, \ldots n$$

(i.e.,

$$\phi_i(t+\Delta t) = x_i + \Delta x_i, \quad x_i := \phi_i(t)).$$

Since ϕ_i, $i = 1, \ldots, n$ are strictly increasing we have that $\Delta x_i > 0$. So as $\Delta t \to 0^+$, then $\Delta x_i \to 0^+$ by continuity of ϕ_i.

We observe that

$$\lim_{\Delta t\to 0^+} D\Bigg(\frac{f(\phi_1(t+\Delta t),\dots,\phi_n(t+\Delta t))-f(\phi_i(t),\dots,\phi_n(t))}{\Delta t},$$
$$\sum_{i=1}^{n}{}^{*} f_{x_i}(x_1,\dots,x_n)\odot x_i'(t)\Bigg)$$
$$=\lim_{\Delta t\to 0^+} D\Bigg(\frac{f(x_1+\Delta x_1,\dots,x_n+\Delta x_n)-f(x_1,\dots,x_n)}{\Delta t},$$
$$\sum_{i=1}^{n}{}^{*} f_{x_i}(x_1,\dots,x_n)\odot x_i'(t)\Bigg)$$
$$=\lim_{\Delta t\to 0^+} D\left(\frac{\sum_{i=1}^{*n}\alpha_i}{\Delta t},\sum_{i=1}^{n}{}^{*} f_{x_i}(x_1,\dots,x_n)\odot x_i'(t)\right)$$
$$\leq \lim_{\Delta t\to 0^+} D\Bigg(\frac{f(x_1+\Delta x_1,x_2+\Delta x_2,\dots,x_n+\Delta x_n)-f(x_1,x_2+\Delta x_2,\dots,x_n+\Delta x_n)}{\Delta t},$$
$$f_{x_1}(x_1,\dots,x_n)\odot x_1'(t)\Bigg)$$
$$+\lim_{\Delta t\to 0^+} D\Bigg(\frac{f(x_1,x_2+\Delta x_2,\dots,x_n+\Delta x_n)-f(x_1,x_2,x_3+\Delta x_3,\dots,x_n+\Delta x_n)}{\Delta t},$$
$$f_{x_2}(x_1,\dots,x_n)\odot x_2'(t)\Bigg)$$
$$+\lim_{\Delta t\to 0^+} D\Bigg(\frac{f(x_1,x_2,x_3+\Delta x_3,\dots,x_n+\Delta x_n)-f(x_1,x_2,x_3,x_4+\Delta x_4,\dots,x_n+\Delta x_n)}{\Delta t},$$
$$f_{x_3}(x_1,\dots,x_n)\odot x_3'(t)\Bigg)$$
$$+\cdots+\lim_{\Delta t\to 0^+} D\Bigg(\frac{f(x_1,x_2,\dots,x_{n-1},x_n+\Delta x_n)-f(x_1,x_2,\dots,x_n)}{\Delta t},$$
$$f_{x_n}(x_1,\dots,x_n)\odot x_n'(t)\Bigg)$$
$$=\lim_{\Delta t\to 0^+} D\Bigg(\left(\frac{f(x_1+\Delta x_1,x_2+\Delta x_2,\dots,x_n+\Delta x_n)-f(x_1,x_2+\Delta x_2,\dots,x_n+\Delta x_n)}{\Delta x_1}\right)$$
$$\odot\frac{\Delta x_1}{\Delta t}, f_{x_1}(x_1,\dots,x_n)\odot x_1'(t)\Bigg)$$

$$+\lim_{\Delta t\to 0^+} D\left(\left(\frac{f(x_1,x_2+\Delta x_2,\ldots,x_n+\Delta x_n)-f(x_1,x_2,x_3+\Delta x_3,\ldots,x_n+\Delta x_n)}{\Delta x_2}\right)\right.$$

$$\left.\odot\frac{\Delta x_2}{\Delta t}, f_{x_2}(x_1,\ldots,x_n)\odot x_2'(t)\right)$$

$$+\lim_{\Delta t\to 0^+} D\left(\left(f(x_1,x_2,x_3+\Delta x_3,\ldots,x_n+\Delta x_n)\right.\right.$$

$$\left.\left.-f(x_1,x_2,x_3,x_4+\Delta x_4,\ldots,x_n+\Delta x_n)/\Delta x_3\right)\odot\frac{\Delta x_3}{\Delta t}, f_{x_3}(x_1,\ldots,x_n)\odot x_3'(t)\right)$$

$$+\cdots+\lim_{\Delta t\to 0^+} D\left(\left(\frac{f(x_1,x_2,\ldots,x_{n-1},x_n+\Delta x_n)-f(x_1,\ldots,x_n)}{\Delta x_n}\right)\right.$$

$$\left.\odot\frac{\Delta x_n}{\Delta t}, f_{x_n}(x_1,\ldots,x_n)\odot x_n'(t)\right)$$

(by Corollary A, [1])
$$=$$

$$\lim_{\Delta t\to 0^+} D\left(\left(\frac{(FR)\int_{x_1}^{x_1+\Delta x_1} f_{x_1}(t,x_2+\Delta x_2,\ldots,x_n+\Delta x_n)dt}{\Delta x_1}\right)\right.$$

$$\left.\odot\frac{\Delta x_1}{\Delta t}, f_{x_1}(x_1,\ldots,x_n)\odot x_1'(t)\right)$$

$$+\lim_{\Delta t\to 0^+} D\left(\left(\frac{(FR)\int_{x_2}^{x_2+\Delta x_2} f_{x_2}(x_1,t,x_3+\Delta x_3,\ldots,x_n+\Delta x_n)dt}{\Delta x_2}\right)\right.$$

$$\left.\odot\frac{\Delta x_2}{\Delta t}, f_{x_2}(x_1,\ldots,x_n)\odot x_2'(t)\right)$$

$$+\lim_{\Delta t\to 0^+} D\left(\left(\frac{(FR)\int_{x_3}^{x_3+\Delta x_3} f_{x_3}(x_1,x_2,t,x_4+\Delta x_4,\ldots,x_n+\Delta x_n)dt}{\Delta x_3}\right)\right.$$

$$\left.\odot\frac{\Delta x_3}{\Delta t}, f_{x_3}(x_1,\ldots,x_n)\odot x_3'(t)\right)$$

$$+\cdots+\lim_{\Delta t\to 0^+} D\left(\left(\frac{(FR)\int_{x_{n-1}}^{x_{n-1}+\Delta x_{n-1}} f_{x_{n-1}}(x_1,\ldots,x_{n-2},t,x_n+\Delta x_n)dt}{\Delta x_{n-1}}\right)\right.$$

$$\left.\odot\frac{\Delta x_{n-1}}{\Delta t}, f_{x_{n-1}}(x_1,\ldots,x_n)\odot x_{n-1}'(t)\right)$$

$$+D(f_{x_n}(x_1,\ldots,x_n)\odot x_n'(t), f_{x_n}(x_1,\ldots,x_n)\odot x_n'(t))$$

(by Lemmas 3 and 4)

$$= x_1'(t) \lim_{\Delta t \to 0^+} \frac{1}{\Delta x_1} D\Bigg((FR) \int_{x_1}^{x_1+\Delta x_1} f_{x_1}(t, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n) dt,$$
$$\Delta x_1 \odot f_{x_1}(x_1, \ldots, x_n) \Bigg)$$
$$+ x_2'(t) \lim_{\Delta t \to 0^+} \frac{1}{\Delta x_2} D\Bigg((FR) \int_{x_2}^{x_2+\Delta x_2} f_{x_2}(x_1, t, x_3 + \Delta x_3, \ldots, x_n + \Delta x_n) dt,$$
$$\Delta x_2 \odot f_{x_2}(x_1, \ldots, x_n) \Bigg)$$
$$+ x_3'(t) \lim_{\Delta t \to 0^+} \frac{1}{\Delta x_3} D\Bigg((FR) \int_{x_3}^{x_3+\Delta x_3} f_{x_3}(x_1, x_2, t, x_4 + \Delta x_4, \ldots, x_n + \Delta x_n) dt,$$
$$\Delta x_3 \odot f_{x_3}(x_1, \ldots, x_n) \Bigg) + \cdots$$
$$+ x_{n-1}'(t) \lim_{\Delta t \to 0^+} \frac{1}{\Delta x_{n-1}} D\Bigg((FR) \int_{x_{n-1}}^{x_{n-1}+\Delta x_{n-1}}$$
$$f_{x_{n-1}}(x_1, x_2, \ldots, x_{n-2}, t, x_n + \Delta x_n) dt, \Delta x_{n-1} \odot f_{x_{n-1}}(x_1, \ldots, x_n) \Bigg)$$
$$= \sum_{i=1}^{n-1} x_i'(t) \lim_{\Delta t \to 0^+} \frac{1}{\Delta x_i} D\Bigg((FR) \int_{x_i}^{x_i+\Delta x_i} f_{x_i}(x_1, x_2, \ldots, x_{i-1},$$
$$t, x_{i+1} + \Delta x_{i+1}, \ldots, x_n + \Delta x_n) dt, (FR) \int_{x_i}^{x_i+\Delta x_i} f_{x_i}(x_1, \ldots, x_n) dt \Bigg)$$
$$\overset{\text{(by Lemma 1 of [1])}}{\leq} \sum_{i=1}^{n-1} x_i'(t) \lim_{\Delta t \to 0^+} \frac{1}{\Delta x_i} \Bigg(\int_{x_i}^{x_i+\Delta x_i} D(f_{x_i}(x_1, x_2, \ldots, x_{i-1},$$
$$t, x_{i+1} + \Delta x_{i+1}, \ldots, x_n + \Delta x_n), f_{x_i}(x_1, \ldots, x_n)) dt \Bigg)$$
$$\leq \sum_{i=1}^{n-1} x_i'(t) \lim_{\Delta t \to 0^+} \frac{1}{\Delta x_i} \Bigg(\sup_{\tau \in [x_i, x_i+\Delta x_i]} (D(f_{x_i}(x_1, x_2, \ldots, x_{i-1}, \tau,$$
$$x_{i+1} + \Delta x_{i+1}, \ldots, x_n + \Delta x_n), f_{x_i}(x_1, \ldots, x_n))) \Bigg) \Delta x_i$$
$$\overset{\text{(by Lemma 1 of [1])}}{=} \quad \text{(for some } \tau_i^* \in [x_i, x_i + \Delta x_i])$$

$$\sum_{i=1}^{n-1} x_i'(t) \lim_{\Delta t \to 0^+} D\big(f_{x_i}(x_1, x_2, \ldots, x_{i-1}, \tau_i^*, x_{i+1} + \Delta x_{i+1}, \ldots, x_n + \Delta x_n), f_{x_i}(x_1, \ldots, x_n)\big)$$

(as $\Delta t \to 0^+$, then all $\Delta x_i \to 0^+$ and thus $\tau_i^* \to x_i$, for all $i = 1, \ldots, n$)

$$= \sum_{i=1}^{n-1} x_i'(t) D(f_{x_i}(x_1, \ldots, x_n), f_{x_i}(x_1, \ldots, x_n))$$

$$= \sum_{i=1}^{n-1} x_i'(t) \cdot 0 = 0,$$

by continuity of f_{x_i}, $i = 1, \ldots, n-1$. I.e., we have proved that

$$\left(\frac{dz}{dt}\right)_+ = \sum_{i=1}^{n}{}^{*} \frac{dz}{dx_i} \odot \frac{dx_i}{dt}.$$

II) Call

$$\beta_1 := f(x_1, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_{n-1}, x_n - \Delta x_n) \in \mathbb{R}_{\mathcal{F}}.$$

That is

$$f(x_1, x_2, \ldots, x_n) = \beta_1 \oplus f(x_1, x_2, \ldots, x_{n-1}, x_n - \Delta x_n).$$

Call

$$\beta_2 := f(x_1, x_2, \ldots, x_{n-1}, x_n - \Delta x_n) - f(x_1, x_2, \ldots x_{n-2}, x_{n-1} - \Delta x_{n-1}, x_n - \Delta x_n) \in \mathbb{R}_{\mathcal{F}}.$$

That is

$$f(x_1, x_2, \ldots, x_{n-1}, x_n - \Delta x_n) = \beta_2 \oplus f(x_1, x_2, \ldots, x_{n-2}, x_{n-1} - \Delta x_{n-1}, x_n - \Delta x_n).$$

Call

$$\beta_3 := f(x_1, x_2, \ldots, x_{n-2}, x_{n-1} - \Delta x_{n-1}, x_n - \Delta x_n) - f(x_1, x_2, \ldots, x_{n-3}, x_{n-2} - \Delta x_{n-2}, x_{n-1} - \Delta x_{n-1}, x_n - \Delta x_n) \in \mathbb{R}_{\mathcal{F}}.$$

That is

$$f(x_1, x_2, \ldots, x_{n-2}, x_{n-1} - \Delta x_{n-1}, x_n - \Delta x_n) = \beta_3 \oplus f(x_1, x_2, \ldots, x_{n-3}, x_{n-2} - \Delta x_{n-2}, x_{n-1} - \Delta x_{n-1}, x_n - \Delta x_n),$$

etc. Call

$$\beta_n := f(x_1, x_2-\Delta x_2, \dots, x_n-\Delta x_n) - f(x_1-\Delta x_1, x_2-\Delta x_2, \dots, x_n-\Delta x_n) \in \mathbb{R}_{\mathcal{F}}.$$

That is

$$f(x_1, x_2 - \Delta x_2, \dots, x_n - \Delta x_n) = \beta_n \oplus f(x_1 - \Delta x_1, x_2 - \Delta x_2, \dots, x_n - \Delta x_n).$$

I.e., it holds

$$\mathbb{R}_{\mathcal{F}} \ni f(x_1, x_2, \dots, x_n) - f(x_1 - \Delta x_1, x_2 - \Delta x_2, \dots, x_n - \Delta x_n) = \sum_{i=1}^{n}{}^{*} \beta_i.$$

Since the partial derivatives f_{x_i} exist, the above H-differences β_i, $i = 1, \dots, n$ exist in $\mathbb{R}_{\mathcal{F}}$ for small $\Delta x_i > 0$. In particular we define $\Delta x_i := \phi_i(t) - \phi_i(t - \Delta t)$, $\Delta t > 0$, $i = 1, \dots, n$ (i.e., $\phi_i(t - \Delta t) = x_i - \Delta x_i$, $x_i := \phi_i(t)$). Since ϕ_i, $i = 1, \dots, n$ are strictly increasing we have that $\Delta x_i > 0$. So as $\Delta t \to 0^+$, then $\Delta x_i \to 0^+$ by continuity of ϕ_i.

We observe that

$$\begin{aligned}
&\lim_{\Delta t \to 0^+} D\Bigg(\frac{f(\phi_1(t), \dots, \phi_n(t)) - f(\phi_1(t-\Delta t), \dots, \phi_n(t-\Delta t))}{\Delta t}, \\
&\qquad \sum_{i=1}^{n}{}^{*} f_{x_i}(x_1, \dots, x_n) \odot x_i'(t) \Bigg) \\
&= \lim_{\Delta t \to 0^+} D\Bigg(\frac{f(x_1, \dots, x_n) - f(x_1 - \Delta x_1, \dots, x_n - \Delta x_n)}{\Delta t}, \\
&\qquad \sum_{i=1}^{n}{}^{*} f_{x_i}(x_1, \dots, x_n) \odot x_i'(t) \Bigg) \\
&= \lim_{\Delta t \to 0^+} D\Bigg(\frac{\sum^{*n}_{i=1} \beta_i}{\Delta t}, \sum_{i=1}^{n}{}^{*} f_{x_i}(x_1, \dots, x_n) \odot x_i'(t) \Bigg) \\
&\le \lim_{\Delta t \to 0^+} D\Bigg(\frac{f(x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_{n-1}, x_n - \Delta x_n)}{\Delta t}, \\
&\qquad f_{x_n}(x_1, \dots, x_n) \odot x_n'(t) \Bigg) + \lim_{\Delta t \to 0^+} \\
&\quad D\Bigg(\frac{f(x_1, x_2, \dots, x_{n-1}, x_n - \Delta x_n) - f(x_1, x_2, \dots, x_{n-2}, x_{n-1} - \Delta x_{n-1}, x_n - \Delta x_n)}{\Delta t}, \\
&\qquad f_{x_{n-1}}(x_1, \dots, x_n) \odot x_{n-1}'(t) \Bigg) + \lim_{\Delta t \to 0^+}
\end{aligned}$$

$$D\left(\frac{\begin{array}{l}f(x_1,x_2,\ldots,x_{n-2},x_{n-1}-\Delta x_{n-1},x_n-\Delta x_n)\\ -f(x_1,x_2,\ldots,x_{n-3},x_{n-2}-\Delta x_{n-2},x_{n-1}-\Delta x_{n-1},x_n-\Delta x_n)\end{array}}{\Delta t},\right.$$

$$\left. f_{x_{n-2}}(x_1,\ldots,x_n)\odot x'_{n-2}(t)\right)+\cdots+\lim_{\Delta t\to 0^+}$$

$$D\left(\frac{f(x_1,x_2-\Delta x_2,\ldots,x_n-\Delta x_n)-f(x_1-\Delta x_1,x_2-\Delta x_2,\ldots,x_n-\Delta x_n)}{\Delta t},\right.$$

$$\left. f_{x_1}(x_1,\ldots,x_n)\odot x'_1(t)\right)$$

$$=\lim_{\Delta t\to 0^+}D\left(\left(\frac{f(x_1,x_2,\ldots,x_n)-f(x_1,x_2,\ldots,x_{n-1},x_n-\Delta x_n)}{\Delta x_n}\right)\right.$$

$$\left.\odot\frac{\Delta x_n}{\Delta t},f_{x_n}(x_1,\ldots,x_n)\odot x'_n(t)\right)+\lim_{\Delta t\to 0^+}$$

$$D\left(\left(\frac{f(x_1,x_2,\ldots,x_{n-1},x_n-\Delta x_n)-f(x_1,x_2,\ldots,x_{n-2},x_{n-1}-\Delta x_{n-1},x_n-\Delta x_n)}{\Delta x_{n-1}}\right)\right.$$

$$\left.\odot\frac{\Delta x_{n-1}}{\Delta t},f_{x_{n-1}}(x_1,\ldots,x_n)\odot x'_{n-1}(t)\right)+\lim_{\Delta t\to 0^+}$$

$$D\left(\left(\frac{\begin{array}{l}f(x_1,x_2,\ldots,x_{n-2},x_{n-1}-\Delta x_{n-1},x_n-\Delta x_n)\\ -f(x_1,x_2,\ldots,x_{n-3},x_{n-2}-\Delta x_{n-2},x_{n-1}-\Delta x_{n-1},x_n-\Delta x_n)\end{array}}{\Delta x_{n-2}}\right)\right.$$

$$\left.\odot\frac{\Delta x_{n-2}}{\Delta t},f_{x_{n-2}}(x_1,\ldots,x_n)\odot x'_{n-2}(t)\right)+\cdots+\lim_{\Delta t\to 0^+}$$

$$D\left(\left(\frac{f(x_1,x_2-\Delta x_2,\ldots,x_n-\Delta x_n)-f(x_1-\Delta x_1,x_2-\Delta x_2,\ldots,x_n-\Delta x_n)}{\Delta x_1}\right)\right.$$

$$\left.\odot\frac{\Delta x_1}{\Delta t},f_{x_1}(x_1,\ldots,x_n)\odot x'_1(t)\right)$$

(by Corollary A, [1])
$$=$$

$$D\big(f_{x_n}(x_1,\ldots,x_n)\odot x'_n(t),f_{x_n}(x_1,\ldots,x_n)\odot x'_n(t)\big)$$

$$+\lim_{\Delta t\to 0^+}D\left(\left(\frac{(FR)\int_{x_{n-1}-\Delta x_{n-1}}^{x_{n-1}}f_{x_{n-1}}(x_1,x_2,\ldots,x_{n-2},t,x_n-\Delta x_n)dt}{\Delta x_{n-1}}\right)\right.$$

$$\left.\odot\frac{\Delta x_{n-1}}{\Delta t},f_{x_{n-1}}(x_1,\ldots,x_n)\odot x'_{n-1}(t)\right)+\lim_{\Delta t\to 0^+}$$

$$D\left(\frac{(FR)\int_{x_{n-2}-\Delta x_{n-2}}^{x_{n-2}} f_{x_{n-2}}(x_1,x_2,\ldots,x_{n-3},t,x_{n-1}-\Delta x_{n-1},x_n-\Delta x_n)dt}{\Delta x_{n-2}}\right)$$

$$\odot \frac{\Delta x_{n-2}}{\Delta t}, f_{x_{n-2}}(x_1,\ldots,x_n)\odot x'_{n-2}(t)\Bigg) + \cdots + \lim_{\Delta t\to 0^+}$$

$$D\left(\left(\frac{(FR)\int_{x_1-\Delta x_1}^{x_1} f_{x_1}(t,x_2-\Delta x_2,\ldots,x_n-\Delta x_n)dt}{\Delta x_1}\right)\right.$$

$$\left.\odot \frac{\Delta x_1}{\Delta t}, f_{x_1}(x_1,\ldots,x_n)\odot x'_1(t)\right)$$

$$\overset{\text{(by Lemmas 3, 4)}}{=} x'_{n-1}(t)\lim_{\Delta t\to 0^+}\frac{1}{\Delta x_{n-1}} D\Bigg((FR)$$

$$\int_{x_{n-1}-\Delta x_{n-1}}^{x_{n-1}} f_{x_{n-1}}(x_1,x_2,\ldots,x_{n-2},t,x_n-\Delta x_n)dt, \Delta x_{n-1}\odot f_{x_{n-1}}(x_1,\ldots,x_n)\Bigg)$$

$$+ x'_{n-2}(t)\lim_{\Delta t\to 0^+}\frac{1}{\Delta x_{n-2}} D\Bigg((FR)\int_{x_{n-2}-\Delta x_{n-2}}^{x_{n-2}} f_{x_{n-2}}(x_1,x_2,\ldots,x_{n-3},$$

$$t,x_{n-1}-\Delta x_{n-1},x_n-\Delta x_n)dt, \Delta x_{n-2}\odot f_{x_{n-2}}(x_1,\ldots,x_n)\Bigg)$$

$$+\cdots + x'_1(t)\lim_{\Delta t\to 0^+}\frac{1}{\Delta x_1}$$

$$D\left((FR)\int_{x_1-\Delta x_1}^{x_1} f_{x_1}(t,x_2-\Delta x_2,\ldots,x_n-\Delta x_n)dt, \Delta x_1\odot f_{x_1}(x_1,\ldots,x_n)\right)$$

$$= \sum_{i=1}^{n-1} x'_i(t)\lim_{\Delta t\to 0^+}\frac{1}{\Delta x_i} D\Bigg((FR)\int_{x_i-\Delta x_i}^{x_i} f_{x_i}\big(x_1,x_2,\ldots,x_{i-1},$$

$$t,x_{i+1}-\Delta x_{i+1},\ldots,x_{n-1}-\Delta x_{n-1},x_n-\Delta x_n\big)dt, (FR)\int_{x_i-\Delta x_i}^{x_i} f_{x_i}(x_1,\ldots,x_n)dt\Bigg)$$

$$\overset{\text{(by Lemma 1 of [1])}}{\le} \sum_{i=1}^{n-1} x'_i(t)\lim_{\Delta t\to 0^+}\frac{1}{\Delta x_i}\Bigg(\int_{x_i-\Delta x_i}^{x_i} D(f_{x_i}(x_1,x_2,\ldots,x_{i-1},t,$$

$$x_{i+1}-\Delta x_{i+1},\ldots,x_{n-1}-\Delta x_{n-1},x_n-\Delta x_n), f_{x_i}(x_1,\ldots,x_n))dt\Bigg)$$

$$\le \sum_{i=1}^{n-1} x'_i(t)\lim_{\Delta t\to 0^+}\frac{1}{\Delta x_i}\Bigg(\sup_{\tau\in[x_i-\Delta x_i,x_i]}\big(D(f_{x_i}(x_1,x_2,\ldots,x_{i-1},\tau,x_{i+1}-\Delta x_{i+1},$$

$$\ldots,x_{n-1}-\Delta x_{n-1},x_n-\Delta x_n), f_{x_i}(x_1,\ldots,x_n))\big)\Bigg)\Delta x_i$$

$$\text{(for some } \tau_i^* \in [x_i - \Delta x_i, x_i])$$

$$\overset{\text{(by Lemma 1 of [1])}}{=} \sum_{i=1}^{n-1} x_i'(t) \lim_{\Delta t \to 0^+} D\big(f_{x_i}(x_1, x_2, \dots, x_{i-1},$$

$$\tau_i^*, x_{i+1} - \Delta x_{i+1}, \dots, x_{n-1} - \Delta x_{n-1}, x_n - \Delta x_n), f_{x_i}(x_1, \dots, x_n)\big)$$

(as $\Delta t \to 0^+$, then all $\Delta x_i \to 0^+$ and thus $\tau_i^* \to x_i$, for all $i = 1, \dots, n$)

$$= \sum_{i=1}^{n-1} x_i'(t) D(f_{x_i}(x_1, \dots, x_n), f_{x_i}(x_1, \dots, x_n))$$

$$= \sum_{i=1}^{n-1} x_i'(t) \cdot 0 = 0,$$

by continuity of f_{x_i}, $i = 1, \dots, n-1$. I.e., we have proved that

$$\left(\frac{dz}{dt}\right)_- = \sum_{i=1}^{n}{}^* \frac{dz}{dx_i} \odot \frac{dx_i}{dt}.$$

When $t = a$, or b, then $\frac{dz}{dt}$ equals $\left(\frac{dz}{dt}\right)_+$, or $\left(\frac{dz}{dt}\right)_-$, respectively. Clearly here

$$\left.\frac{dx_i}{dt}\right|_{t=a} = \left.\left(\frac{dx_i}{dt}\right)_+\right|_{t=a}, \quad \text{and} \quad \left.\frac{dx_i}{dt}\right|_{t=b} = \left.\left(\frac{dx_i}{dt}\right)_-\right|_{t=b},$$

etc., the same proof as before. The theorem now is proved. ■

We need the following

Lemma 5. *Let f be a fuzzy continuous function from the open set $U \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, into $\mathbb{R}_{\mathcal{F}}$. Then $f_{\pm}^{(r)}$ are continuous functions from U into $\mathbb{R}$, for all $r \in [0, 1]$.*

P r o o f. Let $x_m, x \in U$, $m \in \mathbb{N}$, be such that $x_m \to x$ as $m \to +\infty$. Then by continuity of f we get $D(f(x_m), f(x)) \to 0$, as $m \to +\infty$. Hence we have

$$D(f(x_m), f(x)) = \sup_{r \in [0,1]} \max\{|(f(x_m))_-^{(r)} - (f(x))_-^{(r)}|, |(f(x_m))_+^{(r)} - (f(x))_+^{(r)}|\} \to 0.$$

Therefore $|(f(x_m))_-^{(r)} - (f(x))_-^{(r)}| \to 0$ and $|(f(x_m))_+^{(r)} - (f(x))_+^{(r)}| \to 0$, as $m \to +\infty$, for all $r \in [0, 1]$. Consequently $(f(x_m))_{\pm}^{(r)} \to (f(x))_{\pm}^{(r)}$, proving that $f_{\pm}^{(r)} \in C(U, \mathbb{R})$, for all $0 \le r \le 1$. ■

We present the interchange of the order of H-fuzzy differentiation.

Theorem 4. *Let U be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, and $f : U \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy continuous function. Assume that all H-fuzzy partial derivatives of f up to order $m \in \mathbb{N}$ exist and are fuzzy continuous. Let $x := (x_1, \ldots, x_n) \in U$. Then the H-fuzzy mixed partial derivative of order k, $D_{x_{\ell_1},\ldots,x_{\ell_k}} f(x)$ is unchanged when the indices $\ell_1, \ldots, \ell_k$ are permuted. Each ℓ_i is a positive integer $\leq n$. Here some or all of ℓ_i's can be equal. Also $k = 2, \ldots, m$ and there are n^k partials of order k.*

Proof. We only need to demonstrate the proof for the case $n = k = 2$. The rest is true by induction on k, and similarly true for $n > 2$. So here $z = f(x, y) : U \subseteq \mathbb{R}^2 \to \mathbb{R}_{\mathcal{F}}$ and $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ exist and are fuzzy continuous functions from U into $\mathbb{R}_{\mathcal{F}}$. We make use of Theorem 5.2 from [6] repeatedly. Here we have

$$[f(x,y)]^r = [(f(x,y))_-^{(r)}, (f(x,y))_+^{(r)}], \quad 0 \leq r \leq 1.$$

By that theorem and the above assumptions $\frac{\partial}{\partial x}(f(x,y))_{\pm}^{(r)}$ exist and

$$\left[\frac{\partial}{\partial x} f(x,y)\right]^r = \left[\frac{\partial}{\partial x}(f(x,y))_-^{(r)}, \frac{\partial}{\partial x}(f(x,y))_+^{(r)}\right],$$

for all $0 \leq r \leq 1$ and all $(x, y) \in U$. Furthermore, the same way $\frac{\partial^2}{\partial y \partial x}(f(x,y))_{\pm}^{(r)}$ exist and

$$\left[\frac{\partial^2}{\partial y \partial x} f(x,y)\right]^r = \left[\frac{\partial^2}{\partial y \partial x}(f(x,y))_-^{(r)}, \frac{\partial^2}{\partial y \partial x}(f(x,y))_+^{(r)}\right],$$

for all $0 \leq r \leq 1$ and all $(x, y) \in U$. Similarly we obtain

$$\left[\frac{\partial^2}{\partial x \partial y} f(x,y)\right]^r = \left[\frac{\partial^2}{\partial x \partial y}(f(x,y))_-^{(r)}, \frac{\partial^2}{\partial x \partial y}(f(x,y))_+^{(r)}\right],$$

for all $0 \leq r \leq 1$ and all $(x, y) \in U$.

Clearly it also holds that

$$\left[\frac{\partial^2}{\partial x^2} f(x,y)\right]^r = \left[\frac{\partial^2}{\partial x^2}(f(x,y))_-^{(r)}, \frac{\partial^2}{\partial x^2}(f(x,y))_+^{(r)}\right],$$

and

$$\left[\frac{\partial^2}{\partial y^2} f(x,y)\right]^r = \left[\frac{\partial^2}{\partial y^2}(f(x,y))_-^{(r)}, \frac{\partial^2}{\partial y^2}(f(x,y))_+^{(r)}\right],$$

for all $0 \le r \le 1$ and all $(x, y) \in U$. By Lemma 5 we find that

$$\frac{\partial^2}{\partial x^2}(f(x,y))_{\pm}^{(r)}, \frac{\partial^2}{\partial y^2}(f(x,y))_{\pm}^{(r)}, \frac{\partial^2}{\partial x \partial y}(f(x,y))_{\pm}^{(r)}, \frac{\partial^2}{\partial y \partial x}(f(x,y))_{\pm}^{(r)}$$

are all continuous for any $r \in [0, 1]$. But by basic real analysis, Theorem 6-20, p. 121 of [4] we have

$$\frac{\partial^2}{\partial x \partial y}(f(x,y))_{\pm}^{(r)} = \frac{\partial^2}{\partial y \partial x}(f(x,y))_{\pm}^{(r)},$$

for any $r \in [0, 1]$. Thus we get

$$\left[\frac{\partial^2}{\partial x \partial y} f(x,y)\right]^r = \left[\frac{\partial^2}{\partial y \partial x} f(x,y)\right]^r,$$

for all $0 \le r \le 1$. That is the H-fuzzy partial derivatives are equal, $\frac{\partial^2}{\partial x \partial y} f(x, y) = \frac{\partial^2 f(x,y)}{\partial y \partial x}$ for all $(x, y) \in U$. ■

Finally it follows the multivariate Fuzzy Taylor's formula.

Theorem 5. *Let U be an open convex subset of $\mathbb{R}^n$, $n \in \mathbb{N}$ and $f: U \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy continuous function. Assume that all H-fuzzy partial derivatives of f up to order $m \in \mathbb{N}$ exist and are fuzzy continuous. Let $z := (z_1, \dots, z_n)$, $x_0 := (x_{01}, \dots, x_{0n}) \in U$ such that $z_i \ge x_{0i}$, $i = 1, \dots, n$. Let $0 \le t \le 1$, we define $x_i := x_{0i} + t(z_i - x_{0i})$, $i = 1, 2, \dots, n$ and $g_z(t) := f(x_0 + t(z - x_0))$. (Clearly $x_0 + t(z - x_0) \in U$.) Then for $N = 1, \dots, m$ we obtain*

$$g_z^{(N)}(t) = \left[\left(\sum_{i=1}^{n}{}^{*} (z_i - x_{0i}) \odot \frac{\partial}{\partial x_i}\right)^N f\right](x_1, x_2, \dots, x_n). \tag{12}$$

Furthermore it holds the following fuzzy multivariate Taylor formula

$$f(z) = f(x_0) \oplus \sum_{N=1}^{m-1}{}^{*} \frac{g_z^{(N)}(0)}{N!} \oplus \mathcal{R}_m(0, 1), \tag{13}$$

where

$$\mathcal{R}_m(0,1) := (FR)\int_0^1 \left(\int_0^{s_1} \cdots \left(\int_0^{s_{m-1}} g_z^{(m)}(s_m)ds_m\right) ds_{m-1}\right) \cdots \Bigg) ds_1. \tag{14}$$

N o t e . (Explaining formula (12)). When $N = n = 2$ we have ($z_i \geq x_{0i}$, $i = 1, 2$)

$$g_z(t) = f(x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})), \quad 0 \leq t \leq 1.$$

We apply Theorems 3 and 4 repeatedly, etc. Thus we have

$$g_z'(t) = (z_1 - x_{01}) \odot \frac{\partial f}{\partial x_1}(x_1, x_2) \oplus (z_2 - x_{02}) \odot \frac{\partial f}{\partial x_2}(x_1, x_2).$$

Furthermore it holds

$$\text{(15)} \qquad g_z''(t) = (z_1 - x_{01})^2 \odot \frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) \oplus 2(z_1 - x_{01}) \cdot (z_2 - x_{02}) \odot \frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \oplus (z_2 - x_{02})^2 \odot \frac{\partial^2 f}{\partial x_2^2}(x_1, x_2).$$

When $n = 2$ and $N = 3$ we get

$$\text{(16)} \qquad g_z'''(t) = (z_1 - x_{01})^3 \odot \frac{\partial^3 f}{\partial x_1^3}(x_1, x_2) \oplus 3(z_1 - x_{01})^2 (z_2 - x_{02}) \odot \frac{\partial^3 f(x_1, x_2)}{\partial x_1^2 \partial x_2} \oplus 3(z_1 - x_{01})(z_2 - x_{02})^2 \cdot \frac{\partial^3 f(x_1, x_2)}{\partial x_1 \partial x_2^2} \oplus (z_2 - x_{02})^3 \odot \frac{\partial^3 f}{\partial x_2^3}(x_1, x_2).$$

When $n = 3$ and $N = 2$ we obtain ($z_i \geq x_{0i}$, $i = 1, 2, 3$)

$$\text{(17)} \; g_z''(t) = (z_1 - x_{01})^2 \odot \frac{\partial^2 f}{\partial x_1^2}(x_1, x_2, x_3) \oplus (z_2 - x_{02})^2 \odot \frac{\partial^2 f}{\partial x_2^2}(x_1, x_2, x_3) \oplus (z_3 - x_{03})^2 \odot \frac{\partial^2 f}{\partial x_3^2}(x_1, x_2, x_3) \oplus 2(z_1 - x_{01})(z_2 - x_{02}) \odot \frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_1 \partial x_2} \oplus 2(z_2 - x_{02})(z_3 - x_{03}) \odot \frac{\partial^2 f(x_1, x_2, x_3)}{\partial x_2 \partial x_3} \oplus 2(z_3 - x_{03})(z_1 - x_{01}) \odot \frac{\partial^2 f}{\partial x_3 \partial x_1}(x_1, x_2, x_3),$$

etc.

P r o o f o f T h e o r e m 5. Let $z := (z_1, \ldots, z_n)$, $x_0 := (x_{01}, \ldots, x_{0n}) \in U$, $n \in \mathbb{N}$, such that $z_i > x_{0i}$, $i = 1, 2, \ldots, n$. We define

$$x_i := \phi_i(t) := x_{0i} + t(z_i - x_{0i}), \quad 0 \leq t \leq 1; \quad i = 1, 2, \ldots, n.$$

Thus $\frac{dx_i}{dt} = z_i - x_{0i} > 0$. Consider

$$Z := g_z(t) := f(x_0 + t(z - x_0)) = f(x_{01} + t(z_1 - x_{01}), \dots, x_{0n} + t(z_n - x_{0n})) \\ = f(\phi_1(t), \dots, \phi_n(t)).$$

Since by assumptions $f\colon U \to \mathbb{R}_{\mathcal{F}}$ is fuzzy continuous, also f_{x_i} exist and are fuzzy continuous, by Theorem 3 (11) we get

$$\frac{dZ(x_1, \dots, x_n)}{dt} = \sum_{i=1}^{n}{}^{*} \frac{\partial Z(x_1, \dots, x_n)}{\partial x_i} \odot \frac{dx_i}{dt} \\ = \sum_{i=1}^{n}{}^{*} \frac{\partial f(x_1, \dots, x_n)}{\partial x_i} \odot (z_i - x_{0i}).$$

I.e.,

$$g_z'(t) = \sum_{i=1}^{n}{}^{*} \frac{\partial f(x_1, \dots, x_m)}{\partial x_i} \odot (z_i - x_{0i}).$$

Next we see

$$\frac{d^2 Z}{dt^2} = g_z''(t) = \frac{d}{dt}\left(\sum_{i=1}^{n}{}^{*} \frac{\partial f(x_1, \dots, x_n)}{\partial x_i} \odot (z_i - x_{0i})\right) \\ = \sum_{i=1}^{n}{}^{*} (z_i - x_{0i}) \odot \frac{d}{dt}\left(\frac{\partial f(x_1, \dots, x_n)}{\partial x_i}\right) \\ = \sum_{i=1}^{n}{}^{*} (z_i - x_{0i}) \odot \left[\sum_{j=1}^{n}{}^{*} \frac{\partial^2 f(x_1, \dots, x_n)}{\partial x_j \partial x_i} \odot (z_j - x_{0j})\right] \\ = \sum_{i=1}^{n}{}^{*} \sum_{j=1}^{n}{}^{*} \frac{\partial^2 f(x_1, \dots, x_n)}{\partial x_j \partial x_i} \odot (z_i - x_{0i}) \cdot (z_j - x_{0j}).$$

That is

$$g_z''(t) = \sum_{i=1}^{n}{}^{*} \sum_{j=1}^{n}{}^{*} \frac{\partial^2 f(x_1, \dots, x_m)}{\partial x_j \partial x_i} \odot (z_i - x_{0i}) \cdot (z_j - x_{0j}).$$

The last is true by Theorem 3 (11) under the additional assumptions that f_{x_i}; $\frac{\partial^2 f}{\partial x_j \partial x_i}$, $i, j = 1, 2, \dots, n$ exist and are fuzzy continuous.

Working similarly, we find

$$\frac{d^3Z}{dt^3} = g_z'''(t) = \frac{d}{dt}\left(\sum_{i=1}^{n}{}^{*}\sum_{j=1}^{n}{}^{*}\frac{\partial^2 f(x_1,\dots,x_n)}{\partial x_j \partial x_i} \odot (z_i - x_{0i})\cdot(z_j - x_{0j})\right)$$

$$= \sum_{i=1}^{n}{}^{*}\sum_{j=1}^{n}{}^{*}(z_i - x_{0i})\cdot(z_j - x_{0j})\frac{d}{dt}\left(\frac{\partial^2 f(x_1,\dots,x_n)}{\partial x_j \partial x_i}\right)$$

$$= \sum_{i=1}^{n}{}^{*}\sum_{j=1}^{n}{}^{*}(z_i - x_{0i})\cdot(z_j - x_{0j})\left[\sum_{k=1}^{n}{}^{*}\frac{\partial^3 f(x_1,\dots,x_n)}{\partial x_k \partial x_j \partial x_i} \odot (z_k - x_{0k})\right]$$

$$= \sum_{i=1}^{n}{}^{*}\sum_{j=1}^{n}{}^{*}\sum_{k=1}^{n}{}^{*}\frac{\partial^3 f(x_1,\dots,x_n)}{\partial x_k \partial x_j \partial x_i} \odot (z_i - x_{0i})\cdot(z_j - x_{0j})\cdot(z_k - x_{0k}).$$

That is,

$$g_z'''(t) = \sum_{i=1}^{n}{}^{*}\sum_{j=1}^{n}{}^{*}\sum_{k=1}^{n}{}^{*}\frac{\partial^3 f(x_1,\dots,x_n)}{\partial x_k \partial x_j \partial x_i} \odot (z_i - x_{0i})\cdot(z_j - x_{0j})\cdot(z_k - x_{0k}).$$

That last is true by Theorem 3 (11) under the additional assumptions that

$$\frac{\partial^3 f(x_1,\dots,x_n)}{\partial x_k \partial x_j \partial x_i}, \quad i,j,k = 1,\dots,n$$

do exist and are fuzzy continuous, etc. In general, one obtains that for $N = 1,\dots,m \in \mathbb{N}$,

$$g_z^{(N)}(t) = \sum_{i_1=1}^{n}{}^{*}\sum_{i_2=1}^{n}{}^{*}\cdots\sum_{i_N=1}^{n}{}^{*}\frac{\partial^N f(x_1,\dots,x_n)}{\partial x_{i_N}\partial x_{i_{N-1}}\cdots\partial x_{i_1}} \odot \prod_{r=1}^{N}(z_{i_r} - x_{0i_r}),$$

which by Theorem 4 is the same as (12) for the case $z_i > x_{0i}$, see also (15), (16), and (17). The last is true by Theorem 3 (11) under the assumptions that all H-partial derivatives of f up to order m exist and they are all fuzzy continuous including f itself.

Next let $t_{\tilde{m}} \to \tilde{t}$, as $\tilde{m} \to +\infty$, $t_{\tilde{m}}$, $\tilde{t} \in [0,1]$. Consider

$$x_{i\tilde{m}} := x_{0i} + t_{\tilde{m}}(z_i - x_{0i})$$

and

$$\tilde{x}_i := x_{0i} + \tilde{t}(z_i - x_{0i}), \quad i = 1,2,\dots,n.$$

That is

$$x_{\tilde{m}} = (x_{1\tilde{m}}, x_{2\tilde{m}}, \dots, x_{n\tilde{m}}) \text{ and } \tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n) \text{ in } U.$$

Then $x_{\tilde{m}} \to \tilde{x}$, as $\tilde{m} \to +\infty$. Clearly using the properties of D-metric and under the theorem's assumptions, we obtain that

$$g_z^{(N)}(t) \text{ is fuzzy continuous for } N = 0, 1, \dots, m.$$

Then by Theorem 1 [1], from the univariate fuzzy Taylor formula, we obtain

$$g_z(1) = g_z(0) \oplus g_z'(0) \oplus \frac{g_z''(0)}{2!} \oplus \cdots \oplus \frac{g_z^{(m-1)}(0)}{(m-1)!} \oplus \mathcal{R}_m(0,1),$$

where

$$\mathcal{R}_m(0,1) := (FR)\int_0^1 \left(\int_0^{s_1} \cdots \left(\int_0^{s_{m-1}} g_z^{(m)}(s_m) ds_m \right) ds_{m-1} \right) \cdots \Bigg) ds_1.$$

By Lemma 4, [1] and Corollary 13.2, p. 644, [5], the remainder $\mathcal{R}_m(0,1)$ exist in $\mathbb{R}_{\mathcal{F}}$. I.e., we get the multivariate fuzzy Taylor formula

$$f(z) = f(x_0) \oplus g_z'(0) \oplus \frac{g_z''(0)}{2!} \oplus \cdots \oplus \frac{g_z^{(m-1)}(0)}{(m-1)!} \oplus \mathcal{R}_m(0,1),$$

when $z_i > x_{0i}$, $i = 1, 2, \dots, n$.

Finally, we would like to take care of the case that some $x_{0i} = z_i$. Without loss of generality we may assume that $x_{01} = z_1$, and $z_i > x_{0i}$, $i = 2, \dots, n$. In this case we define

$$\tilde{Z} := \tilde{g}_z(t) := f(x_{01}, x_{02} + t(z_2 - x_{02}), \dots, x_{0n} + t(z_n - x_{0n})).$$

Therefore one has

$$\tilde{g}_z'(t) = \sum_{i=2}^{n}{}^{*} \frac{\partial f(x_{01}, x_2, \dots, x_n)}{\partial x_i} \odot (z_i - x_{0i}),$$

and in general we find

$$\tilde{g}_z^{(N)}(t) = \sum_{i_2=2,\dots,i_N=2}^{n}{}^{*} \frac{\partial^N f(x_{01}, x_2, \dots, x_n)}{\partial x_{i_N} \partial x_{N-1} \cdots \partial x_{i_2}} \odot \prod_{r=2}^{N} (z_{i_r} - x_{0i_r}),$$

for $N = 1, \dots, m \in \mathbb{N}$. Notice that all $\tilde{g}_z^{(N)}$, $N = 0, 1, \dots, m$ are fuzzy continuous and

$$\tilde{g}_z(0) = f(x_{01}, x_{02}, \dots, x_{0n}), \quad \tilde{g}_z(1) = f(x_{01}, z_2, z_3, \dots, z_n).$$

Then one can write down a fuzzy Taylor formula, as above, for $\tilde{g}_z$. But $\tilde{g}_z^{(N)}(t)$ coincides with $g_z^{(N)}(t)$ formula at $z_1 = x_{01} = x_1$. That is both Taylor formulae in that case coincide.

At last we remark that if $z = x_0$, then we define $Z^* := g_z^*(t) := f(x_0) =: c \in \mathbb{R}_{\mathcal{F}}$ a constant. Since $c = c + \tilde{o}$, tha tis $c - c = \tilde{o}$, we obtain the H-fuzzy derivative $(c)' = \tilde{o}$. Consequently we have that

$$g_z^{*(N)}(t) = \tilde{o}, \quad N = 1, \ldots, m.$$

The last coincide with the $g_z^{(N)}$ formula, established earlier, if we apply there $z = x_0$. And, of course, the fuzzy Taylor formula now can be applied trivially for g_z^*. Furthermore in that case it coincides with the Taylor formula proved earlier for g_z. We have established a multivariate fuzzy Taylor formula for the case of $z_i \geq x_{0i}$, $i = 1, 2, \ldots, n$. That is (12)–(14) are true. ∎

At last we give the following useful

Corollary 1. *Let U be an open convex subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, and $f\colon U \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy continuous function. Assume that all the first H-fuzzy partial derivatives f_{x_i} of f exist and are fuzzy continuous. Let $z := (z_1, \ldots, z_n)$, $x_0 := (x_{01}, \ldots, x_{0n}) \in U$ such that $z_i \geq x_{0i}$, $i = 1, \ldots, n$. Let $0 \leq t \leq 1$, we define $x_i := x_{0i} + t(z_i - x_{0i})$, $i = 1, 2, \ldots, n$ and $g_z(t) := f(x_0 + t(z - x_0))$. Then*

$$g_z'(t) = \sum_{i=1}^{n}{}^* \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} \odot (z_i - x_{0i}). \tag{18}$$

Furthermore it holds

$$\begin{aligned} f(z) &= f(x_0) \oplus (FR) \int_0^1 g_z'(s)ds \\ &= f(x_0) \oplus \sum_{i=1}^{n}{}^* (z_i - x_{0i}) \odot (FR) \int_0^1 \frac{\partial f(x_1(s), \ldots, x_n(s))}{\partial x_i} ds. \end{aligned} \tag{19}$$

P r o o f. By Theorem 5, case of $m = 1$. The second part of (19) is valid by Theorem 2.6 of [9]. Here $x_i(s) = x_{0i} + s(z_i - x_{0i})$, $s \in [0, 1]$, $i = 1, \ldots, n$ with $z_i \geq x_{0i}$. ∎

C o m m e n t . Theorem 5 and Corollary 1 are still valid when U is a compact convex subset of $\mathbb{R}^n$ such that $U \subseteq W$, where W is an open subset of $\mathbb{R}^n$. Now $f\colon W \to \mathbb{R}_{\mathcal{F}}$ and it has all the properties of f as in Theorem 5 and Corollary 1. Clearly here $x_0, z \in U$.

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