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Comparison of a Pair of Upper Bounds for a Ratio of Gamma Functions [1]

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To Academician Blagovest Sendov, on the occasion of his 70th anniversary, for his outstanding scientific achievements, and for his decades-long contributions to the struggle for peace and justice.

For the ratio $\Gamma(x+1)/\Gamma(x+s)$, $0 < s < 1$, J. D. Kečkič and P. M. Vasič [Publ. Inst. Math. (Beograd), (N.S.) **11** (1971)] have established the upper bound $(x+1)^{x+1/2}/(x+s)^{x+s-1/2}$, $x > \frac{1}{2}$. For the same ratio, D. Kershaw [Math. Comp. **41** (1983)] has proved that $\exp[(1-s)\psi(x+(s+1)/2)]$ is an upper bound. He noted further that numerical evidence supports the view that the latter bound is closer than the one due to Kečkič and Vasič when $x > .79$. Here this is proved and some supplementary results furnished.

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1. Introduction

D. Kershaw [7] has found an upper bound for the ratio $\Gamma(x+1)/\Gamma(x+s)$, $0 < s < 1$ as had, e.g., J. D. Kečkič and P. M. Vasič [6] for $x > \frac{1}{2}$. He pointed out that numerical calculations suggested that his upper bound is smaller than theirs for $x > 0.79$ and noted the lack of a rigorous proof of this relationship.

The main purpose here is to provide such a proof for $x > 0.789548397$, thereby justifying Kershaw's conjecture.

Specifically, this requires proving, for $x > 0.789548397$ and $0 < s < 1$, that

$$(1) \qquad \frac{(x+1)^{x+1/2}}{(x+s)^{x+s-1/2}}\, e^{-(1-s)} - \exp[(1-s)\psi(x+\frac{s+1}{2})] > 0,$$

where

$$\psi(\tau) = \Gamma'(\tau)/\Gamma(\tau).$$

This inequality, as Kershaw demonstrates, is a consequence of the inequality, for $x > 1.289548397$,

$$(2) \qquad f(x) := \int_{x-1/2}^{x+1/2} \left[\ln(t) - \frac{1}{2t}\right] dt - \psi(x) > 0,$$

which can be written equivalently as

$$(3) \qquad f(x) = x \ln\left(1 + \frac{1}{x - 1/2}\right) + \ln\left(x - \frac{1}{2}\right) - 1 - \psi(x) > 0.$$

It is in this form that Kershaw's conjecture will be established here.

Thanks to *Maple 6*, a graph of $f(x)$ both supports his belief and illustrates the situation in greater detail (Figure 1). It crosses the x-axis at $x = x_0 = 1.289548397$, reaches its maximum at $x = x_1 = 1.8791378055$, with $f(x_1) = 0.0043831827$, and has a point of inflection at $x = x_2 = 2.470402669$ with $f(x_2) = 0.0036994893$.

These numbers are all courtesy of *Maple 6*, but could be obtained, so to speak, by hand, as could all the values occurring below. When the specific values are not required, x' will be used to designate the location of $f'(x) = 0$ and x'' that of $f''(x) = 0$, rather than x_1 and x_2.

Somewhat more than (2) or (3) will be established here: Figure 1 (the graph of $f(x)$) will be justified analytically by showing that $f(x)$ is monotonic where indicated, convex or concave where so depicted, that it possesses precisely one zero, one maximum and one point of inflection, that $f(x) \downarrow 0$, $x' < x < \infty$, and that $f'(x) \uparrow 0$, $x'' < x < \infty$.

2. The interval $2.498\ldots < x < \infty$

The main point is to show that $f(x)$ decreases to zero when $x' < x < \infty$. This is a consequence of $f'(x)$ increasing to zero (and so being negative) in that interval. Once this is established, supplementary arguments, similar in character, will verify that $f(x)$ vanishes precisely once (at $x = x_0$) in $\frac{1}{2} < x \leq x''$ so that Kershaw's upper bound for $\Gamma(x+1)/\Gamma(x+s)$ indeed improves that of Kečkič and Vasič in the interval claimed.

Various monotonicity properties lead to this goal. They are, in turn, consequences of the following family of inequalities established by H. Alzer [2, Theorem 9]: .

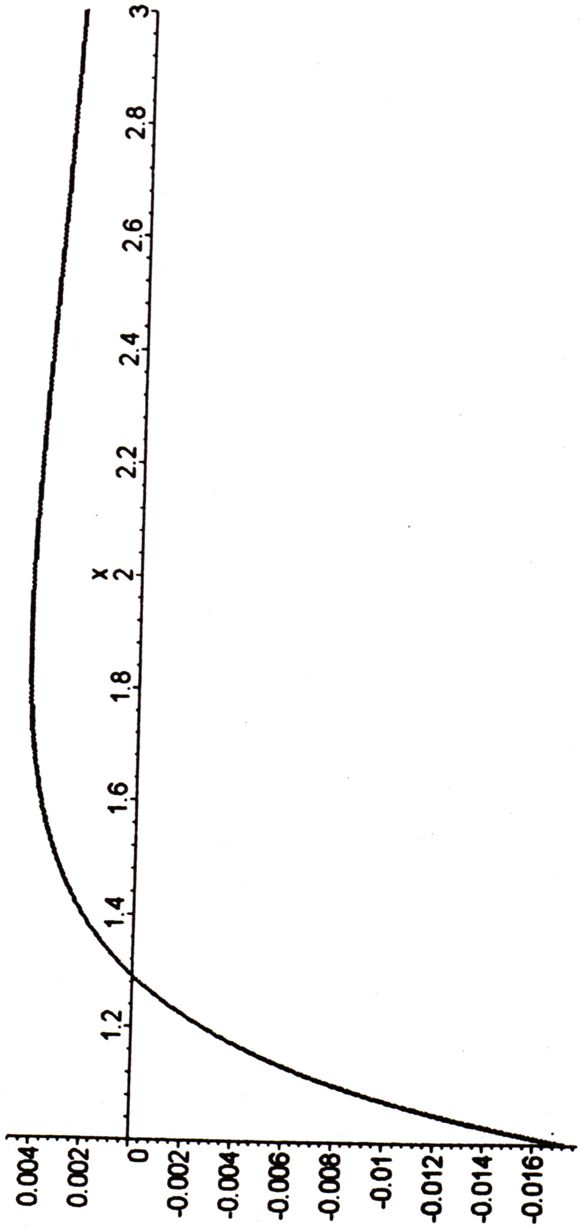

Figure 1: Graph of $f(x)$

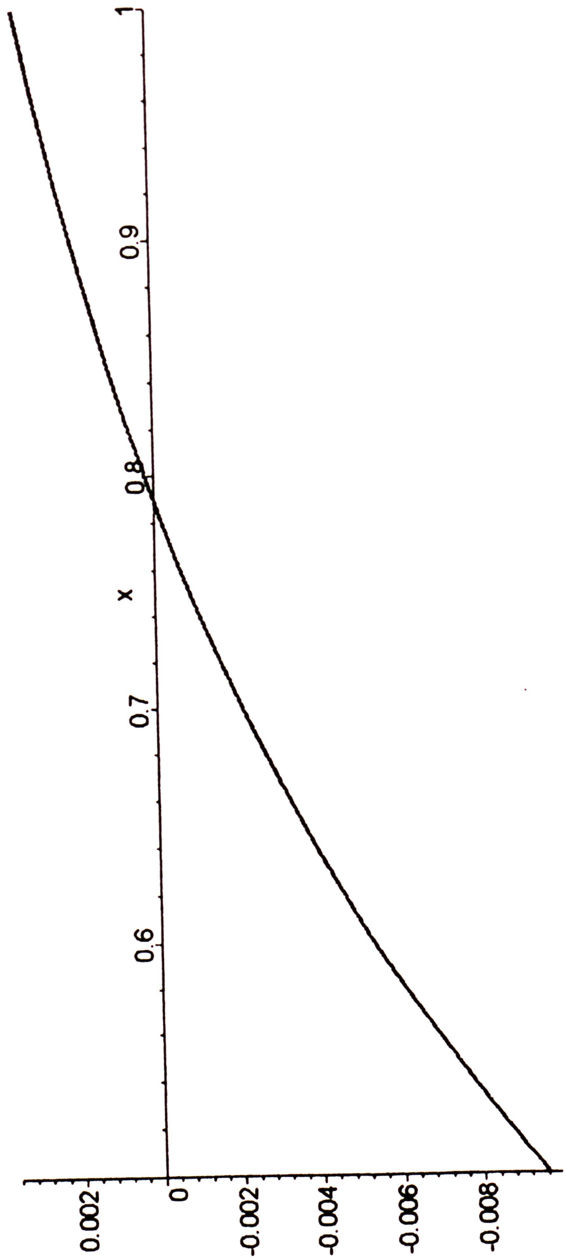

Figure 2: Graph of left-hand side of (1) for $s = 0$

Theorem A. *For* $k = 1, 2, \ldots, n = 0, 1, 2, \ldots$ *and all* $x > 0$,

$$S_k(2n; x) < (-1)^{k+1}\psi^{(k)}(x) < S_k(2n+1; x), \tag{4}$$

where

$$S_k(m; x) = \frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{m} B_{2i}\frac{(2i+k-1)!}{(2i)!\,x^{2i+k}},$$

and B_{2i} *is the* $2i$*th Bernoulli number.*

The reader will recognize $S_k(m; x)$ as the m-th partial sum of the asymptotic expansion [1, p. 260] of $(-1)^{k+1}\psi^{(k)}(x)$. Bearing in mind that the sequence $\{B_{2i}\}$ is alternating, with $B_{4i} < 0$, $B_{4i+2} > 0$, $i = 1, 2, \ldots$, the inequalities (4) tell us that the partial sums (beginning with the third term) approximate $(-1)^{k+1}\psi^{(k)}(x)$ to within an error whose sign is the same as that of the first term neglected and of lesser magnitude.

An alternative path to (4), via the asymptotic series for $\psi^{(k)}(x), k = 1, 2, \ldots$, is available on using [8, §64 or §65].

For the more general ratio $\Gamma(x+a)/\Gamma(x+b)$, C.L. Frenzen [4] has determined rigorously such error estimates.

Our use of (4) is restricted to $k = 1, 2, 3$. The first step is to prove

$$f''(x) > 0, \; 2.498069640 < x < \infty. \tag{5}$$

Putting $k = 2, n = 1$ in (4), we find that

$$f''(x) > \frac{1}{x+\frac{1}{2}} + \frac{1}{2(x+\frac{1}{2})^2} - \frac{1}{x-\frac{1}{2}} - \frac{1}{2(x-\frac{1}{2})^2} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{2x^4} - \frac{1}{6x^5}$$

and so

$$f_2(x) := 96x^6\left(x^2 - \frac{1}{4}\right)^2 f''(x) > 24x^6 - 48x^5 - 34x^4 + 6x^3 + 11x^2 - 1.$$

This, eventually positive, polynomial, has three variations in sign, Hence it cannot have more than three real zeros (in fact at most two, taking into account that its complex zeros must occur in pairs). The larger such zero is the lower bound of the interval specified in (5).

Thus (5) is verified.

This done, it follows that $f'(x)$ is an increasing function in that interval. It is clear that the derivative of the first (integral) term defining $f(x)$ as in (2) approaches zero as $x \to \infty$. It is equally clear [1, p. 260, 6.4.1] that $\psi'(x) \to 0$ as $x \to \infty$.

Hence $f'(x) \to 0$ as $x \to \infty$ and so

(6) $$f'(x) < 0, \quad 2.498069640 < x < \infty.$$

Therefore $f(x)$ is a decreasing function in the same interval. Its limit, as shown immediately below, as $x \to \infty$ is zero and so $f(x)$ is positive, as asserted earlier, in that interval.

Applying [1, p. 259, 6.3.21] to (3),

$$f(x) = x \ln\left(1 + \frac{1}{x - 1/2}\right) + \frac{1}{2}\ln\left(x^2 - \frac{1}{4}\right) - 1 - \ln x + o(1) = o(1), \text{ as } x \to \infty,$$

and so

(7) $$f(x) > 0, \quad 2.498069640 < x < \infty.$$

3. The interval $\frac{1}{2} < x \leq 2.498\ldots$

In sum, it has been established that $f(x)$ is concave, positive and decreasing to zero in $2.498069640 < x < \infty$. It is time to consider the remaining interval $\frac{1}{2} < x \leq 2.490869640$.

Clearly, $f(x)$ is positive near the upper end of that interval. Furthermore, $f(\frac{1}{2}+) = -\infty$, so that $f(x)$ vanishes somewhere earlier in that interval. Indeed, $f(x)$ vanishes somewhere in $1 < x < 2.490869640$, since $f(1) = \ln(1.5) - 1 + C = -0.017119226992$, where C is the Euler-Mascheroni constant. Moreover, $f(x)$ has a positive maximum. To reach objective (2), we begin with $f'''(x)$ whose sign will govern the behaviour of $f''(x)$, thence of $f'(x)$ and finally of $f(x)$. These calculations will reveal the remaining asserted properties, namely that $f(x)$ vanishes precisely once, has exactly one maximum and one point of inflection. Thus, $f(x)$ will be seen to possess the monotonicities, the convexity, concavity and positivity properties suggested by its graph.

With $k = 3, n = 0$, the lower inequality (4) becomes

$$f'''(x) > -\frac{1}{(x+\frac{1}{2})^2} - \frac{1}{(x+\frac{1}{2})^3} + \frac{1}{(x-\frac{1}{2})^2} + \frac{1}{(x-\frac{1}{2})^3} - \left(\frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5}\right) := f_3(x).$$

With $F_3(x) = 64x^5(x^2 - 1/4)^3 f_3(x)$, we have

$$F_3(x) = -64x^6 + 160x^5 + 72x^4 - 36x^3 - 22x^2 + 3x + 2.$$

This eventually negative polynomial has exactly two real zeros, namely -0.3381847866 and 2.814231782, and so

(8) $$f'''(x) > 0, \quad \frac{1}{2} < x \leq 2.814231782.$$

Now we note that $f'(x)$ has a relative maximum for an x' in $1.75 < x < 2$. Standard tables [1, p.270] show that

$$f'(1.75) = 0.00146257 > 0, \quad f'(2) = -0.000751049 < 0.$$

Such an x' falls in the interval for which $f'''(x) > 0$, hence in which $f''(x)$ increases. But $f''(x') < 0$ and so $f''(x) < 0$, $\frac{1}{2} < x \le x'$.

Accordingly, $f'(x) \downarrow f'(x') = 0$, $\frac{1}{2} < x < x'$ and so $f(x)$ increases in $\frac{1}{2} < x < x'$. Hence this x' yields the first maximum of $f(x)$. Moreover, $f(1.75) = 0.004297761$, so that $f(x)$ has one and only one zero in $\frac{1}{2} < x < x'$, since $f(\frac{1}{2}) = -\infty$.

Combining (5) and (8), it is clear that there exists x'', $f''(x'') = 0$ in $\frac{1}{2} < x < 2.814231782$ and that $f''(x) < 0$, $\frac{1}{2} < x < x''$, $f''(x) > 0$, $x > x''$. Thus x'' gives the unique point of inflection in $\frac{1}{2} < x < \infty$.

Hence, $f'(x)$ increases in $x'' < x < \infty$, and so (as argued in connection with (6)), $f'(x) \uparrow 0$ and $f(x) \downarrow 0$ in $x'' < x < \infty$.

Finally, in $x' < x < x''$, $f''(x) < 0$, and so $f'(x)$ decreases there from $f'(x') = 0$, and $f'(x) < 0$. Thus, $f(x)$ decreases to $f(x'')$ in $x' < x \le x''$ and so also in $x' < x < \infty$.

This completes the proof of (2) and of Kershaw's conjecture (1).

4. Remarks

(1) The proof above shows that Kershaw's upper bound in $0 < s < 1$ for $\Gamma(x+1)/\Gamma(x+s)$, is sharper than that of Kečkič and Vasič in $0.7895\ldots < x < \infty$, (although, as Kershaw observed, more cumbersome). They show also that the graph of (1) is justified analytically. The proof permits the identification of x_1 with x' and x_2 with x''.

(2) As Figure 2 illustrates, the Kečkič-Vasič bound is sharper than Kershaw's for sufficiently small positive s. Thus, the interval for which Kershaw foresaw that his bound is closer than theirs uniformly for $0 < s < 1$ is "best possible." Shortening the s-interval at the lower end would lengthen the x-interval in (2).

(3) Alzer provides [2] an extensive bibliography. Relevant contributions subsequent to his are found in [3,5,9,10,11].

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