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On the Growth of Harmonic Polynomials

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This article is dedicated to the 70th anniversary of Acad. Bl. Sendov

In this paper we study the growth of harmonic polynomials. Our results are refinements of certain results of G. Szegö, and W.W. Rogosinski pertaining to the following question. How large can $|\Re f(z)|$ and $|f(z)|$ be on $|z| = R > 1$, if $f(z) := \sum_{\nu=0}^{n} c_\nu z^n$, where $c_\nu \in \mathbb{C}$ for $\nu = 0, \ldots, n$, and $|\Re f(z)| \le 1$ for $|z| = 1$? It was shown by Szegö that $|\Re f(z)|$ cannot be larger than R^n. Although $\max_{|z|=R} |f(z)|$ can be larger than R^n for values of R sufficiently close to 1, *even if* $f(0)$ *is real*, Rogosinski proved that under this additional supposition $|f(z)|$ does not exceed R^n for $|z| = R > 1 + \delta_n$, where $\delta_n \to 0$ as $n \to \infty$. These two results are the main objects of our investigation.

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1. Introduction

For any entire function f, and $r > 0$, let

$$M_f(r) := \max_{|z|=r} |f(z)| \quad \text{and} \quad A_f(r) := \max_{|z|=r} |\Re f(z)| .$$

It is well known (see [8], [12, p. 336], [6]) that if f is a polynomial of degree at most n, then

$$M_f(R) \le M_f(1)\, R^n \qquad (R > 1)\,, \tag{1.1}$$

where equality holds if and only if $f(z)$ is a constant multiple of z^n. This inequality has been generalized and refined in a variety of ways. For example, it is known [3, Theorem 2] that

$$M_f(R) + (R^n - R^{n-2})|f(0)| \le M_f(1)\, R^n \qquad (R > 1)\,, \tag{1.2}$$

where the coefficient of $|f(0)|$ is the best possible one for each R.

It was proved by Szegö [12, p. 336] that if

$$u(r\,,\,\varphi) := a_0 + 2\sum_{\nu=1}^{n} r^{\nu}\left(a_{\nu}\cos\nu\varphi + b_{\nu}\sin\nu\varphi\right)$$

is a harmonic polynomial that is not identically zero and

$$U(r) := \max_{0\le\varphi\le 2\pi} |u(r\,,\,\varphi)|\,,$$

then

$$\frac{U(r_2)}{U(r_1)} \le \left(\frac{r_2}{r_1}\right)^n \qquad (0 < r_1 < r_2)\,, \tag{1.3}$$

where equality holds only for $u(r\,,\,\varphi) \equiv a\,r^n\cos n(\varphi-\alpha)$, $a\in\mathbb{R}$, $\alpha\in\mathbb{R}$, $a\ne 0$. In particular, if f is a polynomial of degree at most n, then

$$A_f(R) < A_f(1)\,R^n \qquad (R > 1) \tag{1.4}$$

unless $f(z)\equiv a\,z^n + \mathrm{i}\beta$, $a\ne 0$, $\beta\in\mathbb{R}$. Subsequently, he showed [13, p. 68] that if f is *any* polynomial of degree at most n, then

$$\left|f\left(R\mathrm{e}^{\mathrm{i}\theta}\right) - f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| \le A_f(1)\,(R^n-1) \qquad (R>1\,,\,\theta\in\mathbb{R})\,, \tag{1.5}$$

which clearly contains (1.4) and also leads to the inequality

$$M_f(R) \le M_f(1) + A_f(1)\,(R^n-1) \qquad (R>1)\,. \tag{1.6}$$

Rogosinski [9] developed a remarkable approach to a fairly large class of extremal problems for polynomials. He, not only, gave another proof of (1.4) but also applied his method to obtain many other results. In particular, he proved [9, pp. 271-272] that if $f(0)$ is real, then

$$M_f(R) \le A_f(1)\,R^n \qquad \left(R \ge \frac{1}{\rho_n}\right)\,, \tag{1.7}$$

where

$$\rho_n := \sup\left\{\rho > 0 \,:\, \Re\left(1 + 2\sum_{\nu=1}^{n-1} z^{\nu} + z^n\right) \ge 0 \quad \text{for} \quad |z|\le\rho\right\}\,. \tag{1.8}$$

It is known (see [10], [7]) that

$$\rho_1 = 1,\ \ \rho_2 = \frac{1}{\sqrt{2}},\ \ \rho_3 \approx 0.7004$$

and that for $n\ge 4$, the number ρ_n is larger than $1-(\log n)/n$.

Remark 1.1. In order to highlight the significance of (1.7) we wish to mention that if f is a polynomial of degree at most n such that $f(0)$ is real and $|\Re f(z)|\le 1$ for $|z|\le 1$, then (see [14]; also see [5])

$$|\Im f(z)| \le \mathcal{M}_n := \frac{2}{n+1} \sum_{\substack{1 \le k \le n \\ k \text{ odd}}} \cot \frac{k\pi}{2(n+1)} \qquad (|z| \le 1)\,,$$

where $\mathcal{M}_n$ cannot be replaced by a smaller number. *The asymptotic value of* $\mathcal{M}_n$ *as* $n \to \infty$ *is* $(2/\pi)\log n$. Thus, $M_f(R)$ may be larger than $A_f(1)R^n$ for values of R sufficiently close to 1.

By Hadamard's three–circles theorem [15, p. 53], $\log M_f(r)$ is a convex function of $\log r$. Hence, for $1 < R < 1/\rho_n$, we have

$$\log M_f(R) \le \log M_f(1) + \frac{\log R}{|\log \rho_n|}\left\{\log M_f(1/\rho_n) - \log M_f(1)\right\}\,.$$

Using (1.7) to estimate $M_f(1/\rho_n)$, we obtain

$$M_f(R) \;\le\; \{M_f(1)\}^{\left(1 - \frac{\log R}{|\log \rho_n|}\right)}\;\{A_f(1)\}^{\left(\frac{\log R}{|\log \rho_n|}\right)}\;(\rho_n)^{\left(n\,\frac{\log R}{\log \rho_n}\right)}\,.$$

Now, note that $(\rho_n)^{n\,(\log R)/\log \rho_n}$ is simply R^n, and so

$$M_f(R) \;\le\; \left\{\frac{M_f(1)}{A_f(1)}\right\}^{\left(1 - \frac{\log R}{|\log \rho_n|}\right)} \cdot\; A_f(1)\,R^n \qquad \left(1 \le R \le \frac{1}{\rho_n}\right). \tag{1.9}$$

Among other things, Rogosinski [9, p. 273] also proved that *if* f *is a polynomial of degree at most* n *such that* $c_0 := f(0)$ *is real and* $|\Re f(z)| \le 1$ *for* $|z| \le 1$, *then for* $n > 1$,

$$|f(z)|^2 \le R^2 - c_0^2 \qquad \left(|z| = R \ge \frac{1}{r_{n-1}}\right), \tag{1.10}$$

where

$$r_k := \sup\left\{\rho > 0 \;:\; \Re\left(1 + 2\sum_{j=1}^{k} z^j\right) \ge 0 \;\text{ for }\; |z| \le \rho\right\}. \tag{1.11}$$

It is known [11, pp. 554-555] that $r_k \ge r_1 = 1/2$ and that $r_k \uparrow 1$ as $k \to \infty$. If k is odd, then r_k is the positive root of $2r^{k+1} + r - 1 = 0$; we do not know any such (simple) characterization of r_k for even k.

In spite of its profoundness, the paper of Rogosinski has not received the attention it deserves. So, we propose to start out by giving a brief account of his theory, which we shall then use to obtain refinements of (1.4) and (1.7), analogous to (1.2). These refinements are contained in Theorems 1.1 and 1.2, respectively. A few observations, independent of Rogosinski's theory, are also included.

Theorem 1.1. *Let f be a polynomial of degree at most n such that $c_0 := f(0)$ is real, and let $A_f(r) := \max_{|z|=r} |\Re f(z)|$. Furthermore, for even $n \in \mathbb{N}$, let*

$$\kappa_n(R) := \frac{(R-1)(R^n-1)}{R+1} = (R-1)^2 \sum_{\nu=0}^{(n-2)/2} R^{2\nu} \qquad (R>1)\,,$$

and for odd $n \in \mathbb{N}$, let

$$\kappa_n(R) := \begin{cases} \dfrac{(R^2-1)(R^n-1)}{R^2+2R\cos(\pi/n)+1} & (1<R\le R_n) \\ \dfrac{(R-1)(R^n+1)}{R+1} & (R \ge R_n)\,, \end{cases}$$

where R_n is the (easily seen) only root of the equation

$$\left(1-\cos\frac{\pi}{n}\right) R^{n+1} = R^2 + \left(1+\cos\frac{\pi}{n}\right) R + 1$$

in $(1\,,\infty)$. Then,

$$A_f(R) + \kappa_n(R)\,|c_0| \le A_f(1)\,R^n \qquad (R>1)\,. \tag{1.12}$$

The coefficient of $|c_0|$ in (1.12) is the best possible one for each R and each n.

Inequality (1.12) bears the same relationship to (1.4) as (1.2) does to (1.1). Note that in the case where n is odd, we may write

$$\kappa_n(R) := \begin{cases} (R+1)(R-1)^2 \displaystyle\prod_{k=1}^{(n-3)/2} \left(R^2 - 2R\cos\frac{2k\pi}{n} + 1\right) & (1<R\le R_n) \\ (R-1) \displaystyle\prod_{k=0}^{(n-3)/2} \left(R^2 - 2R\cos\frac{(2k+1)\pi}{n} + 1\right) & (R\ge R_n)\,. \end{cases}$$

The next result is a refinement of (1.7). Also see how the right-hand side of (1.2) is being replaced by $A_f(1)R^n$, with little change in the coefficient of $|f(0)|$ on its left-hand side, and with only a minor restriction on R.

Theorem 1.2. *Let f be a polynomial of degree at most n such that $f(0)$ is real, and let $A_f(r) := \max_{|z|=r} |\Re f(z)|$ and $M_f(r) := \max_{|z|=r} |f(z)|$. In addition, let R_n^* denote the only root of the equation*

$$R^{n-1} - R^{n-2} - 1 = 0 \tag{1.13}$$

in $(1\,,\infty)$. Then

$$M_f(R) + (R-1)\left(R^{n-1} - R^{n-2} - 1\right)|f(0)| \le A_f(1)\,R^n \qquad (R \ge R_n^*)\,. \tag{1.14}$$

Remark 1.2. It is easily seen that

$$R_2^* = 2, \ \ R_3^* = \frac{1+\sqrt{5}}{2}, \ \ R_4^* \in (1.4655\,,\, 1.4656) \ \text{and} \ R_5^* \in (1.3802\,,\, 1.3803)\,.$$

Note that for $n \geq 6$, we have

$$\begin{aligned}\left\{1+\frac{\log(n-2)}{n-2}\right\}^{n-1} &- \left\{1+\frac{\log(n-2)}{n-2}\right\}^{n-2} - 1 \\ &= \frac{\log(n-2)}{n-2}\left\{1+\frac{\log(n-2)}{n-2}\right\}^{n-2} - 1 \\ &= \frac{\log(n-2)}{n-2}\exp\left\{(n-2)\log\left(1+\frac{\log(n-2)}{n-2}\right)\right\} - 1 \\ &> \frac{\log(n-2)}{n-2}\exp\left\{\log(n-2) - \frac{1}{2}\frac{(\log(n-2))^2}{n-2}\right\} - 1 \\ &= \{\log(n-2)\}\exp\left\{-\frac{1}{2}\frac{(\log(n-2))^2}{n-2}\right\} - 1 > 0\,.\end{aligned}$$

Since $R^{n-1} - R^{n-2} - 1 < 0$ for $R = 1$, this implies that

$$R_n^* < 1 + \frac{\log(n-2)}{n-2} \qquad (n \geq 6)\,.$$

Remark 1.3. Dividing the two sides of (1.14) by R^n and letting R tend to 1, we conclude that *if* $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$ *is a polynomial of degree at most* n *such that* $c_0 = f(0)$ *is real, then*

$$|c_n| + |c_0| \leq A_f(1)\,. \tag{1.15}$$

The example $f(z) := z^n + \mathrm{i}\beta$, $\beta \neq 0$ shows that (1.15) may fail if $f(0)$ is not real. It was proved by Visser [16] that $|c_n|+|c_0| \leq M_f(1)$ for $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$ whether c_0 is real or not.

2. Rogosinski's theory

Let $\mathcal{P}_n$ denote the class of all polynomials of degree at most n, and let $\mathcal{P}_{n,1}$ be the subclass consisting of all polynomials in $\mathcal{P}_n$ which are of the form

$$f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu, \ \ c_\nu = a_\nu - \mathrm{i}b_\nu, \ \ b_0 = 0, \ a_\nu, \ b_\nu \in \mathbb{R} \ \ (\nu = 0, 1, \ldots, n) \tag{2.1}$$

such that

$$|\Re f(z)| \leq 1 \qquad (|z| \leq 1)\,. \tag{2.2}$$

In addition, let $n+1$ numbers $\gamma_0 = \lambda_0 + \mathrm{i}\mu_0, \ldots, \gamma_n = \lambda_n + \mathrm{i}\mu_n$ be given, where $\lambda_\nu = \Re\gamma_\nu$ for $\nu = 0, 1, \ldots, n$, and associate with each f of the form (2.1), the quantity

$$J(f) := \sum_{\nu=0}^{n} \gamma_\nu c_\nu \,.$$

The problem is to maximize $|\Re J(f)|$ when f varies in $\mathcal{P}_{n,1}$.

To start with we note that $\Re f\left(\mathrm{e}^{\mathrm{i}\theta}\right) = a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)$ and that $\Re J(f) = \sum_{\nu=0}^{n} \lambda_\nu a_\nu + \sum_{\nu=1}^{n} \mu_\nu b_\nu$. Hence, setting

$$t(\theta) := a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta) \tag{2.3}$$

and

$$I(t) := \sum_{\nu=0}^{n} \lambda_\nu a_\nu + \sum_{\nu=1}^{n} \mu_\nu b_\nu \,, \tag{2.4}$$

we see that our problem is equivalent to the determination of $\max_{t \in \mathcal{T}_{n,1}} |I(t)|$, where $\mathcal{T}_{n,1}$ is the class of all real trigonometric polynomials t of degree at most n such that $|t(\theta)| \le 1$ for $\theta \in [0\,,\,2\pi)$.

Let $\mathcal{T}_n$ denote the real normed linear space of all real trigonometric polynomials t of the form (2.3) with $\|t\|_{[0,2\pi]} := \max_{0 \le \theta \le 2\pi} |t(\theta)|$. The transformation

$$I: \quad t \mapsto I(t) := \sum_{\nu=0}^{n} \lambda_\nu a_\nu + \sum_{\nu=1}^{n} \mu_\nu b_\nu$$

defines a bounded linear functional on $\mathcal{T}_n$. As observed by Rogosinski [9], there exists a function $\mu(\theta)$ of bounded variation in $[0\,,\,2\pi]$ such that

$$I(t) = \int_0^{2\pi} t(\theta)\,\mathrm{d}\mu(\theta), \quad \text{where} \quad \int_0^{2\pi} |\mathrm{d}\mu(\theta)| = \sup_{t \in \mathcal{T}_{n,1}} |I(t)| =: \|I\| \,. \tag{2.5}$$

This representation is obtained with the help of F. Riesz' theorem [17, p. 105–107] about the form of a bounded linear functional on the space of functions continuous in $[0\,,\,2\pi]$ and the Hahn–Banach extension theorem [1, p. 27-29].

The function $\mu(\theta)$ can be normalized so that

$$\mu(0) = 0,\ \mu(2\pi) = \mu(2\pi - 0) \text{ and } \mu(\theta) = \frac{1}{2}\{\mu(\theta + 0) + \mu(\theta - 0)\} \text{ for } 0 < \theta < 2\pi \,.$$

Rogosinski calls the function $\mu(\theta)$, so normalized, an *extremal kernel* for I.

There exists at least one $T \in \mathcal{T}_{n,1}$ with

$$I(T) = \|I\| \,. \tag{2.6}$$

Any such T is said to be an *extremal trigonometric polynomial* for I.

The following result of Rogosinski [9, Theorem 1] contains complete information about extremal kernels and extremal trigonometric polynomials.

Theorem A. (i) *If* $T \equiv 1$ *is an extremal trigonometric polynomial for* I*, then the extremal kernel increases. Conversely, an increasing* $\mu(\theta)$ *is extremal for the* I *of* (2.5)*, and* $T \equiv 1$ *is an associated extremal trigonometric polynomial. A similar correspondence holds between* $T \equiv -1$ *and a decreasing* $\mu(\theta)$*.*

(ii) *If* T *is neither* $\equiv 1$ *nor* $\equiv -1$*, and if it is an extremal trigonometric polynomial for* I*, then the associated extremal kernel* $\mu(\theta)$ *is uniquely determined as a non-constant step function having at most* n *points* α_j *of positive jump and at most* n *points of negative jump, where* $0 \le \alpha_j,\ \beta_j < 2\pi$ *and*

$$T(\alpha_j) = 1, \quad T(\beta_j) = -1\,. \tag{2.7}$$

Conversely, if $\mu(\theta)$ *is such a step function, then it is extremal for the* I *of* (2.5) *if, and only if, there exist trigonometric polynomials* $T \in \mathcal{T}_{n,1}$ *satisfying* (2.7)*. The extremal trigonometric polynomials for* I *are then exactly these* T*.*

Remark 2.1. If $T \equiv 1$ or $T \equiv -1$ is extremal for the functional I of (2.5), then the extremal kernel μ need not be unique.

Theorem A is of little avail if we wish to determine the norm of a given functional I. Nevertheless, it completely determines the class of all I that have a given trigonometric polynomial T as an extremal. There are many interesting functionals for which the trigonometric polynomial $T(\theta) := \cos n\theta$ is extremal. The following result (see [9, Theorem 3] or [7, Theorem 12.3.9]) contains a useful characterization of such functionals.

Theorem B. *In order for* $\cos n\theta$ *to be extremal for the* $I(\not\equiv 0)$ *of* (2.4)*, it is necessary and sufficient that the* $\gamma_k = \lambda_k + \mathrm{i}\mu_k$*, where* $\mu_0 = 0$*, should satisfy* $\mu_n = 0$ *and*

$$\lambda_n + 2\Re \sum_{\nu=1}^{n-1} \gamma_{n-\nu} \left(\mathrm{e}^{\mathrm{i}j\pi/n}\right)^{\nu} + (-1)^j \lambda_0 \ge 0 \qquad (j = 0, 1, \ldots, 2n-1)\,. \tag{2.8}$$

Furthermore, if the left-hand side of (2.8) *is positive for each* j*, then* $\cos n\theta$ *is the only extremal trigonometric polynomial for* I*.*

Theorem B can be reformulated as follows (see [9, p. 269] or [7, Theorem 12.3.10]).

Theorem B′. *Let* γ_0 *be real and let* $\{\gamma_1, \ldots, \gamma_n\}$ *be any set of* n *numbers in* $\mathbb{C}$ *such that* $\sum_{\nu=0}^{n} |\gamma_\nu| > 0$*. Then, in order that*

$$\left| \Re \sum_{\nu=0}^{n} \gamma_\nu c_\nu z^\nu \right| \le \Re \gamma_n \qquad (|z| \le 1) \tag{2.9}$$

should hold for every polynomial $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$, *such that*

$$c_0 \in \mathbb{R} \quad \text{and} \quad |\Re f(z)| \le 1 \qquad (|z| \le 1)\,,$$

it is necessary and sufficient that $\gamma_n > 0$, *and that*

$$\Re\left\{\gamma_n + 2\sum_{\nu=1}^{n-1} \gamma_{n-\nu} z_j^\nu + \gamma_0 z_j^n\right\} \ge 0 \tag{2.10}$$

for $z_j = \mathrm{e}^{\mathrm{i}j\pi/n}$, *where* $j = 0, 1, \ldots, 2n-1$. *Furthermore, if the left-hand side of* (2.10) *is positive for all the* z_j, *then equality cannot hold in* (2.9) *unless* $f(z) \equiv z^n$.

Rogosinski carried the argument substantially further. First, he observed that if $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree at most n such that $c_0 \in \mathbb{R}$, and $|\Re f(z)| \le 1$ for $|z| \le 1$, then the inequality

$$\left|\sum_{\nu=0}^{n} \gamma_\nu c_\nu z^\nu\right| \le \Re\gamma_n \qquad (|z| \le 1) \tag{2.11}$$

holds if and only if the inequality

$$\left|\Re\left(\mathrm{e}^{\mathrm{i}\alpha}\sum_{\nu=0}^{n} \gamma_\nu c_\nu z^\nu\right)\right| \le \Re\gamma_n \qquad (|z| \le 1)$$

is satisfied for all real α. Next, he noted that

$$\mathrm{e}^{\mathrm{i}\alpha}\sum_{\nu=0}^{n} \gamma_\nu c_\nu z^\nu = \sum_{\nu=0}^{n} \gamma_\nu(\alpha) c_\nu \left(z\mathrm{e}^{\mathrm{i}\alpha/n}\right)^\nu\,,$$

where $\gamma_\nu(\alpha) = \gamma_\nu\, \mathrm{e}^{\mathrm{i}(n-\nu)\alpha/n}$ for $\nu = 0, 1, \ldots, 2n-1$. Since $\gamma_n(\alpha)$ is equal to γ_n and $\left|\Re f\left(z\mathrm{e}^{\mathrm{i}\alpha/n}\right)\right| \le 1$ for $|z| \le 1$, he could apply Theorem B′ to deduce the following result (see [9, Theorem 2] or [7, Theorem 12.3.12]).

Theorem C. *Let* $\{\gamma_0, \ldots, \gamma_n\}$ *be any set of* $n+1$ *numbers in* $\mathbb{C}$, *which are not all zero. Then, in order that we may have* $\max_{|z|=1} |\sum_{\nu=0}^{n} \gamma_\nu c_\nu z^\nu| \le \Re\gamma_n$ *for every polynomial* $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$, *where* $c_0 \in \mathbb{R}$ *and* $|\Re f(z)| \le 1$ *for* $|z| \le 1$, *it is necessary and sufficient that* $\gamma_n > 0$ *and*

$$\Re\left\{\gamma_n + 2\sum_{\nu=1}^{n-1} \gamma_{n-\nu} z^\nu + \gamma_0 z^n\right\} \ge 0 \qquad (|z| \le 1)\,. \tag{2.12}$$

3. A lemma

For the proof of Theorem 1.2 we need the following auxiliary result due to Fejér ([2, Theorem II]; also see [11, p. 557] and [10, p. 75]).

Lemma 3.1. *If* $\lambda_n \geq 0$, $\lambda_{n-1} - 2\lambda_n \geq 0$ *and* $\lambda_{j-1} - 2\lambda_j + \lambda_{j+1} \geq 0$ *for* $j = 1, \ldots, n-1$, *then*

$$\lambda_0 + 2\sum_{j=1}^{n} \lambda_j \cos j\theta \geq 0 \qquad (\theta \in \mathbb{R}).$$

4. Proofs of the main results

We shall now present the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Throughout this proof, λ is supposed to be real. Note that $|\Re f(z) \pm \lambda c_0| \leq R^n$ for $|z| \leq R^n$ if and only if

$$\left| \Re \left((1 \pm \lambda)c_0 + \sum_{\nu=1}^{n} R^{\nu} c_{\nu} z^{\nu} \right) \right| \leq R^n \qquad (|z| \leq 1),$$

that is, if and only if

$$\left| \Re \left((1 \pm \lambda)R^{-n}c_0 + \sum_{\nu=1}^{n} R^{-n+\nu} c_{\nu} z^{\nu} \right) \right| \leq 1 \qquad (|z| \leq 1).$$

Hence, applying Theorem B$'$, with $\gamma_0 = (1 \pm \lambda)R^{-n}$ and $\gamma_\nu = R^{-(n-\nu)}$ for $\nu = 1, \cdots, n$, we see that the desired inequality holds if and only if

$$\Re \left\{ 1 + 2\sum_{\nu=1}^{n-1} \left(R^{-1} \mathrm{e}^{\mathrm{i}j\pi/n} \right)^{\nu} + (1 \pm \lambda) \left(R^{-1} \mathrm{e}^{\mathrm{i}j\pi/n} \right)^{n} \right\} \geq 0$$

for $j = 0, 1, \ldots, 2n-1$. In order to see if this is true or not we find it more convenient to consider the function

$$\varphi(z) := |1 - z|^2 \left(1 + 2\sum_{\nu=1}^{n-1} z^{\nu} + (1 \pm \lambda) z^n \right),$$

and examine the sign of $\Re\varphi\left(R^{-1}\mathrm{e}^{\mathrm{i}j\pi/n}\right)$ for $j = 0, 1, \ldots, 2n-1$. Since

$$\begin{aligned} \varphi(z) &= |1 - z|^2 \left(2\sum_{\nu=0}^{n} z^{\nu} - (1 + z^n) \pm \lambda z^n \right) \\ &= 2\left(1 - \overline{z}\right)\left(1 - z^{n+1}\right) - |1 - z|^2 \left(1 + z^n \mp \lambda z^n\right), \end{aligned}$$

we have

$$\begin{aligned}
\Re\varphi\left(R^{-1}e^{i\vartheta}\right) &= \Re\left\{2\left(1-R^{-1}e^{-i\vartheta}\right)\right\} - 2R^{-n-1}\cos(n+1)\vartheta + 2R^{-n-2}\cos n\vartheta \\
&\qquad -\left(1-2R^{-1}\cos\vartheta + R^{-2}\right)\left\{1+(1\mp\lambda)R^{-n}\cos n\vartheta\right\} \\
&= 1 - R^{-2} - R^{-n}\cos n\vartheta + R^{-(n+2)}\cos n\vartheta \\
&\qquad \pm\lambda\left(R^{-n}+R^{-(n+2)}\right)\cos n\vartheta \\
&\qquad -2R^{-(n+1)}\left\{\cos(n+1)\vartheta - (1\mp\lambda)\cos n\vartheta\cos\vartheta\right\} \\
&= \left(1-R^{-2}\right)\left(1-R^{-n}\cos n\vartheta\right) + 2R^{-(n+1)}\sin n\vartheta\sin\vartheta \\
&\qquad \pm\lambda\left(1-2R^{-1}\cos\vartheta+R^{-2}\right)R^{-n}\cos n\vartheta\,.
\end{aligned}$$

In particular,

$$\begin{aligned}
\Re\varphi\left(R^{-1}e^{ij\pi/n}\right) &= \left(1-R^{-2}\right)\left(1-(-1)^j R^{-n}\right) \\
&\qquad \pm(-1)^j\left(1-2R^{-1}\cos\frac{j\pi}{n}+R^{-2}\right)\lambda R^{-n}\,.
\end{aligned}$$

Hence, $\Re\varphi\left(R^{-1}e^{ij\pi/n}\right)\geq 0$ for $j=0,1,\ldots,2n-1$, if and only if

$$\left(R^2-2R\cos\frac{j\pi}{n}+1\right)|\lambda| \leq \left(R^2-1\right)\left(R^n-(-1)^j\right) \qquad (j=0,1,\ldots,2n-1)\,,$$

and so, if and only if

$$|\lambda| \leq \frac{\left(R^2-1\right)\left(R^n-(-1)^j\right)}{R^2-2R\cos(j\pi/n)+1} \qquad (j=0,1,\ldots,2n-1)\,. \tag{4.1}$$

Now let $j = 2m$ $(0\leq m\leq n-1)$. Then, (4.1) is seen to require, in particular, that

$$|\lambda| \leq \frac{(R^2-1)(R^n-1)}{R^2-2R\cos(2m\pi/n)+1} \qquad (m=0,1,\ldots,n-1)\,.$$

Note that if $0\leq m\leq n-1$, then

$$R^2-2R\cos\frac{2m\pi}{n}+1 \leq \begin{cases} R^2+2R+1 & \text{if } n \text{ is even} \\ R^2+2R\cos(\pi/n)+1 & \text{if } n \text{ is odd}\,. \end{cases}$$

Hence, $\Re\varphi\left(R^{-1}e^{ij\pi/n}\right)\geq 0$ for $j=0,2,\ldots,2n-2$, if and only if

$$|\lambda| \leq \begin{cases} \dfrac{(R-1)(R^n-1)}{R+1} & \text{if } n \text{ is even} \\[2ex] \dfrac{(R^2-1)(R^n-1)}{R^2+2R\cos(\pi/n)+1} & \text{if } n \text{ is odd}\,. \end{cases}$$

By taking $j=2m+1$ $(0\leq m\leq n-1)$ we see (4.1) also require that

$$|\lambda| \le \frac{(R^2-1)(R^n+1)}{R^2 - 2R\cos(2m+1)\pi/n + 1} \qquad (m = 0, 1, \ldots, n-1)\,.$$

But, for $0 \le m \le n-1$, we have

$$R^2 - 2R\cos\frac{(2m+1)\pi}{n} + 1 \le \begin{cases} R^2 + 2R\cos(\pi/n) + 1 & \text{if } n \text{ is even} \\ R^2 + 2R + 1 & \text{if } n \text{ is odd}\,. \end{cases}$$

Hence, $\Re\varphi\left(R^{-1}\mathrm{e}^{\mathrm{i}j\pi/n}\right) \ge 0$ for $j = 1, 3, \ldots, 2n-1$, if and only if

$$|\lambda| \le \begin{cases} \dfrac{(R^2-1)(R^n+1)}{R^2+2R\cos(\pi/n)+1} & \text{if } n \text{ is even} \\ \dfrac{(R-1)(R^n+1)}{R+1} & \text{if } n \text{ is odd}\,. \end{cases}$$

We conclude that $\Re\varphi\left(R^{-1}\mathrm{e}^{\mathrm{i}j\pi/n}\right) \ge 0$ for $j = 0, 1, \ldots, 2n-1$, if and only if

$$|\lambda| \le \begin{cases} \min\left\{\dfrac{(R-1)(R^n-1)}{R+1}, \dfrac{(R^2-1)(R^n+1)}{R^2+2R\cos(\pi/n)+1}\right\} & \text{if } n \text{ is even} \\ \min\left\{\dfrac{(R^2-1)(R^n-1)}{R^2+2R\cos(\pi/n)+1}, \dfrac{(R-1)(R^n+1)}{R+1}\right\} & \text{if } n \text{ is odd}\,. \end{cases}$$

It is easily checked that

$$\frac{(R-1)(R^n-1)}{R+1} < \frac{(R^2-1)(R^n+1)}{R^2+2R\cos(\pi/n)+1}$$

for all $R > 1$. Therefore, if n is even, then $\Re\varphi\left(R^{-1}\mathrm{e}^{\mathrm{i}j\pi/n}\right) \ge 0$ for $j = 0, 1, \ldots, 2n-1$, if and only if $|\lambda| \le (R-1)(R^n-1)/(R+1)$. It follows that if $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$, $c_0 \in \mathbb{R}$ is a polynomial of degree at most n, where n is *even*, and $|\Re f(z)| \le 1$ for $|z| \le 1$, then

$$|\Re f(z)| + \frac{(R-1)(R^n-1)}{(R+1)}|c_0| \le R^n \qquad (R > 1)\,.$$

If n is *odd* and $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$, $c_0 \in \mathbb{R}$ is a polynomial of degree at most n such that $|\Re f(z)| \le 1$ for $|z| \le 1$, then for any $R > 1$, we have

$$|\Re f(z)| + |c_0| \min\left\{\frac{(R^2-1)(R^n-1)}{R^2+2R\cos(\pi/n)+1}, \frac{(R-1)(R^n+1)}{R+1}\right\} \le R^n\,. \tag{4.2}$$

■

Remark 4.1. Let n be *odd*. The coefficient of $|c_0|$ in (4.2) is clearly equal to $R-1$ if $n = 1$. For $n \ge 3$ and any $R > 1$,

$$\frac{(R^2-1)(R^n-1)}{R^2+2R\cos(\pi/n)+1} \le \frac{(R-1)(R^n+1)}{R+1}$$

if and only if

$$\left(1-\cos\frac{\pi}{n}\right)R^{n+1}\le R^2+\left(1+\cos\frac{\pi}{n}\right)R+1\,.$$

It follows that if n is odd and R_n denotes the only root of the equation

$$\left(1-\cos\frac{\pi}{n}\right)R^{n+1}= R^2+\left(1+\cos\frac{\pi}{n}\right)R+1$$

in $(1\,,\,\infty)$, then

$$\min\left\{\frac{(R^2-1)(R^n-1)}{R^2+2R\cos(\pi/n)+1}\,,\,\frac{(R-1)(R^n+1)}{R+1}\right\} = \begin{cases}\dfrac{(R^2-1)(R^n-1)}{R^2+2R\cos(\pi/n)+1} & \text{if } 1<R\le R_n\\[2mm] \dfrac{(R-1)(R^n+1)}{R+1} & \text{if } R\ge R_n\,.\end{cases}$$

P r o o f o f T h e o r e m 1.2. Note that $R^{-n}\{f(Rz)+\lambda c_0\}=\sum_{\nu=0}^{n}\gamma_\nu c_\nu z^\nu$, where

$$\gamma_0=(1+\lambda)R^{-n},\ \ \gamma_\nu=R^{-(n-\nu)}\quad(\nu=1,\dots,n)\,.$$

According to Theorem C, the desired inequality holds if for $R\ge R_n^*$ and $|\lambda|\le(R-1)(R^{n-1}-R^{n-2}-1)$, we have

$$\Re\left\{R^n+2\sum_{\nu=1}^{n-1}R^{n-\nu}\,\mathrm{e}^{\mathrm{i}\nu\theta}+(1+\lambda)\,\mathrm{e}^{\mathrm{i}n\theta}\right\}\ge 0\qquad(\theta\in\mathbb{R})\,. \tag{4.3}$$

To see that (4.3) is indeed satisfied for

$$R\ge R_n^*\quad\text{and}\quad |\lambda|\le(R-1)(R^{n-1}-R^{n-2}-1)\,,$$

let us write

$$R^n+2\sum_{\nu=1}^{n-1}R^{n-\nu}\,\cos\nu\theta+(1+\lambda)\,\cos n\theta=C_1(\theta)+C_2(\theta)\,,$$

where

$$C_1(\theta):=R^n-(R-1)(R^{n-1}-R^{n-2}-1)+2\sum_{\nu=1}^{n-1}R^{n-\nu}\,\cos\nu\theta+\cos n\theta$$

and

$$C_2(\theta):=(R-1)(R^{n-1}-R^{n-2}-1)+\lambda\,\cos n\theta\,.$$

Now, let $R\ge R_n^*$, so that $R^{n-1}-R^{n-2}-1\ge 0$. Then, applying Lemma 3.1 with

$$\lambda_0:=R^n-(R-1)(R^{n-1}-R^{n-2}-1),\ \lambda_j:=R^{n-j}\quad(j=1,\dots,n-1),\ \lambda_n:=\frac{1}{2}\,,$$

we see that $C_1(\theta)\ge 0$ for all real θ. In addition, it is clear that $C_2(\theta)\ge 0$ for all real θ if $|\lambda|\le(R-1)(R^{n-1}-R^{n-2}-1)$. Hence, (4.3) is certainly satisfied for any such value of λ. ■

5. An addendum to Theorem 1.2

In view of Remark 1.1, inequality (1.14) fails for values of R sufficiently close to 1, even though the coefficient of $|f(0)|$ appearing on its left-hand side is negative for $R \in (1\,,\,R_n^*)$. This may be overcome by modifying the right-hand side appropriately, such as adding to it $M_f(1) - A_f(1)$, as we do in the following result. It may be noted that the coefficient of $|f(0)|$ in (5.1) is positive for $R > 1$, and the inequality can therefore be seen as a 'counterpart' of (1.14) for $R \in (1\,,\,R_n^*)$.

Theorem 1.2′ *Let f be a polynomial of degree at most n such that $f(0)$ is real, and let $M_f(r) := \max_{|z|=r}|f(z)|$ and $A_f(r) := \max_{|z|=r}|\Re f(z)|$. In addition, let κ_n be the function appearing in Theorem* (1.12). *Then, for any $R > 1$, we have*

$$M_f(R) + n\left(\int_1^R t^{-1}\kappa_n(t)\,\mathrm{d}t\right)|f(0)| \le A_f(1)R^n + M_f(1) - A_f(1)\,. \tag{5.1}$$

P r o o f. A well-known extension of Bernstein's inequality for the derivative of a polynomial says that if g is a polynomial of degree at most n such that $|\Re g(z)| \le \mu$ for $|z| = 1$, then (see [13, p. 61], [9, p. 273], [4, p. 60])

$$|g'(z)| \le \mu n \qquad (|z| = 1)\,. \tag{5.2}$$

For any $t \ge 1$, let $g(z) := f(tz)$. Then, by (1.12),

$$\max_{|z|=1}|\Re g(z)| = A_f(t) \le \mu := A_f(1)t^n - \kappa_n(t)|f(0)|\,.$$

and so, (5.2) implies that

$$|f'(tz)| \le nA_f(1)t^{n-1} - nt^{-1}\kappa_n(t)|f(0)| \qquad (|z| = 1\,,\ t \ge 1)\,.$$

Since $f(Rz) - f(z) = \int_1^R zf'(tz)\,\mathrm{d}t$ we see that for any real θ the inequality

$$\left|f\left(R\mathrm{e}^{\mathrm{i}\theta}\right) - f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right| \le A_f(1)(R^n - 1) - n|f(0)|\int_1^R t^{-1}\kappa_n(t)\,\mathrm{d}t\,, \tag{5.3}$$

holds for all $R > 1$. It clearly implies (5.1), and is also a refinement of (1.5). ■

Remark 5.1. Using Minkowski's inequality [15, p. 384] in conjunction with (5.3), we see that for any $p \ge 1$,

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f\left(R\mathrm{e}^{\mathrm{i}\theta}\right)\right|^p \mathrm{d}\theta\right)^{1/p} + n\left(\int_1^R t^{-1}\kappa_n(t)\,\mathrm{d}t\right)|f(0)|$$
$$\le A_f(1)\,R^n + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f\left(\mathrm{e}^{\mathrm{i}\theta}\right)\right|^p \mathrm{d}\theta\right)^{1/p} - A_f(1)\,.$$

This inequality is an extension of (5.1) since

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f\left(r\,\mathrm{e}^{\mathrm{i}\theta}\right)\right|^{p}\,\mathrm{d}\theta\right)^{1/p}\to M_f(r)\quad\text{as}\quad p\to\infty\,.$$

Remark 5.2. Let f be an *arbitrary* polynomial of degree at most n. In view of (1.2) and (5.2), we have

$$M_{f'}(t)+\left(t^{n-1}-t^{n-3}\right)|f'(0)|\le M_{f'}(1)t^{n-1}\le A_f(1)\,n\,t^{n-1}\qquad(t>1)\,,$$

from which we conclude that

$$M_f(R)+\left\{\frac{1}{n}(R^n-1)-\frac{1}{n-2}(R^{n-2}-1)\right\}|f'(0)|$$
$$\le A_f(1)R^n+M_f(1)-A_f(1)\qquad(R>1)\,,$$

whether $f(0)$ is real or not.

6. Two further observations and a couple of questions

6.1. An L^2– analogue of (1.12)

Let $f(z):=\sum_{\nu=0}^{n}c_\nu z^\nu$, $c_0\in\mathbb{R}$. Then,

$$\begin{aligned}\Re f(r\mathrm{e}^{\mathrm{i}\theta}) &= c_0+\sum_{\nu=1}^{n}\frac{1}{2}\left(c_\nu r^\nu\mathrm{e}^{\mathrm{i}\nu\theta}+\overline{c}_\nu r^\nu\mathrm{e}^{-\mathrm{i}\nu\theta}\right)\\ &= \sum_{\nu=-n}^{-1}\frac{1}{2}\,\overline{c}_{|\nu|}r^{|\nu|}\mathrm{e}^{\mathrm{i}\nu\theta}+c_0+\sum_{\nu=1}^{n}\frac{1}{2}\,c_\nu r^\nu\mathrm{e}^{\mathrm{i}\nu\theta}\,.\end{aligned}$$

Hence,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\Re f(r\mathrm{e}^{\mathrm{i}\theta})\right|^2\,\mathrm{d}\theta=|c_0|^2+\sum_{\nu=1}^{n}\frac{1}{2}\,|c_\nu|^2\,r^{2\nu}\qquad(r>0)\,,\tag{6.1}$$

and so for any $R>1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\Re f(R\mathrm{e}^{\mathrm{i}\theta})\right|^2\,\mathrm{d}\theta+(R^{2n}-1)|c_0|^2=R^{2n}|c_0|^2+\frac{1}{2}\sum_{\nu=1}^{n}|c_\nu|^2R^{2\nu}$$

$$\le R^{2n}\left(|c_0|^2+\frac{1}{2}\sum_{\nu=1}^{n}|c_\nu|^2\right)=R^{2n}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\Re f(\mathrm{e}^{\mathrm{i}\theta})\right|^2\,\mathrm{d}\theta\right)\,.$$

Thus, if

$$\mathcal{N}_f(p\,;\,r):=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\Re f(r\mathrm{e}^{\mathrm{i}\theta})\right|^p\,\mathrm{d}\theta\right)^{1/p}\qquad(r>0\,,\;p>0)\,,$$

then, analogously to (1.12), we have

$$(\mathcal{N}_f(2\,;\,R))^2 + (R^{2n}-1)|f(0)|^2 \le (\mathcal{N}_f(2\,;\,1))^2\ R^{2n} \qquad (R>1)\,. \tag{6.2}$$

Problem 1. *Find the* L^p *– analogue of* (6.2), *for any positive number* p.

6.2. An L^2– analogue of (1.14)

Let $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$, $c_0 \in \mathbb{R}$. Then, by (6.1)

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(\mathrm{e}^{\mathrm{i}\theta})\right|^2\,\mathrm{d}\theta &= |c_0|^2 + \sum_{\nu=1}^{n}|c_\nu|^2 \\ &= 2\left\{|c_0|^2 + \frac{1}{2}\sum_{\nu=1}^{n}|c_\nu|^2\right\} - |c_0|^2 = 2\,(\mathcal{N}_f(2\,;1))^2 - |c_0|^2\,.\end{aligned}$$

Hence, for any $R>1$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(R\mathrm{e}^{\mathrm{i}\theta})\right|^2\,\mathrm{d}\theta &= \sum_{\nu=1}^{n}|c_\nu|^2R^{2\nu} + |c_0|^2\,R^{2n} - (R^{2n}-1)|c_0|^2 \\ &\le \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(\mathrm{e}^{\mathrm{i}\theta})\right|^2\,\mathrm{d}\theta\right)R^{2n} - (R^{2n}-1)|c_0|^2 \\ &\le \left\{2\,(\mathcal{N}_f(2\,;\,1))^2 - |c_0|^2\right\}R^{2n} - (R^{2n}-1)|c_0|^2\,,\end{aligned}$$

that is, if

$$\mu_f(p\,;\,r) := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(r\mathrm{e}^{\mathrm{i}\theta})\right|^p\,\mathrm{d}\theta\right)^{1/p} \qquad (r>0\,,\ p>0)\,,$$

then, analogously to (1.14), we have

$$(\mu_f(2\,,\,R))^2 + (2R^{2n}-1)\,|f(0)|^2 \le 2\,(\mathcal{N}_f(2\,;\,1))^2\ R^{2n} \quad (R>1)\,. \tag{6.3}$$

Problem 2. *Find the* L^p*– analogue of* (6.3), *for any positive number* p.

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