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Whitney's Constants and Sendov's Conjectures

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Dedicated to Professor Blagovest Sendov on the occasion of his 70th anniversary

In this paper we review the history and the current state of Whitney's constants problem.

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1. Introduction

Let C be a space of continuous functions f on $I := [0,1]$ with the uniform norm

$$\|f\| := \max_{x\in I} |f(x)|.$$

For a function $f \in C$, denote the k–th difference with step h by

$$\Delta_h^k f(x) := \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh)$$

and the k-th modulus of continuity by

$$\omega_k(f) := \sup_{x, x+kh\in I} |\Delta_h^k f(x)|.$$

Let $\mathbf{P}_k$ be a space of algebraic polynomials of degree $\leq k$. Whitney's constants are defined by

$$W_k := \sup_{f\in C\setminus \mathbf{P}_{k-1}} \inf_{p\in \mathbf{P}_{k-1}} \frac{\|f-p\|}{\omega_k(f)}.$$

Let $L_{k-1}(f,x)$ be the Lagrange polynomial of degree $\le k-1$, which interpolates f at equidistant points $x_m := m/(k-1)$:

$$f(x_m) = L_{k-1}(f,x_m), \quad m = 0,\dots,k-1.$$

Whitney's interpolation constants are defined by

$$W'_k := \sup_{f\in C\setminus \mathbf{P}_{k-1}} \frac{\|f - L_{k-1}(f,\cdot)\|}{\omega_k(f)} = \sup_{f\in C,\ f(x_m)=0} \frac{\|f\|}{\omega_k(f)}.$$

In this paper we are mainly interested in estimates of W_k and W'_k. Namely, we intend to describe the current situation with the following Sendov's conjectures, see [17]:

First Sendov's conjecture : $W_k \le 1$.

Second Sendov's conjecture : $W'_k \le 2$.

2. Historical remarks

2.1. Results of Burkill, Whitney, Beurling and Brudnyi. It is clear, that $W_1 = 1/2$ and $W'_1 = 1$. This is the case of approximation of a continuous function by constant. The results $W_2 = 1/2$ and $W'_2 = 1$ are due to H. Burkill [4] and H. Whitney [27].

Burkill's lemma. *If $f \in C$ and $f(0) = f(1) = 0$, then $\|f\| \le \omega_2(f)$.*

Proof. Suppose that $|f(a)| = \|f\|$ and $a \le 1/2$. Then $f(a) = -1/2(f(0) - 2f(a) + f(2a)) + 1/2f(2a)$ and $\|f\| \le |f(0) - 2f(a) + f(2a)| \le \omega_2(f)$. In the case $a > 1/2$ we have the same conclusion by symmetry. ■

H. Burkill conjectured, that $W_k \le W'_k < \infty$. This conjecture was proved in 1957 by H. Whitney [27]. The main ingradient of his proof is the following Lemma.

Whitney's lemma. *Let $k,\nu \in N,\ k,\nu \ge 2,\ X = \{0,1,\dots,\nu(k-1)\}$. Then for $f(s),\ s \in X$, there exist numbers*

$$a_i = a_i(s,\nu,k), \quad i = 0,\dots,\nu(k-1)-k,$$

$$b_j = b_j(s,\nu,k),\ j = 0,\dots,k-1,$$

such that

$$f(s) = \sum_{j=0}^{\nu(k-1)-k} a_j \Delta_1^k f(j) + \sum_{j=0}^{k-1} b_j f(j\nu).$$

Moreover, for $s = 1$ *and for arbitrary positive* $\varepsilon > 0$ *there exists* ν*, such that*

$$\sum_{j=1}^{k-1} b_j(s, \nu, k) < \varepsilon.$$

This lemma can be used in various generalizations of Whitney's inequality $W_k' \leq \infty$. For example, it was used in proofs of analogues of Whitney's estimate for functions in L^p [24], for functions on complex arcs [26, 15], for Chebyshev approximations [28, 16]. But we can not obtain good estimates of W_k' and W_k in this way. We can not write any, even a bad estimate, for all k. By using special identities, Whitney proved the inequalities

$$\frac{8}{15} \leq W_3 \leq \frac{7}{10}, \quad \frac{16}{15} \leq W_3' \leq \frac{14}{9}.$$

Whitney noted, that "the problem of finding the W_k, W_k' is probably extremely difficult ...".

Another proof of Whitney's theorem with estimates $W_k \leq Ck^{2k}$ was obtained by Yu. Brudnyi [2]. A modified Brudnyi's proof with estimate $W_k \leq (k+1)k^k$, one can find in Sendov's paper [19].

On the other hand, the situation is not difficult for integrable on $[0, +\infty)$ functions. We have the following Whitney–Beurling identity [27]: For $A > 0$, $0 \leq y < x$,

$$1/A \left\{ \int_0^A \Delta_h^k f(x)\, dh - \int_0^A \Delta_h^k f(y) dh \right\}$$

$$= (-1)^k (f(x) - f(y)) + 1/A \left\{ \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \left(\int_{y+jA}^{x+jA} f - \int_y^x f \right) \right\}.$$

This identity implies (we may consider $A >> 1$) the estimate

$$|f(x) - f(y)| \leq 2 \sup_{x,\, h>0} |\Delta_h^k f(x)|.$$

2.2. Results of Ivanov, Takev, Binev and Sendov. In the papers [17,18,19], Sendov attracted attention to the problem of Whitney's constants.

In 1985, as a result of the works [7,1,20], the following remarkable inequality was obtained [21,22]:

$$W_k \le \text{ Const} \le 6. \tag{1}$$

The use of the Ivanov–Takev integral operators [7]

$$\psi_i(f,x) := \frac{(-1)^{k-i}}{h\binom{k}{i}} \int_0^h \Delta_y^k f(x-iy)\,dy,$$

$$h = (k+1)^{-1},\ t \in [0,h],\ x = ih+t,\ i = 0,1,\dots,k,$$

was essential for proving (1). These operators can be considered as a convenient tool for transplantation of Whitney–Beurling proof from $[0,\infty)$ to $I = [0,1]$. Sendov deduced (1) from the following lemma.

Sendov's lemma. *Let $f \in C$, $k \in N$. Then there exists $p \in \mathbf{P_{k-1}}$ such, that*

$$f(x) = p(x) + \psi_i(f,x) + \sum_{j=0}^{k} \frac{1}{h} \int_0^t \psi_j(f, jh+y) l'_{k,j}(\frac{x-y}{h})\,dy,$$

where

$$l_{k,j}(x) := \prod_{m=0, m\ne j}^{k} (x-m)/(j-m).$$

An analysis of Sendov's lemma leads [9,10] to the following

Modified Sendov's lemma. *Let $f \in C$, $k \in N$ and $\int_0^{i/k} f(t)\,dt = 0$, $i = 1,\dots,k$. Then for $x \in (0,1/k]$ we have*

$$f(ix) = \varphi_i(f,x) + \int_{1/k}^{x} \sum_{j=1}^{k} \varphi_j(f,y) \cdot \frac{j}{i} \cdot \left[l_{k,j}(\frac{x}{y} i) \right]'_x dy,$$

where

$$\varphi_i(f,x) := \frac{(-1)^{k-i}}{\binom{k}{i}} \frac{1}{x} \int_0^x \Delta_y^k f(i(x-y))\,dy.$$

The modified Sendov's lemma implies the inequality

$$W_k \le 3. \tag{2}$$

Inequality (2) was independently announced by Yu. Brudnyi [3], B. Sendov [23] and the author [8].

The estimate

$$W_k' < \quad \text{Const} \quad < 36$$

was obtained by M. Takev [25]. Combination of Takev's method with the modified Sendov's lemma produced the inequality [13]

$$W_k' < 5.$$

3. Recent developments

3.1. Estimates of W_k. In the modified Sendov's lemma we can replace $f(x)$ by $f(x) - q(x)$ and remove the condition $\int_0^{i/k} f = 0$, by a special choice of $q \in \mathbf{P_{k-1}}$:

$$\int_0^{i/k} f(t) - q(t)\, dt = 0, \quad i = 1, \ldots, k.$$

It is natural to define the corresponding constants:

$$W_k^* := \sup_{f \in C \setminus \mathbf{P}_{k-1}} \frac{\|f - q\|}{\omega_k(f)}\,.$$

It is clear that $W_k \le W_k^*$.

Theorem 1. ([12, 5])

$$W_k^* \le 2 \qquad \textit{for} \qquad k \le 82000.$$
$$W_k^* \le 2 + \exp(-2) \qquad \textit{for} \qquad k > 82000.$$

To prove Theorem 1, we need another modification of the Whitney–Beurling idea. To this end, put $F(x) := \int_0^x f(u)\, du$. The following identity can be checked directly.

Lemma 1. ([12]) *If $m \in \{0, 1, \ldots, k\}$, $x \in I$ and $\delta > 0$ are such that $[x - m\delta, x + (k - m)\delta] \subset I$, then*

$$(-1)^{k-m}\binom{k}{m} f(x) = \int_0^1 \Delta_{t\delta}^k f(x - m\delta t)\, dt$$

$$-(-1)^{k-m}\frac{1}{\delta}\binom{k}{m}(\sigma_{k-m}-\sigma_m)F(x)$$

$$-\frac{1}{\delta}\sum_{j=0,j\neq m}^{k}(-1)^{k-j}\binom{k}{j}\frac{1}{j-m}F(x+(j-m)\delta),$$

where

$$\sigma_0 := 0, \quad \sigma_m := \sum_{j=1}^{m}\frac{1}{j}, \qquad m = 1, 2, \ldots .$$

The estimates of F in Lemma 1 provide the following Zhuk–Natanson identity.

Lemma 2. ([31, 12]) *If* $F(i/k) = 0, \quad i = 1, \ldots, k,$ *then*

$$F(x) = A_k(x)\int_0^1 \Delta_{t/k}^k f(x(1-t))\,dt, \quad x \in I,$$

where

$$A_k(x) := \frac{k^k}{k!}x\left(x-\frac{1}{k}\right)\left(x-\frac{2}{k}\right)\ldots\left(x-\frac{k}{k}\right) = x(-1)^k\prod_{j=1}^{k}\left(1-\frac{kx}{j}\right).$$

Combining Lemma 1 and Lemma 2, one can obtain the inequality $W_k^* <$ Const . For relatively small $k < 1000$, one can use PC for estimating the constants. But for $k >> 1$ we need something else. The next Shevchuk's lemma is the appropriate tool for the estimates in case of large k.

Lemma 3. ([5]) *Let* $g := f - q$. *Suppose that* $\omega_k(g) \le 1$, $m < k/2$, $x \in [m/k, (m+1)/k]$, $\delta := (1-x)/(k-m)$. *Then*

$$\binom{k}{m}|g(x)|$$

$$\le 1 + (k\delta)^k - (-1)^{k-m}\binom{k}{m}A_k'(x) + \frac{2}{\delta}\sum_{j=0}^{m-1}\binom{k}{j}\frac{1}{m-j}\left(|A_k(x+\delta(j-m))|\right).$$

Note, that the modified Sendov's lemma implies the inequality $|g(x)| \le 1$ for $x \in [1/k, 1-1/k]$. So, to prove Theorem 1, we need only the following

Lemma 4. ([5]) *For $x \in [0, 1/k)$ we have*

$$(1-x)^k - (-1)^k A_k'(x) \le 1 + \frac{1}{e^2},$$

and

$$(1-x)^k - (-1)^k A_k'(x) \le 1, \quad k \le 82000.$$

Remark. Theorem 1 corrects an arithmetical mistake in [12], where it was claimed that $W_k^* \le 2$ for all k.

Theorem 2. ([11, 14, 29])

$$W_k^* = 1, \quad k \le 7.$$

It is not hard to prove that $W_k^* \ge 1, k \ge 1$ (see [10]). An example can be constructed by smoothing the function $f(x) = 0$, $x \ne 0$, $f(1) = 1$. The inequality $W_1^* \le 1$ is trivial. With the intention to make the idea of the proof as clear as possible, let us consider the simple case $k = 2$. Put

$$G(x,y) := \frac{1}{y-x}\int_x^y g(t)\,dt, \quad G(x,x) := g(x)$$

and

$$\Delta^k_{h1,h2}G(x,y) := \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}G(x+jh1, y+jh2).$$

It easy to check that

$$\Delta^k_{h1,h2}G(x,y) = \int_0^1 \Delta^k_{h1+t(h2-h1)}g(x+t(y-x))\,dt.$$

We need to prove that if $\int_0^{i/2} g = \int_0^{i/2}(f-q) = 0$, $i = 1,2$ and $\omega_2(g) \le 1$, then $|g(x)| \le 1$, or in other notation: if $G(0, j/2) = 0$, $j = 1,2$ and $|\Delta^2_{h1,h2}G(x,y)| \le 1$, then we have $|g(x)| \le 1$.

Suppose that $\max|g(y)| = g(x)$ and $x < 1/3$ (the case of $x \in [1/3, 1/2]$ is more simple; in the case $\max|g(y)| = -g(x)$, we can consider the function $g_1 = -g$). We have the identities:

$$g(x) = \Delta^2_{(1-x)/2,x}G(x,x) - \frac{x}{1-3x}(6G(0,2x) - G(2x,3x) - 2G(1/2, 1/2 + x/2))$$

and

$$g(x) = -\frac{1}{2}\Delta^2_{x,0}G(0,x) + G(0,2x).$$

The first identity is a global estimate (with big step $h = (1-x)/2$). The second identity is a local estimate (with step $h = x$). Now from the local estimate we deduce

$$G(0,2x) \geq g(x) - 1/2.$$

Combining last inequality with the global estimate, we find that

$$1 \geq \left|\Delta^2_{(1-x)/2,x}G(x,x)\right|$$

$$\geq \left|g(x) + \frac{x}{1-3x}(6G(0,2x) - G(2x,3x) - 2G(1/2,1/2+x/2))\right|$$

$$\geq \left|g(x) + \frac{x}{1-3x}\left(6(g(x) - 1/2) - g(x) - 2g(x)\right)\right|,$$

or

$$g(x) \leq 1.$$

The proof of Theorem 2 for $2 < k < 8$ is not simple. The main idea of the proofs, presented in [14, 29], is due to H. Whitney [27]. Put $G(u) := \int_0^u g(t)\,dt$ and suppose, that $G(i/k) = 0$, $i = 1,\dots,k$. Let $\max|g(y)| = g(x) > \omega_k(f)$. Consider the identity

$$(-1)^k \int_0^1 \Delta^k_{th+(1-t)\alpha x/2}\, g(x)\,dt - g(x)$$

$$= \frac{1}{h-\alpha x/2}\left(\sum_{j=1}^k \frac{(-1)^j\binom{k}{j}}{j}G(x+jh) - \sum_{j=1}^k \frac{(-1)^j\binom{k}{j}}{j}G(x+j\frac{\alpha x}{2})\right),$$

$$h = (1-x)/k,\ \alpha : 0 < \frac{\alpha x}{2} < h.$$

Since the left hand side of this identity is non-positive, then

$$M_\alpha(x) := -\frac{1}{x}\sum_{j=1}^k \frac{(-1)^j\binom{k}{j}}{j}G(x+j\frac{\alpha x}{2}) \leq -\frac{1}{x}\sum_{j=1}^k \frac{(-1)^j\binom{k}{j}}{j}G(x+jh).$$

Lemma 2 implies (see [14, 29])

$$M_\alpha(x) \leq \sigma_k - 1.$$

The core of the proof is the following identity

$$Ag(x) = \sum a_i g_i + \sum b_j \Delta_j^k + \sum c_l M_{\alpha(l)}(x), \tag{3}$$

where $a_i, b_j \in R,\ c_k \in R_+,\ g_i := G(\frac{x(i-1)}{2}, \frac{xi}{2}),\ \Delta_j^k :=$ means of finite differences. We will use (3) for x near the origin (the difficult case of Theorem 2). For other x, we can use some modification of (3) (see [14, 29]). We can suppose, that $|\Delta_j^k| \le 1$. To prove Theorem 2, it is sufficient to construct identity (3) with the constraint

$$A > \sum |a_i| + \sum |b_j| + (\sigma_k - 1) \sum c_l.$$

Identity for $\mathbf{k = 2}$, $x \in [0, 1/3]$.

$$g(x) = \frac{1}{12}(g_5 + g_6) - \frac{1}{2}\Delta_{x,0}^2 G(0, x) + \frac{1}{3} M_2(x).$$

Identity for $\mathbf{k = 3}$, $x \in [0, 1/6]$, $g_0 := g(0)$.

$$Ag(x) = \sum_{i=0}^{12} a_i g_i + \sum_{j=1}^{4} b_j \Delta_j^3 + c M_2(x),$$

$$\Delta_1^3 = \Delta_{x/2,0}^3 G(x/2, x), \quad \Delta_2^3 = \Delta_{x/2,0}^3 G(0, x),$$

$$\Delta_3^3 = \Delta_{x,x/2}^3 G(0, x/2), \quad \Delta_4^3 = \Delta_{2x,3x/2} G(0, 0).$$

$$A = 396/7, \quad c = 12,$$

$$\mathbf{a} = [4/3, 0, 0, 0, 148/7, 5/7, 5/7, 0, 0, 0, -4/9, -4/9, -4/9],$$

$$\mathbf{b} = [22/7, -22/7, 88/7, 4/3].$$

We have $396/7 = A > \sum |a_i| + \sum |b_j| + (\sigma_3 - 1) \cdot 12 = 388/7$.

Identity for $\mathbf{k = 4}$, $x \in [0, 1/12]$.

$$Ag(x) = \sum_{i=1}^{22} a_i g_i + \sum_{j=1}^{12} b_j \Delta_j^4 + \sum_{l=1}^{4} c_l M_{(l+1)}(x),$$

$$\Delta_1^4 = \Delta_{\frac{x}{2},0}^4 G(0, x), \quad \Delta_2^4 = \Delta_{x,\frac{x}{2}}^4 G(0, \frac{x}{2}), \quad \Delta_3^4 = \Delta_{\frac{x}{2},\frac{x}{2}}^4 G(0, \frac{x}{2}),$$

$$\Delta_4^4 = \Delta_{x,x}^4 G(0, \frac{x}{2}), \quad \Delta_5^4 = \Delta_{\frac{3x}{2},\frac{3x}{2}} G(0, \frac{x}{2}), \quad \Delta_6^4 = \Delta_{\frac{x}{2},\frac{x}{2}}^4 G(\frac{x}{2}, x),$$

$$\Delta_7^4 = \Delta_{x,x}^4 G(\frac{x}{2}, x),\ \Delta_8^4 = \Delta_{\frac{3x}{2},\frac{3x}{2}} G(\frac{x}{2}, x),\ \Delta_9^4 = \Delta_{x,x}^4 G(3x, \frac{7x}{2}),$$

$$\Delta_{10}^4 = \Delta_{\frac{x}{2},\frac{x}{2}}^4 G(\frac{9x}{2}, 5x),\ \Delta_{11}^4 = \Delta_{\frac{3x}{2},\frac{3x}{2}}^4 G(\frac{9x}{2}, 5x),\ \Delta_{12}^4 = \Delta_{\frac{x}{2},\frac{x}{2}}^4 G(\frac{15x}{2}, 8x).$$

The coefficients a_i, b_j, c_l, for the case $k = 4$ and the identities for x, which are separated from the intervals endpoints, can be found in [14]. Appropriate identities for $k = 5, 6, 7$ were constructed by O. Zhelnov [29].

3.2. Estimates of W_k'. The method, proposed by M. Takev [25], intermediate approximation by polynomials $q : \int_0^{i/k}(f - q) = 0, i = 1, \dots k$, and estimates of Lemma 3, led to the following theorem.

Theorem 3. ([5])

$$W_k' \le 3.$$

Since the inequality

$$|f(x) - L_{k-1}(f, x)| \le \omega_k(f), \quad x \in [1/k, 1 - 1/k],$$

is known (see for example, the estimates in [13]), we only need to prove that

$$|f(x) - L_{k-1}(f, x)| \le 3\, \omega_k(f), \quad x \in [0, 1/k).$$

By using the notation $g(x) := f(x) - q(x)$, we get

$$|f(x) - L_{k-1}(f; x)| \le |f(x) - q(x) - L_{k-1}(f, x) + q(x))|$$

$$\le |f(x) - q(x)| + |L_{k-1}(f - q, x)| = |g(x)| + \left|\sum_{m=0}^{k-1} g(x_m) l_{k-1,m}((k-1)x)\right|.$$

To estimate the value of $|g(x)|$ for $x \in [0, 1/k)$ and at the points $x_m,\ m = 0, \dots, k-1$, we shall use Lemma 3.

For $x \in [0, 1/k)$ we have the inequality

$$|g(x)| \le 1 + (1 - x)^k - (-1)^k A_k'(x).$$

To estimate

$$|L_{k-1}(g, x)| = \left|\sum_{m=0}^{k-1} g(x_m) l_{k-1,m}((k-1)x)\right|, \qquad x_m = \frac{m}{k-1},$$

we can use the following Lemma 5.

Lemma 5. ([5]) *Suppose that* $f \in C$, $\omega_k(f) \le 1$. *Then for each* $m = 0, \dots, k-1$, *we have*

$$|g(x_m)| \le \binom{k-1}{m}^{-1} + 2\,(k-1)\,\sigma_{k-1}\,|A_k(x_m)|\,.$$

The proof of Lemma 5 is the most technical part of the paper [5]. One can use Lemma 5 to deduce Lemma 6.

Lemma 6. ([5]) *Let* $f \in C,\ k > 7,\ \omega_k(f) \le 1,\ x \in [0, 1/k)$. *Then*

$$|f(x) - L_{k-1}(f, x)| \le 2 + e\,(k-1)\,\sigma_{k-1}|A_{k-1}(x)|.$$

Since

$$e\,(k-1)\,\sigma_{k-1}|A_{k-1}(x)| \le 1,$$

we have Theorem 3 for $k > 7$.

For $k \le 7$, the second Sendov's conjecture follows from Theorem 2. It was proved by Danilenko ($k = 4$) and O. Zhelnov ($k = 5, 6, 7$).

Theorem 4. ([6, 30]).

$$W'_k \le 2, \quad k = 4, 5, 6, 7.$$

References

[1] P. Binev. $O(n)$ bound of Whitney's constants, *C.R. Acad. Bulg. Sci.*, **38**, No 10, 1985, 1303-1305.

[2] Yu.A. Brudnyi. On a theorem on local best approximations, *Kazan Gos. Univ. Ucen. Zap.*, **124**, No 6, 1964, 43-49.

[3] Yu.A. Brudnyi. Whitney's constants, In: *Modern Problems of the Theory of Functions* (Abstracts of All-Union School-Conference, Baku, 1989 (In Russian).

[4] H. Burkill. Cesaro–Perron almost periodic functions, *Proc. Lond. Math. Soc.,* **3**, No 2, 1952, 150-174.

[5] J. Gilewicz, Yu. Kryakin, I. Shevchuk. Boundedness of the interpolation Whitney's constants by 3, Submitted to: *Journ. of Appr. Theory.*

[6] I.G. Danilenko. On the Sendov problem for the Whitney interpolation constant, *Ukr. Math. J.,* **50**, No 5, 1998, 831-833; *Ukr. Mat. Zh.*, **50**, No 5, 1998, 732-734 (Russian original).

[7] K. Ivanov, M. Takev. $O(n \ln(n))$ bounds Whitney constants, *C.R. Acad. Bulg. Sci.*, **38**, No 9, 1985, 1129-1131.

[8] Yu.V. Kryakin. On Whitney Constants, In: *Modern Problems of the Theory of Functions* (Abstracts of All-Union School-Conference, Baku, 1989 (In Russian).

[9] Yu.V. Kryakin. Whitney constants, *Mat. Zam.*, **46**, No 2, 1989, 155-157 (In Russian).

[10] Yu.V. Kryakin. On the theorem of H. Whitney in spaces L_p, $1 \le p \le \infty$, *Math. Balkanika*, **4**, No 3, 1990, 258-271.

[11] Yu.V. Kryakin. On the exact constants in the Whitney theorem, *Math. Notes*, **54**, 1993, 688-700; *Mat. Zametki.*, **53**, No 1, 1993, 34-51 (Russian original).

[12] Yu.V. Kryakin. On Whitney's theorem and constants, *Russ. Acad. Sci., Sb. Math.,* **81**, No 2, 1995, 281-295; *Math. Sb.*, **185**, No 3, 1994, 25-40 (Russian original).

[13] Yu.V. Kryakin, M.D. Takev. Whitney interpolation constants, *Ukr. Mat. Zh.*, **47**, No 8, 1995, 1038-1043 (In Russian).

[14] Yu.V. Kryakin. On functions with a bounded n-th difference, *Izv. Math.*, **61**, No 2, 1997, 331-346; *Izv. Ross. Akad. Nauk. Ser. Mat.*, **161**, No 2, 1997, 95-110 (Russian original).

[15] Yu.V. Kryakin. On approximation by splines in $L_p, p > 0$, In: *Fourier Series: Theory and Applications (Kamenets-Podol'sky, 1997), Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos.*, **20**, Kiev 1998, 132-141 (In Russian).

[16] Yu.V. Kryakin. On the approximation by Chebyshev splines in the metric L_p, $p > 0$. *Ukr. Math. J.*, **50**, No 9, 1998, 1365-1375; *Ukr. Mat. Zh.*, **50**, No 9, 1998, 1193-1201 (Russian original).

[17] Bl. Sendov. On the constants of H. Whitney, *C.R. Acad. Bulg. Sci.*, **35**, 1982, 431-434.

[18] Bl. Sendov. A new proof of H. Whitney's theorem, *C.R. Acad. Bulg. Sci.,* **35**, 1982, 609-611.

[19] Bl. Sendov. Modifying Steklov means, *C.R. Acad. Bulg. Sci.,* **36**, 1983, 315-317 (In Russian).

[20] Bl. Sendov. The constants of H. Whitney are bounded, *C.R. Acad. Bulg.*

Sci., **38**, No 10, 1985, 1299-1302.

[21] Bl. Sendov. On a theorem of Whitney, *Sov. Math. Dokl.*, **34**, 1987, 603-606: *DAN SSSR*, **291**, 1986, 1296-1300 (Russian original).

[22] Bl. Sendov. On the theorem and constants of H. Whitney, *Constr. Approx.*, **3**, 1987, 1-11.

[23] Bl. Sendov, V. Popov. *The Averaged Moduli of Smoothness*, Wiley, Chichester, 1988.

[24] E.A. Storozhenko, Yu.V. Kryakin. Whitney's theorem in the L^p–metric, $0 < p < \infty$, *Sb. Math.*,, **186**, No 3, 1995, 281-295; *Mat. Sbornik*, **186**, No 3, 1995, 131-142 (Russian original).

[25] M.D. Takev. On the theorem of Whitney–Sendov, In: *Constructive Theory of Functions*, Sofia, 1992, 269-275.

[26] P.M. Tamrazov. Divided differences and moduli of smoothness of functions, function superposition and their application, In: *Constructive Theory of Functions*, Sofia, 1984, 840-851.

[27] H. Whitney. On functions with bounded n–differences, *J. Math. Pures. Appl.,* **36**, No 9, 1957, 67-95.

[28] Z. Wronicz. Chebyshevian splines, *Diss. Math.*, **305**, 1990, 1-98.

[29] O. Zhelnov. Whitney constants are bounded by 1 for $k = 5, 6, 7$, *East Journal for Approximation,* Submitted.

[30] O. Zhelnov. Whitney interpolation constants W_k' are bounded by 2 for $k = 5, 6, 7$, *Ukr. Mat. Zh.*, To appear.

[31] V.V. Zhuk, G.I. Natanson. On the theory of cubic periodic splines with equidistant nodes, *Vest. Leningr. Univ.*, **37**, No 1, 1984, 5-11 (In Russian).

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