Whitney’s Constants and Sendov’s Conjectures

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Dedicated to Professor Blagovest Sendov on the occasion of his 70th anniversary

In this paper we review the history and the current state of Whitney’s constants problem.

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1. Introduction

Let $C$ be a space of continuous functions $f$ on $I := [0, 1]$ with the uniform norm

$$
\|f\| := \max_{x \in I} |f(x)|.
$$

For a function $f \in C$, denote the $k$-th difference with step $h$ by

$$
\Delta_k^h f(x) := \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh)
$$

and the $k$-th modulus of continuity by

$$
\omega_k(f) := \sup_{x,x+kh \in I} |\Delta_k^h f(x)|.
$$

Let $P_k$ be a space of algebraic polynomials of degree $\leq k$. Whitney’s constants are defined by

$$
W_k := \sup_{f \in C \setminus P_{k-1}} \inf_{p \in P_{k-1}} \frac{\|f - p\|}{\omega_k(f)}.
$$
Let $L_{k-1}(f, x)$ be the Lagrange polynomial of degree $\leq k - 1$, which interpolates $f$ at equidistant points $x_m := m/(k - 1)$:

$$f(x_m) = L_{k-1}(f, x_m), \quad m = 0, \ldots, k - 1.$$  

Whitney’s interpolation constants are defined by

$$W_k := \sup_{f \in C \setminus \mathbb{P}_{k-1}} \frac{\|f - L_{k-1}(f, \cdot)\|}{\omega_k(f)} = \sup_{f \in C, f(x_m) = 0} \frac{\|f\|}{\omega_k(f)}.$$  

In this paper we are mainly interested in estimates of $W_k$ and $W_k'$. Namely, we intend to describe the current situation with the following Sendov’s conjectures, see [17]:

**First Sendov’s conjecture**: $W_k \leq 1$.

**Second Sendov’s conjecture**: $W_k' \leq 2$.

### 2. Historical remarks

#### 2.1. Results of Burkill, Whitney, Beurling and Brudnyi

It is clear, that $W_1 = 1/2$ and $W_1' = 1$. This is the case of approximation of a continuous function by constant. The results $W_2 = 1/2$ and $W_2' = 1$ are due to H. Burkill [4] and H. Whitney [27].

**Burkill’s lemma.** If $f \in C$ and $f(0) = f(1) = 0$, then $\|f\| \leq \omega_2(f)$.

**Proof.** Suppose that $|f(a)| = \|f\|$ and $a \leq 1/2$. Then $f(a) = -1/2(f(0) - 2f(a) + f(2a)) + 1/2f(2a)$ and $\|f\| \leq |f(0) - 2f(a) + f(2a)| \leq \omega_2(f)$. In the case $a > 1/2$ we have the same conclusion by symmetry.

H. Burkill conjectured, that $W_k \leq W_k' < \infty$. This conjecture was proved in 1957 by H. Whitney [27]. The main ingredient of his proof is the following Lemma.

**Whitney’s lemma.** Let $k, \nu \in \mathbb{N}$, $k, \nu \geq 2$, $X = \{0, 1, \ldots, \nu(k - 1)\}$. Then for $f(s), s \in X$, there exist numbers

$$a_i = a_i(s, \nu, k), \quad i = 0, \ldots, \nu(k - 1) - k,$$

$$b_j = b_j(s, \nu, k), \quad j = 0, \ldots, k - 1,$$
such that
\[ f(s) = \sum_{j=0}^{\nu(k-1)-k} a_j \Delta_j^k f(j) + \sum_{j=0}^{k-1} b_j f(j\nu). \]
Moreover, for \( s = 1 \) and for arbitrary positive \( \varepsilon > 0 \) there exists \( \nu \), such that
\[ \sum_{j=1}^{k-1} b_j(s,\nu,k) < \varepsilon. \]

This lemma can be used in various generalizations of Whitney’s inequality \( W_k' \leq \infty \). For example, it was used in proofs of analogues of Whitney’s estimate for functions in \( L^p \) \([24]\), for functions on complex arcs \([26, 15]\), for Chebyshev approximations \([28, 16]\). But we can not obtain good estimates of \( W_k' \) and \( W_k \) in this way. We can not write any, even a bad estimate, for all \( k \). By using special identities, Whitney proved the inequalities
\[ \frac{8}{15} \leq W_3 \leq \frac{7}{10}, \quad \frac{16}{15} \leq W_3' \leq \frac{14}{9}. \]
Whitney noted, that “the problem of finding the \( W_k, W_k' \) is probably extremely difficult …”.

Another proof of Whitney’s theorem with estimates \( W_k \leq C k^{2k} \) was obtained by Yu. Brudnyi \([2]\). A modified Brudnyi’s proof with estimate \( W_k \leq (k+1)^k \), one can find in Sendov’s paper \([19]\).

On the other hand, the situation is not difficult for integrable on \([0, +\infty)\) functions. We have the following Whitney–Beurling identity \([27]\): For \( A > 0 \), \( 0 \leq y < x \),
\[
\frac{1}{A} \left\{ \int_0^A \Delta_h^k f(x) dh - \int_0^A \Delta_h^k f(y) dh \right\} = (-1)^k (f(x) - f(y)) + 1/A \left\{ \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \left( \int_{y+ja}^{x+ja} f - \int_{y}^{x} f \right) \right\}.
\]
This identity implies (we may consider \( A >> 1 \)) the estimate
\[
|f(x) - f(y)| \leq 2 \sup_{x, h>0} |\Delta_h^k f(x)|.
\]

2.2. Results of Ivanov, Takev, Binev and Sendov. In the papers \([17,18,19]\), Sendov attracted attention to the problem of Whitney’s constants.
In 1985, as a result of the works [7,1,20], the following remarkable inequality was obtained [21,22]:

\[ W_k \leq \text{Const} \leq 6. \]


\[ \psi_i(f,x) := \frac{(-1)^{k-i}}{h(k)} \int_0^h \Delta_y^k f(x - iy) dy, \]

\[ h = (k + 1)^{-1}, \ t \in [0,h], \ x = ih + t, \ i = 0,1,\ldots,k; \]

was essential for proving (1). These operators can be considered as a convenient tool for transplantation of Whitney–Beurling proof from \([0,\infty)\) to \(I = [0,1]\). Sendov deduced (1) from the following lemma.

**Sendov’s lemma.**  Let \( f \in C, \ k \in N. \) Then there exists \( p \in P_{k-1} \) such, that

\[ f(x) = p(x) + \psi_i(f,x) + \sum_{j=0}^{k} \frac{1}{h} \int_0^t \psi_j(f,jh + y)l_{k,j}(x - y) dy, \]

where

\[ l_{k,j}(x) := \prod_{m=0,m\neq j}^{k} (x - m)/(j - m). \]

An analysis of Sendov’s lemma leads [9,10] to the following

**Modified Sendov’s lemma.**  Let \( f \in C, \ k \in N \) and \( \int_0^{1/k} f(t) \, dt = 0, \ i = 1,\ldots,k. \) Then for \( x \in (0,1/k] \) we have

\[ f(ix) = \varphi_i(f,x) + \int_{1/k}^{x} \sum_{j=1}^{k} \varphi_j(f,y) \frac{j}{k} \cdot \left[ l_{k,j}(x/y)\right]_x dy, \]

where

\[ \varphi_i(f,x) := \frac{(-1)^{k-i}}{k} \int_0^x \Delta_y^k f(i(x - y)) dy. \]

The modified Sendov’s lemma implies the inequality

\[ (2) \quad W_k \leq 3. \]
Inequality (2) was independently announced by Yu. Brudnyi [3], B. Sendov [23] and the author [8].

The estimate

\[ W'_k < \text{Const} < 36 \]

was obtained by M. Takev [25]. Combination of Takev’s method with the modified Sendov’s lemma produced the inequality [13]

\[ W'_k < 5. \]

3. Recent developments

3.1. Estimates of \( W_k \). In the modified Sendov’s lemma we can replace \( f(x) \) by \( f(x) - q(x) \) and remove the condition \( f^{i/k}_0 f = 0 \), by a special choice of \( q \in P_{k-1} \):

\[ \int_0^{t} f(t) - q(t) \, dt = 0, \quad i = 1, \ldots, k. \]

It is natural to define the corresponding constants:

\[ W^*_k := \sup_{f \in C \setminus P_{k-1}} \frac{\|f - q\|}{\omega_k(f)}. \]

It is clear that \( W_k \leq W^*_k \).

**Theorem 1.** ([12, 5])

\[ W^*_k \leq 2 \quad \text{for} \quad k \leq 82000. \]

\[ W^*_k \leq 2 + \exp(-2) \quad \text{for} \quad k > 82000. \]

To prove Theorem 1, we need another modification of the Whitney–Beurling idea. To this end, put \( F(x) := \int_0^x f(u) \, du \). The following identity can be checked directly.

**Lemma 1.** ([12]) If \( m \in \{0, 1, \ldots, k\}, \ x \in I \) and \( \delta > 0 \) are such that \([x - m\delta, x + (k - m)\delta] \subset I, \) then

\[ (-1)^{k-m} \binom{k}{m} f(x) = \int_0^1 \Delta^k_{t \delta} f(x - m\delta t) \, dt \]
\[-(-1)^{k-m}\frac{1}{\delta}\binom{k}{m}(\sigma_{k-m} - \sigma_m)F(x)\]

\[-\frac{1}{\delta} \sum_{j=0, j\neq m}^{k} (-1)^{k-j}\binom{k}{j} \frac{1}{j-m} F(x + (j-m)\delta),\]

where

\[\sigma_0 := 0, \quad \sigma_m := \sum_{j=1}^{m} \frac{1}{j}, \quad m = 1, 2, \ldots.\]

The estimates of \(F\) in Lemma 1 provide the following Zhuk–Natanson identity.

**Lemma 2.** ([31, 12]) If \(F(i/k) = 0, \quad i = 1, \ldots, k\), then

\[F(x) = A_k(x) \int_{0}^{1} \Delta_{i/k} f(x(1-t)) \, dt, \quad x \in I,\]

where

\[A_k(x) := \frac{k^k}{k!} x \left( x - \frac{1}{k} \right) \left( x - \frac{2}{k} \right) \ldots \left( x - \frac{k}{k} \right) = x(-1)^k \prod_{j=1}^{k} \left( 1 - \frac{kx}{j} \right).\]

Combining Lemma 1 and Lemma 2, one can obtain the inequality \(W_k^* < \text{Const.}\). For relatively small \(k < 1000\), one can use PC for estimating the constants. But for \(k \gg 1\) we need something else. The next Shevchuk’s lemma is the appropriate tool for the estimates in case of large \(k\).

**Lemma 3.** ([5]) Let \(g := f - q\). Suppose that \(\omega_k(g) \leq 1, \quad m < k/2, \quad x \in [m/k, (m+1)/k], \quad \delta := (1-x)/(k-m)\). Then

\[\binom{k}{m} |g(x)| \leq 1 + (k\delta)^k - (-1)^{k-m}\binom{k}{m} A_k(x) + \frac{2}{\delta} \sum_{j=0}^{m-1} \binom{k}{j} \frac{1}{m-j} (|A_k(x + \delta(j-m))|).\]

Note, that the modified Sendov’s lemma implies the inequality \(|g(x)| \leq 1\) for \(x \in [1/k, 1-1/k]\). So, to prove Theorem 1, we need only the following...
Lemma 4. ([5]) For $x \in [0, 1/k)$ we have

$$(1 - x)^k - (-1)^k A_k^*(x) \leq 1 + \frac{1}{e^x},$$

and

$$(1 - x)^k - (-1)^k A_k^*(x) \leq 1, \quad k \leq 82000.$$

Remark. Theorem 1 corrects an arithmetical mistake in [12], where it was claimed that $W_k^* \leq 2$ for all $k$.

Theorem 2. ([11, 14, 29])

$$W_k^* = 1, \quad k \leq 7.$$  

It is not hard to prove that $W_k^* \geq 1$, $k \geq 1$ (see [10]). An example can be constructed by smoothing the function $f(x) = 0$, $x \neq 0$, $f(1) = 1$. The inequality $W_k^* \leq 1$ is trivial. With the intention to make the idea of the proof as clear as possible, let us consider the simple case $k = 2$. Put

$$G(x, y) := \frac{1}{y - x} \int_x^y g(t) \, dt, \quad G(x, x) := g(x)$$

and

$$\Delta_{h_1, h_2}^k G(x, y) := \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} G(x + jh_1, y + jh_2).$$

It easy to check that

$$\Delta_{h_1, h_2}^k G(x, y) = \int_0^1 \Delta_{h_1 + (t h_2 - h_1)}^k g(x + t(y - x)) \, dt.$$

We need to prove that if $\int_0^{i/2} g = \int_0^{i/2} (f - q) = 0$, $i = 1, 2$ and $\omega_2(g) \leq 1$, then $|g(x)| \leq 1$, or in other notation: if $G(0, j/2) = 0$, $j = 1, 2$ and $|\Delta_{h_1, h_2}^2 G(x, y)| \leq 1$, then we have $|g(x)| \leq 1$.

Suppose that max $|g(y)| = g(x)$ and $x < 1/3$ (the case of $x \in [1/3, 1/2]$ is more simple; in the case max $|g(y)| = -g(x)$, we can consider the function $g_1 = -g$). We have the identities:

$$g(x) = \Delta_{(1-x)/2, x}^2 G(x, x) - \frac{x}{1 - 3x} (6G(0, 2x) - G(2x, 3x) - 2G(1/2, 1/2 + x/2))$$
and
\[ g(x) = -\frac{1}{2} \Delta^2_{x,0} G(0, x) + G(0, 2x). \]

The first identity is a global estimate (with big step \( h = (1 - x)/2 \)). The second identity is a local estimate (with step \( h = x \)). Now from the local estimate we deduce
\[ G(0, 2x) \geq g(x) - 1/2. \]
Combining last inequality with the global estimate, we find that
\[ 1 \geq \left| \Delta^2_{(1-x)/2, x} G(x, x) \right| \]
\[ \geq \left| g(x) + \frac{x}{1 - 3x} (6G(0, 2x) - G(2x, 3x) - 2G(1/2, 1/2 + x/2)) \right| \]
\[ \geq \left| g(x) + \frac{x}{1 - 3x} (6(g(x) - 1/2) - g(x) - 2g(x)) \right|, \]
or
\[ g(x) \leq 1. \]

The proof of Theorem 2 for \( 2 < k < 8 \) is not simple. The main idea of the proofs, presented in [14, 29], is due to H. Whitney [27]. Put \( G(u) := \int_0^u g(t) \, dt \) and suppose, that \( G(i/k) = 0, i = 1, \ldots, k \). Let \( \max |g(y)| = g(x) > \omega_k(f) \).

Consider the identity
\[ (-1)^k \int_0^1 \Delta^k_{th + (1-t)\alpha x/2} g(x) \, dt - g(x) \]
\[ = \frac{1}{h - \alpha x/2} \left( \sum_{j=1}^{k} \frac{(-1)^j}{j} G(x + jh) - \sum_{j=1}^{k} \frac{(-1)^j}{j} G(x + j \alpha x/2) \right), \]
\[ h = (1 - x)/k, \alpha : 0 < \frac{\alpha x}{2} < h. \]

Since the left hand side of this identity is non-positive, then
\[ M_\alpha(x) := -\frac{1}{x} \sum_{j=1}^{k} \frac{(-1)^j}{j} G(x + j \alpha x/2) \leq -\frac{1}{x} \sum_{j=1}^{k} \frac{(-1)^j}{j} G(x + jh). \]

Lemma 2 implies (see [14, 29])
\[ M_\alpha(x) \leq \sigma_k - 1. \]
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The core of the proof is the following identity

\[
Ag(x) = \sum a_ig_i + \sum b_j\Delta_j^k + \sum c_lM_{\alpha(l)}(x),
\]

where \(a_i, b_j \in \mathbb{R}, \ c_k \in \mathbb{R}_+, \ g_i := G\left(\frac{x(i-1)}{2}, \frac{x}{2}\right), \ \Delta_j^k := \text{means of finite differences.} \)

We will use (3) for \(x\) near the origin (the difficult case of Theorem 2). For other \(x\), we can use some modification of (3) (see [14, 29]). We can suppose, that \(|\Delta_j^k| \leq 1\). To prove Theorem 2, it is sufficient to construct identity (3) with the constraint

\[
A > \sum |a_i| + \sum |b_j| + (\sigma_k - 1) \sum c_l.
\]

**Identity for \(k = 2\), \(x \in [0, 1/3]\),**

\[
g(x) = \frac{1}{12}(g_5 + g_6) - \frac{1}{2}\Delta_{x,0}^2G(0, x) + \frac{1}{3}M_2(x).
\]

**Identity for \(k = 3\), \(x \in [0, 1/6]\), \(g_0 := g(0)\).**

\[
Ag(x) = \sum_{i=0}^{12} a_ig_i + \sum_{j=1}^{4} b_j\Delta_j^3 + cM_2(x),
\]

\[
\Delta_1^3 = \Delta_{x/2,0}^3G(x/2, x), \quad \Delta_2^3 = \Delta_{x/2,0}^3G(0, x),
\]
\[
\Delta_3^3 = \Delta_{x,x/2}^3G(0, x/2), \quad \Delta_4^3 = \Delta_{2x,3x/2}^3G(0, 0).
\]

\[
A = \frac{396}{7}, \quad c = 12,
\]

\[
a = [4/3, 0, 0, 0, 148/7, 5/7, 5/7, 0, 0, 0, -4/9, -4/9, -4/9],
\]

\[
b = [22/7, -22/7, 88/7, 4/3].
\]

We have \(396/7 = A > \sum |a_i| + \sum |b_j| + (\sigma_3 - 1) \cdot 12 = 388/7\).

**Identity for \(k = 4\), \(x \in [0, 1/12]\).**

\[
Ag(x) = \sum_{i=1}^{22} a_ig_i + \sum_{j=1}^{12} b_j\Delta_j^4 + \sum_{l=1}^{4} c_lM_{\alpha(l+1)}(x),
\]

\[
\Delta_1^4 = \Delta_{x,0}^4G(0, x), \quad \Delta_2^4 = \Delta_{x,x/2}^4G(0, \frac{x}{2}), \quad \Delta_3^4 = \Delta_{2x,3x/2}^4G(0, \frac{x}{2}),
\]
\[
\Delta_4^4 = \Delta_{x,2x}^4G(0, \frac{x}{2}), \quad \Delta_5^4 = \Delta_{2x,3x}^4G(0, \frac{x}{2}), \quad \Delta_6^4 = \Delta_{2x,4x}^4G(\frac{x}{2}).
\]
\[ \Delta_7^4 = \Delta_{x,\bar{x}}^4 G \left( \frac{x}{2}, x \right), \quad \Delta_8^4 = \Delta_{3\frac{x}{2}, 3\frac{x}{2}}^4 G \left( \frac{x}{2}, x \right), \quad \Delta_9^4 = \Delta_{x,\bar{x}}^4 G \left( 3x, \frac{7x}{2} \right), \]

\[ \Delta_{10}^4 = \Delta_{\frac{9x}{2}, \frac{5x}{2}}^4 G \left( \frac{9x}{2}, 5x \right), \quad \Delta_{11}^4 = \Delta_{\frac{9x}{2}, \frac{5x}{2}}^4 G \left( \frac{9x}{2}, 5x \right), \quad \Delta_{12}^4 = \Delta_{\frac{15x}{2}, \frac{8x}{2}}^4 G \left( \frac{15x}{2}, 8x \right). \]

The coefficients \( a_i, b_j, c_l \), for the case \( k = 4 \) and the identities for \( x \), which are separated from the intervals endpoints, can be found in [14]. Appropriate identities for \( k = 5, 6, 7 \) were constructed by O. Zhelnov [29].

### 3.2. Estimates of \( W'_k \)

The method, proposed by M. Takev [25], intermediate approximation by polynomials \( q : \int_{0}^{1/k} (f - q) = 0, i = 1, \ldots, k \), and estimates of Lemma 3, led to the following theorem.

**Theorem 3.** ([5])

\[ W'_k \leq 3. \]

Since the inequality

\[ |f(x) - L_{k-1}(f, x)| \leq \omega_k(f), \quad x \in \left[ \frac{1}{k}, 1 - \frac{1}{k} \right], \]

is known (see for example, the estimates in [13]), we only need to prove that

\[ |f(x) - L_{k-1}(f, x)| \leq 3 \omega_k(f), \quad x \in \left[ 0, \frac{1}{k} \right]. \]

By using the notation \( g(x) := f(x) - q(x) \), we get

\[ |f(x) - L_{k-1}(f; x)| \leq |f(x) - q(x) - L_{k-1}(f, x) + q(x)| \]

\[ \leq |f(x) - q(x)| + |L_{k-1}(f - q, x)| = |g(x)| + \left| \sum_{m=0}^{k-1} g(x_m) l_{k-1, m}((k - 1)x) \right|. \]

To estimate the value of \( |g(x)| \) for \( x \in [0, 1/k) \) and at the points \( x_m, m = 0, \ldots, k - 1 \), we shall use Lemma 3.

For \( x \in [0, 1/k) \) we have the inequality

\[ |g(x)| \leq 1 + (1 - x)^k - (-1)^k A'_k(x). \]

To estimate

\[ |L_{k-1}(g, x)| = \left| \sum_{m=0}^{k-1} g(x_m) l_{k-1, m}((k - 1)x) \right|, \quad x_m = \frac{m}{k - 1}, \]
we can use the following Lemma 5.

**Lemma 5.** ([5]) Suppose that $f \in C$, $\omega_k(f) \leq 1$. Then for each $m = 0, \ldots, k - 1$, we have

$$|g(x_m)| \leq \binom{k-1}{m}^{-1} + 2(k-1)\sigma_{k-1}|A_k(x_m)|.$$

The proof of Lemma 5 is the most technical part of the paper [5]. One can use Lemma 5 to deduce Lemma 6.

**Lemma 6.** ([5]) Let $f \in C$, $k > 7$, $\omega_k(f) \leq 1$, $x \in [0, 1/k]$. Then

$$|f(x) - L_{k-1}(f, x)| \leq 2 + e(k-1)\sigma_{k-1}|A_{k-1}(x)|.$$

Since

$$e(k-1)\sigma_{k-1}|A_{k-1}(x)| \leq 1,$$

we have Theorem 3 for $k > 7$.

For $k \leq 7$, the second Sendov’s conjecture follows from Theorem 2. It was proved by Danilenko ($k = 4$) and O. Zhelnov ($k = 5, 6, 7$).

**Theorem 4.** ([6, 30]).

$$W'_k \leq 2, \quad k = 4, 5, 6, 7.$$

**References**


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[29] O. Zhelnov. Whitney constants are bounded by 1 for k = 5, 6, 7, East Journal for Approximation, Submitted.


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