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On Gauss-Lucas Theorem Concerning the Location of the Critical Points of a Polynomial

Abdul Aziz * *and B. A. Zargar* **

This article is dedicated to the 70th anniversary of Acad. Bl. Sendov

In this paper we present an interesting generalization of the Gauss-Lucas Theorem. Besides, we prove an analogous result for polynomials not vanishing in a disk. We also establish a special case of the well-known Sendov conjecture.

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1. Introduction and statement of results

Let

$$P(z) = \prod_{j=1}^{n} (z - z_j)$$

be a polynomial of degree n with complex coefficients, then concerning the location of the critical points of $P(z)$, we have the following celebrated result known as the Gauss-Lucas Theorem (for references see [3, p.22]).

Theorem A. *If all the zeros of a polynomial $P(z)$ lie in $|z - c| \leq R$, then all the zeros of $P'(z)$ lie in $|z - c| \leq R$.*

In the literature there exist some extensions and generalizations of this famous result (for example, see [2], [5], [6]). Here we first present the following generalization of Theorem A which includes the Gauss-Lucas Theorem as a special case.

Theorem 1. *If all the zeros of a polynomial $P(z)$ of degree n lie in the disk $|z-c| \leq R$ and if w is any real or complex number satisfying the inequality,*

$$|(w - c)P'(w)| \leq |(w - c)P'(w) - nP(w)|, \tag{1}$$

then

$$|w-c| \leq R. \tag{2}$$

If a strict inequality holds in (1), then the strict inequality holds in (2) also.

R e m a r k. The Gauss-Lucas Theorem is a special case of Theorem 1. If all the zeros of $P(z)$ lie in the circle $|z-c| \leq R$ and w is a critical point of $P(z)$, then $P'(w) = 0$ so that inequality (1) is trivially satisfied. Hence by Theorem 1 we have $|w-c| \leq R$. This shows that all the zeros of $P'(z)$ also lie in $|z-c| \leq R$.

Next, we prove the following result for polynimoials not vanishing in a disk.

Theorem 2. *If all the zeros of a polynomial $P(z)$ of degree n lie in the region $|z-c| \geq R$ and if w is any real or complex number satisfying the inequality*

$$|(w-c)P'(w)| \geq |(w-c)P'(w) - nP(w)|, \tag{3}$$

then

$$|w-c| \geq R. \tag{4}$$

If a strict inequality holds in (3), then the strict inequality holds in (4) also.

The Gauss-Lucas Theorem pertains to the regional location of the critical points of a polynomial $P(z)$ when all zeros of $P(z)$ are known. Concerning the location of critical points of a polynomial $P(z)$ relative to each individual zero of $P(z)$, we have the following conjecture.

Conjecture. *If*

$$P(z) = (z-a)\prod_{j=1}^{n-1}(z-z_j), \quad 0 \leq a \leq 1$$

is a polynomial of degree n, such that $|z_j| \leq 1$, $j = 1, 2, \ldots, n-1$, then the disk $\{z : |z-a| \leq 1\}$ must contain at least one of zero of $P'(z)$.

The above conjecture is due to Bulgarian mathematician Blagovest Sendov. This conjecture entered the literature by mistake as "Ilief's conjecture". The shortest explanation of its origin is given by M. Marden [4]. In his autobiographical note, Sendov [8] explains how it has been formulated and why it has been attributed to L. Ilief. In this paper, Sendov gives a comprehensive review of problems from the Geometry of Polynomials related to this conjecture.

In full generality, the conjectured result has been proved only for polynomials of degree ≤ 8. It has also been verified for some special classes of polynomials (see [8]). It is known (see [1, Theorem 4, with $\alpha = a$]) that if $P(z) = (z-a)Q(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$ and if

$$\Re \frac{aQ'(a)}{Q(a)} \geq \frac{n-1}{2},$$

then $P'(z)$ has at least one zero in $|z - a/2| \leq |a/2|$, and hence Sendov's conjecture is true in this case. Here we present the following result which is an interesting special case of Sendov's conjecture.

Theorem 3. *Let*

$$P(z) = (z-a)Q(z), \quad 0 \leq a \leq 1$$

be a polynomial of degree n having all its zeros in $|z| \leq 1$, if

$$\Re \frac{aQ'(a)}{Q(a)} \leq \frac{(n-1)a}{4}, \tag{5}$$

or

$$\left| \frac{Q'(a)}{Q(a)} \right| \geq \frac{n-1}{2}, \tag{6}$$

then $P'(z)$ has at least one zero in the disk $|z-a| \leq 1$.

For Theorem 3, we need the following result which is of independent interest.

Lemma. *If $P(z)$ is a polynomial of degree n and $z_1, z_2, \ldots, z_n$ are the zeros of $P(z)$, then for every real or complex number $t \neq z_j$, $j = 1, 2, \ldots, n$, we have*

$$2\Re \frac{tP'(t)}{P(t)} = n + \sum_{j=1}^{n} \frac{|t|^2 - |z_j|^2}{|t - z_j|^2}.$$

2. Proofs

Proof of Lemma. The lemma is trivial if $t = 0$. We assume $t \neq 0$. Since

$$P(z) = c \prod_{j=1}^{n} (z - z_j)$$

therefore, if $t \neq z_j$, $j = 1, 2, \ldots, n$, we have

$$\frac{tP'(t)}{P(t)} = \sum_{j=1}^{n} \frac{t}{t - z_j} = \sum_{j=1}^{n} \frac{t - z_j + z_j}{t - z_j}$$

$$= n + \sum_{j=1}^{n} \frac{z_j}{t - z_j} = n + \sum_{j=1}^{n} \frac{z_j(\bar{t} - \bar{z}_j)}{|t - z_j|^2}$$

$$= n + \sum_{j=1}^{n} \frac{z_j \bar{t}}{|t - z_j|^2} - \sum_{j=1}^{n} \frac{|z_j|^2}{|t - z_j|^2} . \tag{7}$$

Also

$$\overline{\frac{tP'(t)}{P(t)}} = \sum_{j=1}^{n} \overline{\frac{t}{t - z_j}} = \sum_{j=1}^{n} \frac{\bar{t}}{\bar{t} - \bar{z}_j}$$

$$= \sum_{j=1}^{n} \frac{\bar{t}(t - z_j)}{|t - z_j|^2} = \sum_{j=1}^{n} \frac{|t|^2 - \bar{t} z_j}{|t - z_j|^2}$$

$$= \sum_{j=1}^{n} \frac{|t|^2}{|t - z_j|^2} - \sum_{j=1}^{n} \frac{\bar{t} z_j}{|t - z_j|^2} . \tag{8}$$

Adding (7) and (8), we get

$$2\Re \frac{tP\prime(t)}{P(t)} = \frac{tP'(t)}{P(t)} + \overline{\frac{tP'(t)}{P(t)}}$$

$$= n + \sum_{j=1}^{n} \frac{|t|^2}{|t - z_j|^2} - \sum_{j=1}^{n} \frac{|z_j|^2}{|t - z_j|^2}$$

$$= n + \sum_{j=1}^{n} \frac{|t|^2 - |z_j|^2}{|t - z_j|^2} .$$

This completes the proof of lemma. ■

P r o o f o f T h e o r e m 1. Let w be any real or complex number satisfying the inequality

$$|(w - c)P'(w)| \leq |nP(w) - (w - c)P'(w)|. \tag{9}$$

If w is a zero of $P(z)$, then clearly $|w-c| \leq R$. Henceforth, we suppose that $P(w) \neq 0$. Now inequality (9) can be written in the form

$$\left|\frac{(w-c)P'(w)}{nP(w)}\right| \leq \left|1 - \frac{(w-c)P'(w)}{nP(w)}\right|$$

which implies

$$\Re\frac{(w-c)P'(w)}{P(w)} \leq \frac{n}{2}. \tag{10}$$

Let $z_1, z_2, \ldots, z_n$ be the zeros of $P(z)$, then by hypothesis,

$$|z_j - c| \leq R \quad \text{for all } j = 1, 2, \ldots, n$$

and we have

$$\frac{(w-c)P'(w)}{P(w)} = \sum_{j=1}^{n} \frac{(w-c)}{w-z_j}.$$

This implies with the help of (10) that

$$\sum_{j=1}^{n} \Re\left[\frac{w-c}{w-z_j}\right] = \Re\sum_{j=1}^{n} \frac{w-c}{w-z_j}$$

$$= \Re\frac{(w-c)P'(w)}{P(w)} \leq \frac{n}{2},$$

from which it follows that

$$\Re\frac{w-c}{w-z_j} \leq \frac{1}{2} \quad \text{for at least one } j = 1, 2, \ldots, n. \tag{11}$$

This gives

$$\left|\frac{w-c}{w-z_j}\right| \leq \left|1 - \frac{w-c}{w-z_j}\right| = \left|\frac{z_j - c}{w-z_j}\right| \quad \text{for at lest one } j = 1, 2, \ldots, n,$$

which gives

$$|w-c| \leq |z_j - c| \quad \text{for at least one } j = 1, 2, \ldots, n.$$

Using now the fact that

$$|z_j - c| \leq R \quad \text{for all } j = 1, 2, \ldots, n,$$

we get

$$|w - c| \leq R. \tag{12}$$

If strict inequality holds in (9), then clearly strict inequality holds in (10) and hence strict inequality holds in (11) also. This shows that the strict inequality holds in (12), too. This completes the proof of Theorem 1. ∎

P r o o f o f T h e o r e m 2. Let w be any real or complex number satisfying the inequality

$$|(w - c)P'(w)| \geq |nP(w) - (w - c)P'(w)|. \tag{13}$$

If w is a zero of $P(z)$, then clearly $|w - c| \geq R$. Henceforth, we suppose that $P(w) \neq 0$. Now inequality (13) can be written in the form

$$\left|\frac{(w - c)P'(w)}{nP(w)}\right| \geq \left|1 - \frac{(w - c)P'(w)}{nP(w)}\right|$$

which implies,

$$\Re\frac{(w - c)P'(w)}{P(w)} \geq \frac{n}{2}. \tag{14}$$

Let $z_1, z_2, \ldots, z_n$ be the zeros of $P(z)$, then by hypothesis,

$$|z_j - c| \geq R \quad \text{for all } j = 1, 2, \ldots, n$$

and we have

$$\frac{(w - c)P'(w)}{P(w)} = \sum_{j=1}^{n} \frac{(w - c)}{w - z_j}.$$

This implies with the help of (14) that

$$\sum_{j=1}^{n} \Re\left|\frac{w - c}{w - z_j}\right| = \Re\sum_{j=1}^{n} \frac{w - c}{w - z_j}$$

$$= \Re\frac{(w - c)P'(w)}{P(w)} \geq \frac{n}{2},$$

$$\Re\frac{w - c}{w - z_j} \geq \frac{1}{2} \quad \text{for at least one } j = 1, 2, \ldots, n. \tag{15}$$

This gives,

$$\left|\frac{w-c}{w-z_j}\right| \geq \left|1-\frac{w-c}{w-z_j}\right|$$

$$= \left|\frac{z_j-c}{w-z_j}\right| \qquad \text{for at least one } j=1,2,\ldots,n,$$

which gives

$$|w-c| \geq |z_j-c| \qquad \text{for at least one } j=1,2,\ldots,n.$$

Using now the fact that

$$|z_j-c| \geq R \qquad \text{for all } j=1,2,\ldots,n,$$

we get

$$|w-c| \geq R. \tag{16}$$

As in the proof of Theorem 1, it can be easily seen that if the strict inequality holds in (13), then the strict inequality also holds in (16). This completes the proof of Theorem 2. ■

Proof of Theorem 3. If $a=0$, the result follows by the Gauss-Lucas Theorem and for $a=1$, the result has been proved by Rubinstein [7]. Henceforth we assume $0<a<1$. Since

$$P(z)=(z-a)Q(z),$$

we have

$$P'(z)=(z-a)Q'(z)+Q(z)$$

and

$$P''(z)=(z-a)Q''(z)+2Q'(z).$$

This gives,

$$P'(a)=Q(a) \quad \text{and } P''(a)=2Q'(a). \tag{17}$$

If $P'(a)=0$, then $z=a$ is a zero of $P'(z)$ also. Since this zero lies in the circle defined by (6), the result follows in this case. We may now suppose that $z=a$ is a simple zero of $P(z)$ so that $P'(a)\neq 0$ which implies $Q(a)\neq 0$ and thus, $a\neq z_j$ for any $j=1,2,\ldots,n-1$.

If $w_1, w_2, \dots, w_{n-1}$ are the zeros of $P'(z)$, then clearly $a \neq z_j$, $j = 1, 2, \dots, n-1$ and by the above lemma we have

$$2\Re\frac{aP''(a)}{P'(a)} = (n-1) + \sum_{j=1}^{n-1} \frac{|a|^2 - |w_j|^2}{|a-w_j|^2}. \tag{18}$$

By hypothesis,

$$\Re\frac{aQ'(a)}{Q(a)} \leq \frac{(n-1)}{4}a^2$$

and therefore

$$\Re\frac{2aQ'(a)}{Q(a)} \leq \frac{n-1}{2}a^2.$$

This gives,

$$\Re\frac{aP''(a)}{P'(a)} \leq \frac{n-1}{2}a^2.$$

Using this in (18), it follows that

$$(n-1) + \sum_{j=1}^{n-1} \frac{(a)^2 - |w_j|^2}{|a-w_j|^2} \leq (n-1)(a)^2$$

which gives

$$\sum_{j=1}^{n-1} \frac{|w_j|^2 - a^2}{|a-w_j|^2} \geq (n-1)(1+a^2).$$

This implies,

$$\frac{|w_j|^2 - a^2}{|w_j - a|^2} \geq 1 - a^2 \quad \text{for at least one } \ j = 1, 2, \dots, n-1$$

from which it follows that

$$(1-a^2)|w_j - a|^2 + a^2 \leq |w_j|^2 \quad \text{for at least one } \ j = 1, 2, \dots, n-1. \tag{19}$$

From (19), we conclude with the help of the Gauss-Lucas Theorem that

$$|w_j - a| \leq 1, \quad \text{for at least one } \ j = 1, 2, \dots, n-1.$$

This shows that $P(z)$ has at least one zero in

$$|z-a| \leq 1$$

and this proves the first part of Theorem 3. To establish the second part of Theorem 3, assume that

$$\left|\frac{Q'(a)}{Q(a)}\right| \geq \frac{n-1}{2}.$$

Since $w_1, w_2, \ldots, w_{n-1}$ are the zeros of $P'(z)$, we have

$$\left|\frac{1}{a-w_1} + \frac{1}{a-w_2} + \ldots + \frac{1}{a-w_{n-1}}\right|$$

$$= \left|\frac{P''(a)}{P'(a)}\right| = \left|\frac{2Q'(a)}{Q(a)}\right| \geq n-1.$$

Which gives,

$$\left|\frac{1}{a-w_j}\right| \geq 1 \quad \text{for at least one} \quad j = 1, 2, \ldots, n-1.$$

Evidently,

$$|w_j - a| \leq 1 \quad \text{for at least one} \quad j = 1, 2, \ldots, n-1.$$

This proves Theorem 3 completely. ■

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* *Post-Graduate Dept. of Mathematics & Statistics*
University of Kashmir
Srinagar, 190 006 INDIA

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** *Islamia College of Science and Commerce,*
Srinagar, INDIA