Mathematica
Balkanica

New Series Vol. 16, 2002, Fasc. 1-4

On the Compactness Theorem for Sequences of Closed Sets

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*Dedicated to Professor Blagovest Sendov on his 70th anniversary
in recognition of his contributions
to mathematical analysis and approximation theory*

We display the equivalence of the standard compactness theorem for sequences of closed sets in a separable metric space with a version of the Arzela-Ascoli Theorem valid in that setting. This in turn leads to a study of pointwise convergence of distance functions more generally and its relation to both Kuratowski convergence and Wijsman convergence.

AMS Subj. Classification: Primary 54B20

Key Words: Kuratowski convergence, Wijsman convergence, Arzela-Ascoli theorem, metric spaces with nice closed balls, graph convergence

1. Introduction

Let $A_1, A_2, A_3, \ldots$ be a sequence of closed sets in a metric space $\langle X, d\rangle$. Define the *upper* and *lower closed limits* $\mathrm{Ls}A_n$ and $\mathrm{Li}A_n$ of the sequence as follows:

$$\mathrm{Ls}A_n = \{x \in X : \text{ each ball with center } x \text{ hits } A_n \text{ frequently}\};$$

$$\mathrm{Li}A_n = \{x \in X : \text{ each ball with center } x \text{ hits } A_n \text{ eventually}\}.$$

Clearly, $\mathrm{Li}A_n \subset \mathrm{Ls}A_n$ and it is easy to show that both the upper and lower closed limits are closed subsets of X (though possibly empty). When the reverse inclusion also holds, we get $\mathrm{Li}A_n = \mathrm{Ls}A_n$ and in this case $\langle A_n\rangle$ is declared *Kuratowski convergent* to their common value. Denoting this common value by A, we write $A = K - \lim A_n$. Alternatively, in the literature, convergence in this sense may be called Painlevé-Kuratowski convergence, topological convergence, or closed convergence.

The following classical compactness theorem for sequences of closed sets in a separable metric space has been attributed to Zarankiewicz [16]:

Set Compactness Theorem. *Let $\langle A_n \rangle$ be a sequence of closed sets in a separable metric space $\langle X, d\rangle$. Then $\langle A_n \rangle$ has a Kuratowski convergent subsequence.*

The most frequently cited proof of this result involves a diagonalization process (see, e.g., [1, 11]). There is, however, a less known extremely clever proof, due to Mrowka [14], based on the observation that $\langle A_n \rangle$ is Kuratowski convergent provided whenever $\langle A_n \rangle$ hits a basic open set V frequently, then $\langle A_n \rangle$ hits V eventually. The argument goes like this. Each set A_n is associated with a function $f_n : Z^+ \to \{0, 1\}$ defined by

$$f_n(j) = \begin{cases} 1 & \text{if } B_j \cap A_n \neq \emptyset, \\ 0 & \text{if } B_j \cap A_n = \emptyset, \end{cases}$$

where $\{B_j : j \in Z^+\}$ is a countable base for the topology, and the sequential compactness of a countable product of copies of $\{0, 1\}$ is invoked to immediately obtain the desired result. If we accept the continuum hypotheses, one can construct a sequence with no convergent subsequence in each nonseparable metric space (see e.g., [4, page 150]). For results involving general topological spaces, the reader may consult [6].

A main purpose of this note is to exhibit the fundamental equivalence of this sequential compactness theorem for sets with the usual version of the Arzela-Ascoli Theorem valid in separable metric spaces. We also gain some further insight into the relationship between Kuratowski convergence and pointwise convergence of distance functions for the sets in the sequence.

2. Notation and terminology

Let $\langle X, d\rangle$ be a metric space. We denote the family of nonempty closed subsets of X by $\mathrm{CL}(X)$. If $x \in X$ and $\varepsilon > 0$, we write $S_\varepsilon[x]$ for the open ball with center x in X and radius ε. For $A \in \mathrm{CL}(X)$, we write $d(\cdot, A)$ for the distance function for the set A defined by $d(x, A) = \inf\{d(x, a) : a \in A\}$. A sequence $\langle A_n \rangle$ in $\mathrm{CL}(X)$ is declared *Wijsman convergent* to $A \in \mathrm{CL}(X)$ provided $\langle d(\cdot, A_n)\rangle$ converges pointwise to $d(\cdot, A)$. As is well-known, Wijsman convergence in $\mathrm{CL}(X)$ forces Kuratowski convergence [2,3,4,7].

If $\langle Y, \rho\rangle$ is another metric space, we write $C(X, Y)$ for the continuous functions from X to Y. For definiteness, we equip the metrizable product topology on $X \times Y$ with the *product metric*

$$D((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), \rho(y_1, y_2)\}.$$

Recall that a family Ω of functions from a metric space $\langle X, d\rangle$ to $\langle Y, \rho\rangle$ is called *pointwise equicontinuous* provided for each $x \in X$ and $\varepsilon > 0$, there exists $\delta = \delta(x, \varepsilon) > 0$ such that for each $f \in \Omega$, we have $d(x, w) < d \Rightarrow \rho(f(x), f(w)) < \varepsilon$.

With this in mind, the classical Arzela-Ascoli theorem for sequences may now be stated (see e.g., [13]):

Arzela-Ascoli Theorem. *Let $\langle f_n\rangle$ be a sequence of continuous functions from a separable metric space $\langle X, d\rangle$ to a metric space $\langle Y, \rho\rangle$. Suppose $\{f_n : n \in Z^+\}$ has these properties:*
(1) $\{f_n : n \in Z^+\}$ is pointwise equicontinuous;
(2) $\forall x \in X, \{f_n(x) : n \in Z^+\}$ has compact closure in Y.
Then $\langle f_n\rangle$ has a subsequence convergent uniformly on compacta to a continuous function f from X to Y.

3. Results

We first obtain the Arzela-Ascoli Theorem from the compactness theorem for sequences of sets. Of course, the sets in question will be the graphs of our functions!

Lemma 3.1. *Let $\{f_n : n \in Z^+\}$ be a family of continuous functions from a separable metric space $\langle X, d\rangle$ to a metric space $\langle Y, \rho\rangle$. Then $E = \{(x, y) : x \in X, y \in \{f_n(x) : n \in Z^+\}\}$ is a separable subspace of $X \times Y$.*

Proof. Let $A = \{a_i : i \in Z^+\}$ be a countable dense subset of X. We claim that the countable set $\{(a_i, f_n(a_i)) : i \in Z^+, n \in Z^+\}$ is dense in E. Fix $(x, f_k(x)) \in E$ and let $\varepsilon > 0$. Choose $\delta < \varepsilon$ such that $d(x, w) < \delta \Rightarrow \rho(f_k(x), f_k(w)) < \varepsilon$. Choose $a_i \in A$ with $d(a_i, x) < \delta$ so that $\rho(f_k(x), f_k(a_i)) < \varepsilon$. The definition of the product metric D now gives $D((a_i, f_k(a_i)), (x, f_k(x))) < \max\{\delta, \varepsilon\} = \varepsilon$. ■

It is known that for pointwise equicontinuous subsets of $C(X, Y)$, pointwise convergence, uniform convergence on compacta, and Kuratowski convergence all coincide for sequences [10]. But there is no need to assume continuity of the limit in this result.

We sketch a proof.

Lemma 3.2. *Let $\{f_n : n \in Z^+\}$ be a pointwise equicontinuous family of functions from $\langle X, d\rangle$ to $\langle Y, \rho\rangle$ and let $f : X \to Y$ be a function. Let $G_n = \{(x, f_n(x)) : x \in X\}$ and let $G = \{(x, f(x)) : x \in X\}$. The following are equivalent:*
(1) $\langle f_n\rangle$ converges pointwise to f;
(2) $G = K - \lim G_n$;

(3) $\langle f_n \rangle$ converges uniformly to f on compact subsets.
Furthermore, in each case, the limit function f is continuous.

Proof. The implication (1) $\Rightarrow$ (3) is established in the course of the standard proof of the Arzela-Ascoli Theorem as well as continuity of the limit [13, pp. 505-506]. For (3) $\Rightarrow$ (2), since uniform convergence on compacta preserves continuity, it is equivalent to *continuous convergence* [9, p.268]: whenever $\langle x_n \rangle \to x$ we have $\langle f_n(x_n) \rangle \to f(x)$. It now easily follows that $\text{Ls}G_n \subset G \subset \text{Li}G_n$. For (2) $\Rightarrow$ (1), fix $x \in X$and let $\varepsilon > 0$. Choose $d < \varepsilon/2$ such that $d(x,w) < \delta \Rightarrow \rho(f_n(x), f_n(w)) < \varepsilon/2$ for all indices n. Since $(x, f(x)) \in G = \text{Li}G_n$, there exists $j \in Z^+$ and for each $n \geq j$ a point $x_n \in X$ with $D((x, f(x)), (x_n, f_n(x_n)) < \delta$. It now follows from the triangle for ρ that for all $n \geq j$, $\rho(f(x), f_n(x)) < \delta + \varepsilon/2 < \varepsilon$. ■

A noteworthy application of (1) $\Longleftrightarrow$ (2) in Lemma 3.2 occurs in functional analysis (for a different approach, see [4, §5.2]).

Theorem 3.3. *Let X be a Banach space and let $f, f_1, f_2, f_3, \ldots$ be continuous linear functionals on X. Then $\langle f_n \rangle$ is weak* convergent to f if and only if the associated sequence of graphs $\langle G_n \rangle$ is Kuratowski convergent to the graph G of f.*

Proof. Weak* convergence is nothing but pointwise convergence, in which case for each x the set of values $\{f_n(x) : n \in Z^+\}$ is bounded. By the uniform boundedness principle [15] $\{\|f_n\| : n \in Z^+\}$ is bounded by some number $\alpha > 0$. Then α serves as a Lipschitz constant for all the functions, whence $\{f_n : n \in Z^+\}$ is uniformly equicontinuous. On the other hand, if $G = K - \lim G_n$ yet $\{\|f_n\| : n \in Z^+\}$ was not bounded above, we could find $n(1) < n(2) < n(3) < \ldots$ and for each k vector x_k of norm at most one with $f_{n(k)}(x_k) = k$. As a result, for all k $(k^{-1}x_k, 1) \in G_{n(k)}$ which puts $(\theta, 1) \in \text{Ls}G_n = G$, where θ is the origin of the Banach space. This can not happen as $(\theta, 0) \in G$. Again, the sequence of linear functionals is equicontinuous. The result now follows from the equivalence of conditions (1) and (2) under pointwise equicontinuity. ■

Proof of the Arzela-Ascoli theorem from the set compactness theorem

Proof. For each function f_n in our sequence let $G_n = \{(x, f_n(x)) : x \in X\}$. Each of these closed sets sits in the separable subspace E described in the statement of Lemma 3.1. Since cl$E$ is separable and contains each G_n, we can find a subsequence $\langle G_{n(k)} \rangle$ Kuratowski convergent not just in cl E but also in the larger space $X \times Y$ to a closed subset G of cl E.

First, we observe that for each $x \in X$, $G = \text{Ls}G_{n(k)}$ contains each cluster point of the sequence $\langle (x, f_{n(k)}(x)) \rangle$, and the set of such cluster points is nonempty because $\{f_n(x) : n \in Z^+\}$ has compact closure. Equicontinuity now ensures that for each $x \in X$ there is a unique $y \in Y$ with $(x, y) \in G$. To see this, suppose to the contrary that $(x, y_1) \in G$ and $(x, y_2) \in G$ with $y_1 \neq y_2$. Set $\varepsilon = \rho(y_1, y_2)$ and choose $\delta < \varepsilon/3$ by pointwise equicontinuity such that

$$\{w_1, w_2\} \in S_\delta[x] \Rightarrow \forall n \;\; \rho(f_n(w_1), f_n(w_2)) < \varepsilon/3.$$

Since $(x, y_i) \in \mathrm{Li}G_{n(k)}$ for $i = 1, 2$ there exists a large integer j and points w_1, w_2 such that both

$$D((x, y_1), (w_1, f_j(w_1)) < \delta \text{ and } D((x, y_2), (w_2, f_j(w_2)) < \delta.$$

It follows from the triangle inequality for ρ that $\rho(y_1, y_2) < \varepsilon$, a contradiction.

We now know that G is the graph of a function f. Applying Lemma 3.2, the limit function is continuous and uniform convergence on compacta ensues. ■

To obtain the compactness theorem for closed sets from the Arzela-Ascoli Theorem, we need a third lemma.

Lemma 3.4. *Let $\langle X, d\rangle$ be a metric space and let $A_1, A_2, A_3, \ldots$ be a sequence of nonempty closed subsets. Suppose $\langle d(\cdot, A_n)\rangle$ is pointwise convergent to a finite-valued function. Then $\langle A_n\rangle$ is Kuratowski convergent.*

Proof. We need only show $\mathrm{Ls}A_n \subset \mathrm{Li}A_n$. Fix $x \in \mathrm{Ls}A_n$ and let $\varepsilon > 0$. By Cauchyness of $\langle d(x, A_n)\rangle$ choose $K \in Z^+$ such that $n > m \geq K \Rightarrow |d(x, A_n) - d(x, A_m)| < \varepsilon/2$. Then choose $j > K$ such that $S_{\varepsilon/2}[x] \cap A_j \neq \emptyset$. For this j we have $d(x, A_j) < \varepsilon/2$. As a result for all $n \geq K$ we have $d(x, A_n) \leq d(x, A_j) + |d(x, A_n) - d(x, A_j)| < \varepsilon$. This means $\forall n \geq K$, we have $S_\varepsilon[x] \cap A_n \neq \emptyset$. By definition we get $x \in \mathrm{Li}A_n$ and this completes the proof. ■

For a sequence $\langle A_n\rangle$ in $\mathrm{CL}(X)$ we have

$$\mathrm{Ls}A_n = \{x : \liminf_{n\to\infty} d(x, A_n) = 0\},$$

and

$$\mathrm{Li}A_n = \{x : \limsup_{n\to\infty} d(x, A_n) = 0\}.$$

Thus, the pointwise convergence of $\langle d(\cdot, A_n)\rangle$ to a (continuous) function f makes $\{x : f(x) = 0\}$ the Kuratowski limit of $\langle A_n\rangle$. This does not mean that Wijsman convergence of $\langle A_n\rangle$ to $\{x : f(x) = 0\}$ occurs whenever the zero set is nonempty. We look more closely at this situation in §4 below.

Proof of the set compactness theorem from the Arzela-Ascoli Theorem

Proof. If infinitely many of the sets A_n are empty, then there is a constant subsequence each term of which is the empty set. This is obviously convergent. Thus, we may assume by passing to a subsequence that $\forall n A_n \neq \emptyset$. Fix $x_0 \in X$. Now for this sequence of sets, there are two mutually exclusive possibilities:

(i) $\forall j, k$ there exists $n \geq k$ with $S_j[x_0] \cap A_n = \emptyset$;

(ii) $\exists j, k$ such that for all $n \geq k$ we have $S_j[x_0] \cap A_n \neq \emptyset$.

In the case (i) we can find $n(1) < n(2) < \ldots$ such that for each j, $S_j[x_0] \cap A_{n(j)} = \emptyset$. Then actually $\langle A_{n(j)} \rangle$ is eventually outside each ball in X, because each ball lies inside $S_j[x_0]$ for some sufficiently large j. Thus, $\mathrm{Ls} A_{n(j)} = \emptyset$ and so $\emptyset = K - \lim A_{n(j)}$. In the case (ii) we verify that the associated sequence of distance functions satisfies the hypotheses of the Arzela-Ascoli Theorem.

(1) The sequence $\langle d(\cdot, A_n) \rangle$ is equicontinuous because each distance function is 1-Lipschitz: $\forall \varepsilon > 0 \;\; \forall n \;\; d(x, w) < \varepsilon \Rightarrow |d(x, A_n) - d(w, A_n)| \leq d(x, w) < \varepsilon$.

(2) The sequence $\langle d(\cdot, A_n) \rangle$ is pointwise bounded. To see this, fix $x \in X$. Since $S_j[x_0] \cap A_n \neq \emptyset$ for all $n \geq k$, we have for all such n $d(x, A_n) \leq d(x, x_0) + j$. As a result $\{d(x, A_n) : n \in Z^+\}$ is bounded above by

$$d(x, x_0) + j + \max\{d(x, A_n) : n < k\},$$

and so the values of $\{d(x, A_n) : n \in Z^+\}$ lie in an interval.

Applying the Arzela-Ascoli Theorem, $\langle d(\cdot, A_n) \rangle$ has a pointwise convergent subsequence (in fact, uniformly convergent on compacta) to a finite-valued function. From Lemma 3.4, the associated subsequence of sets is Kuratowski convergent. ∎

4. On pointwise convergence of distance functions

Pointwise convergence of $\langle d(\cdot, A_n) \rangle$ for a sequence in $\mathrm{CL}(X)$ to a finite limit forces Kuratowski convergence of the underlying sequence of sets. Also, if for some x we have $\lim_{n \to \infty} d(x, A_n) = \infty$, then this is true for all x, and we have $\emptyset = K - \lim A_n$. But Kuratowski convergence need not force pointwise convergence of the distance functions, and even when pointwise convergence occurs to a finite limit, the limit need not be a distance function. One of the more remarkable results in the theory of set convergence, due to Costantini, Levi and Pelant [8], says that given a particular sequence $\langle A_n \rangle$ Kuratowski convergent to a nonempty set A, one can always find an equivalent metric for the space so that Wijsman convergence occurs, and when $A = \emptyset$, one can find an equivalent metric ρ such that $\langle \rho(., A_n) \rangle$ converges pointwise to the function identically equal to one. More recently, in the case that $A = \emptyset$, this author has shown that an equivalent metric ρ exists for which $\lim_{n\to\infty} \rho(x, A_n) = \infty$ for each $x \in X$, [5].

In this section we further clarify the situation. First we give some simple but illuminating examples that show what can go wrong.

Example. Let $X = \{x_1, x_2, x_3, \ldots\}$ and $Y = \{y_1, y_2, y_3, \ldots\}$ be two disjoint countable sets. On their union W, define a metric d according to these rules:

$$d(x_i, x_j) = 1 \quad \text{if} \quad i \neq j; \quad d(y_i, y_j) = 2 \quad \text{if} \quad i \neq j; \quad d(x_i, y_j) = 3.$$

For the record, we note the space is locally compact, complete and separable. Let $\langle A_n \rangle$ be the following sequence of singleton subsets:

$$A_n = \begin{cases} \{x_{n/2}\} & \text{if } n \text{ is even }, \\ \{y_{(n+1)/2}\} & \text{if } n \text{ is odd }. \end{cases}$$

Clearly, $\langle A_n \rangle$ is Kuratowski convergent to the empty set whereas $\langle d(\cdot, A_n) \rangle$ converges nowhere pointwise.

Next in the same space, define a sequence $\langle B_n \rangle$ by

$$B_n = \begin{cases} Y \cup \{x_n\} & \text{if } n \text{ is even }, \\ Y & \text{if } n \text{ is odd }. \end{cases}$$

As $\langle B_n \rangle$ is Kuratowski convergent to Y, $\langle d(\cdot, B_n) \rangle$ must minimally converge pointwise on Y. For all points x not in Y, $d(x, B_n)$ eventually oscillates between 1 and 3, so pointwise convergence only occurs on Y.

Finally, to illustrate pointwise convergence of distance functions to a finite limit that is not a distance function, for each $n \in Z^+$ let $C_n = \{y_1, x_n, x_{n+1}, x_{n+2}, \ldots\}$. The sequence $\langle d(\cdot, C_n) \rangle$ converges poinwise to a function f defined by

$$f(x) = \begin{cases} 1 & \text{if } x = x_n \text{ for some } n, \\ 0 & \text{if } x = y_1 \text{ if } x = y_1, \\ 2 & \text{if } x = y_n \text{ for some } n > 1. \end{cases}$$

If f were a distance function for a closed set C, then $C = \{y_1\}$. But for each n we have $d(x_n, \{y_1\}) = 3 \neq f(x_n)$. Thus the pointwise limit of a sequence of distance functions need not be a distance function, even if the associated sequence of sets is Kuratowski convergent to a nonempty set.

Example. If X is not complete, stronger forms of convergence of distance functions do not ensure that the limit function is a distance function even when the zero set of the limit is nonempty. In $(0,1]$ as a metric subspace of $\mathbf{R}$, let $A_n = \{1/n, 1\}$. For each n we have

$$d(x, A_n) = \begin{cases} |x - \frac{1}{n}| & \text{if } x \leq \dfrac{n+1}{2n}, \\ 1 - x & \text{if } x \geq \dfrac{n+1}{2n}. \end{cases}$$

Clearly, $\langle d(\cdot, A_n)\rangle$ converges uniformly to the function $f(x) = -|x - \frac{1}{2}| + \frac{1}{2}$ which is not a distance function on $(0,1]$.

Let $\langle X, d\rangle$ be a metric space. It is known that Kuratowski convergence of sequences in $\mathrm{CL}(X)$ to a nonempty limit forces Wijsman convergence if and only if $\langle X, d\rangle$ has *nice closed balls* [3]: if B is a noncompact closed ball in X, then $B = X$. It is obvious that a metric space with nice closed balls is locally compact. Conversely, given a locally compact metrizable space X, using a paracompactness argument we can construct a compatible metric d such that for each $x \in X$ the ball $S_1[x]$ has compact closure [4; Lemma 3.3.11]. Then $\rho = \min\{d, 1\}$ is also a compatible metric with nice closed balls.

A related characterization of spaces with nice closed balls is given by the following theorem.

Theorem 4.1. *Let $\langle X, d\rangle$ be a metric space. The following are equivalent:*

(1) $\langle X, d\rangle$ has nice closed balls;

(2) Whenever $\langle A_n\rangle$ is a sequence in CL(X) Kuratowski convergent to a possibly empty set A, then $\langle d(\cdot, A_n)\rangle$ converges pointwise to an extended real valued function;

(3) Whenever $\langle A_n\rangle$ is a sequence in CL(X) Kuratowski convergent to a possibly empty set A and $\liminf d(x_0, A_n) < \infty$ *for some $x_0 \in X$, then $\langle d(\cdot, A_n)\rangle$ converges pointwise to a finite-valued function.*

(4) Whenever $\langle A_n\rangle$ is a sequence in CL(X) Kuratowski convergent to $A \in$ CL(X), then $\langle d(\cdot, A_n)\rangle$ converges pointwise.

Proof. (1) $\Rightarrow$ (2). For each $x \in X$ let $\lambda(x) = \sup\{d(w, x) : w \in X\}$. If $\mu < \lambda(x)$ and A_n hit $S_\mu[x]$ frequently - a set with compact closure - then $\mathrm{Ls}A_n \neq \emptyset$. Thus, $\langle A_n\rangle$ is Kuratowski convergent to a nonempty set, and Wijsman convergence of $\langle A_n\rangle$ follows [3, Theorem 2.3]. The only other possiblity is for $\lim d(x, A_n) = \lambda(x)$ and we still have pointwise convergence of distance functions.

(2) ⇒ (3). The condition $\liminf d(x_0, A_n) < \infty$ for some $x_0 \in X$ implies $\liminf d(x, A_n) < \infty$ for all $x \in X$. Thus, the limit function whose existence is guaranteed by (2) must be finite valued.

(3) ⇒ (4). This is obvious taking $x_0 \in A$.

(4) ⇒ (1). Suppose X does not have nice closed balls. Then X has a noncompact ball $B = \{x : d(x, x_0) \leq \lambda\}$ that is a proper subset of X. Take $y \in B^c$ and a sequence $\langle x_n \rangle$ in B with no cluster point. For each $n \in Z^+$ set $A_n = \{y, x_n\}$ if n is even and $A_n = \{y\}$ if n is odd. Then each A_n is closed and $\{y\} = K - \lim A_n$. However, the sequence $\langle d(x_0, A_n) \rangle$ fails to converge pointwise as $\limsup d(x_0, A_n) = d(x_0, y) > \lambda$ whereas $\liminf d(x_0, A_n) = \liminf d(x_0, x_{2n}) \leq \lambda$. Thus if (1) fails, then (4) fails. ■

It is not the case, however, that in spaces with nice closed balls that a finite pointwise limit of distance functions must be a distance function.

Example. Equip the positive integers Z^+ with the zero-one metric, and let $A_n = \{n, n+1, n+2, \ldots\}$. Although the space has nice closed balls, the sequence $\langle d(\cdot, A_n) \rangle$ converges to the function identically equal to one on X. This is not a distance function.

The answer to the question *in which spaces is a finite pointwise limit of distance functions always a distance function?* is provided by the next result. Another proof can be given using Theorem 3.4 of [12].

Theorem 4.2. *Let $\langle X, d \rangle$ be a metric space. The following are equivalent:*

(1) Closed and bounded subsets of X are compact;

(2) Whenever $\langle A_n \rangle$ is a sequence in $CL(X)$ for which $\langle d(\cdot, A_n) \rangle$ converges pointwise to a finite-valued limit, then $\langle A_n \rangle$ is Wijsman convergent.

Proof. (1) ⇒ (2). Suppose $\langle d(\cdot, An) \rangle$ converges pointwise to a finite-valued limit f. By Lemma 3.4, $\langle A_n \rangle$ is Kuratowski convergent. Fix $x_0 \in X$; since $\{d(x_0, A_n) : n \in Z^+\}$ is a bounded set, each set A_n hits some fixed closed ball with center x_0. Since closed balls are compact, we get $\mathrm{Ls} A_n \neq \emptyset$. Thus the Kuratowski limit is nonempty, and since X has nice closed balls, Wijsman convergence follows.

(2) ⇒ (1). If (1) fails, then for some $x_0 \in X$ and $\lambda > 0$, the closed ball B with center x_0 and radius λ is not compact. Let $\langle x_n \rangle$ be a sequence in B with no cluster point, and for each $n \in Z^+$ set $A_n = \{x_k : k \geq n\}$. Then $\langle A_n \rangle$ is a decreasing sequence of nonempty closed sets, and so for all $x \in X$ and $n \in Z^+$ we have $d(x, A_{n+1}) \geq d(x, A_n)$. Furthermore, for all n we have $d(x, A_n) \leq d(x, x_0) + \lambda$. Thus, the sequence $\langle d(\cdot, A_n) \rangle$ is pointwise convergent

to a finite-valued limit defined by

$$f(x) = \sup_{n \in Z^+} d(x, A_n).$$

Since $\cap_{n=1}^{\infty} A_n = \emptyset$, the values of f are strictly positive, and so f cannot be a distance function. Thus (2) fails. ■

Suppose X has nice closed balls. The proof of Theorem 4.1 shows that whenever $\langle A_n \rangle$ is a sequence in $\mathrm{CL}(X)$ Kuratowski convergent to $\emptyset$, then for each $x \in X$ we have

$$\lim_{n \to \infty} d(x, A_n) = \sup\{d(x, w) : w \in X\}.$$

Indeed, by [3, Lemma 2.2] this property characterizes spaces with nice closed balls. One might guess that spaces with nice closed balls are exactly those for which the Kuratowski convergence to the empty set implies pointwise convergence of distance functions in the extended sense. However, this property characterizes a larger class of spaces.

Example. Let $\langle Y, d \rangle$ be a compact metric space that has an accumulation point y_0. Let $X = Y - \{y_0\}$ as a subspace of Y. Such a space does not have nice closed balls because for all $x \in X$ sufficiently close to y_0, there will exist a proper noncompact closed ball in X with center x. Now if $\langle A_n \rangle$ is a sequence in $\mathrm{CL}(X)$ Kuratowski convergent to $\emptyset$, then for each $\varepsilon > 0$ we have $A_n \subset S_\varepsilon[y_0]$ for all n sufficiently large, else A_n would hit the compact complement of the ball infinitely often. As result, we have for all $x \in X$ $\lim_{n \to \infty} d(x, A_n) = d(x, y_0)$.

Theorem 4.3. *Let $\langle X, d \rangle$ be a metric space. The following conditions are equivalent:*

(1) Whenever $\langle A_n \rangle$ is a sequence in $\mathrm{CL}(X)$ Kuratowski convergent to $\emptyset$, then $\langle d(\cdot, A_n) \rangle$ converges pointwise to an extended real-valued function;

(2) Whenever $\langle a_n \rangle$ is a sequence in X without a cluster point, then for each $x \in X$, $\lim_{n \to \infty} d(x, a_n)$ exists as an extended real number.

Proof. (1) $\Rightarrow$ (2). This is trivial, letting $A_n = \{a_n\}$.

(2) $\Rightarrow$ (1). Suppose (1) fails for a sequence $\langle A_n \rangle$ in $\mathrm{CL}(X)$. Choose $x_0 \in X$, where $\langle d(x_0, A_n) \rangle$ fails to converge in the extended sense. Set $\alpha = \limsup_{n \to \infty} d(x_0, A_n)$ and $\beta = \liminf_{n \to \infty} d(x_0, A_n)$. By passing to a subsequence we may assume

$$\lim d(x_0, A_{2n}) = \alpha \quad \text{and} \quad \lim d(x_0, A_{2n-1}) = \beta.$$

Choosing $a_n \in A_n$ with $d(x_0, a_n) \langle d(x_0, A_n) + \frac{1}{n}$, it is clear that $\lim_{n\to\infty} d(x_0, a_n)$ does not exist. Finally, $\langle a_n \rangle$ has no cluster point because $\mathrm{Ls}A_n = \emptyset$. Thus, (2) fails. ■

The first example of this section shows that in a complete metric space, the pointwise convergence of a sequence of distance functions to a finite-valued function with a nonempty zero set does not ensure that the limit is a distance function. We close with a consideration of the role of completeness with respect to stronger forms of convergence of distance functions.

Theorem 4.4. *Let $\langle X, d \rangle$ be a metric space. The following are equivalent:*

(1) d is a complete metric;

(2) Whenever $\langle d(\cdot, A_n) \rangle$ is uniformly convergent to a finite-valued function, then the limit function is a distance function;

(3) Whenever $\langle d(\cdot, A_n) \rangle$ is uniformly convergent on bounded subsets of X to a finite-valued function, then the limit function is a distance function.

Proof. (2) $\Rightarrow$ (1). Let $\langle x_n \rangle$ be a Cauchy sequence in X with no cluster point. Then with $A_n = \{x_n\}$, we have for all n and m,

$$\sup_{x\in X} |d(x, A_n) - d(x, A_m)| = d(x_n, x_m)$$

so that $\langle d(\cdot, A_n) \rangle$ is uniformly Cauchy and thus is uniformly convergent to a continuous function f. Clearly, $f(x)$ is never zero and so f cannot be a distance function.

(1) $\Rightarrow$ (3). This is Lemma 3.1.1 of [4].

(3) $\Rightarrow$ (2). This is trivial. ■

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Received: 29.05.2002