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# Nonlinear Piecewise Polynomial Approximation of Functions from Besov Spaces [1]

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*Dedicated to Professor Blagovest Sendov on the occasion of his 70th anniversary*

We present two results on the $N$-term piecewise polynomial approximation in $L_q(\Omega)$ of functions from the Besov space $B_p^{\lambda,\theta}(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain, and $\theta = p$, if $\frac{\lambda}{d} = \frac{1}{p} - \frac{1}{q}$, and $\theta = \infty$, if $\frac{\lambda}{d} > \frac{1}{p} - \frac{1}{q}$.

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## 1. Introduction

We begin with the description of an approximating set of piecewise polynomials $\mathcal{P}_k^N(\mathcal{T})$. The basic component of its definition is a *tree* $\mathcal{T}$ *of partitions for a bounded measurable set* $\Omega$. A *partition* $\pi$ of $\Omega$ is a collection of nonoverlapping subsets[2] of $\Omega$ such that

$$\operatorname{supp}\pi := \bigcup_{\omega\in\pi} \omega = \Omega. \tag{1.1}$$

**Definition 1.1.** $\mathcal{T}$ is said to be a *tree of partitions for* $\Omega$, if $\mathcal{T} = \{\pi_j : j \in \mathbb{Z}_+\}$, where

(a) $\pi_0 := \{\Omega\}$ and $\pi_j$ is a <u>finite</u> partition of $\Omega$, $j \in \mathbb{Z}_+$;

(b) $\pi_{j+1}$ is a <u>refinement</u> of $\pi_j$, i.e., every $\omega \in \pi_{j+1}$ is a subset of a <u>unique</u> $\omega' \in \pi_j$.

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[2]Sets of $\mathbb{R}^d$ *nonoverlap*, if their intersection is a set of $d$-measure zero.

The *set of vertices* of $\mathcal{T}$ is

$$\pi_{\mathcal{T}} := \bigcup_{j\in\mathbb{Z}_+} \pi_j. \tag{1.2}$$

If $\omega, \omega'$ are as in (b), we will say that they are *connected by edge directed from* $\omega'$ *to* $\omega$, written $\omega' \to \omega$.

Hence $\mathcal{T}$ is an ordered tree with the root $\Omega$, and $\pi_j$ is the $j$*-th level* of $\mathcal{T}$. We define $h : \pi_{\mathcal{T}} \to \mathbb{Z}_+$ by

$$h(\omega) := j, \text{ if } \omega \in \pi_j. \tag{1.3}$$

**Definition 1.2.** Given $k, N \in \mathbb{N}$ we define the set $\mathcal{P}_k^N(\mathcal{T})$ as a collection of $N$*-term sums* $\sum\limits_{\omega} p_\omega \mathbf{1}_\omega$, where $p_\omega$ are polynomials of degree $k-1$ and $\omega \in \pi_{\mathcal{T}}$.

Hereafter $\mathbf{1}_\omega$ stands for the characteristic function of $\omega$.

Remark 1.3. Assuming that $p_\omega$ of this definition is a polynomial of *vector* degree $\bar{k} - \bar{e} := (k_1 - 1, \dots, k_d - 1)$, $\bar{k} \in \mathbb{N}^d$, we introduce the set $\mathcal{P}_{\bar{k}}^N(\mathcal{T})$.

The main results of the present paper imply for a given $f \in B_p^{\lambda\theta}(\Omega)$ and a suitable $f_N \in \mathcal{P}_k^N(\mathcal{T})$ the inequality

$$\|f - f_N\|_{L_q(\Omega)} \le CN^{-\frac{\lambda}{d}} \, |f|_{B_p^{\lambda\theta}(\Omega)}.$$

Here the parameters $\lambda, \theta, p, q$ have been introduced above, $k > \lambda$ and a constant $C$ independent of $N$ and $f$. Besides, $\mathcal{T}$ is assumed to be *quasidyadic*, i.e., it satisfies the condition

(c) For every $\omega \in \pi_j$, $j \in \mathbb{Z}_+$, there is a cube $Q_\omega$ such that

$$\omega \subset Q_\omega, \tag{1.4}$$

and

$$|\omega| \approx |Q_\omega| \approx 2^{-jd}|\Omega|. \tag{1.5}$$

Hereafter $|S|$ stands for the $d$-measure of $S \subset \mathbb{R}^d$, and $F \approx G$ means that

$$C_1 F \le G \le C_2 F$$

with positive constants independent of the arguments of functions $F$ and $G$. Specially, the constants in (1.5) are independent of $\omega$ and $j$.

The aforementioned approximation is provided by the so called Up-and-Down Alforithm developed in collaboration with Dr. Inna Kozlov (see [4] and [7]). Since this algorithm has a relatively simple computer realization, the results presented may be of interest both for Approximation Theory and Numerical Analysis, in particular, for the finite element methods.

## 2. Formulations of the main results

We deal with an extended scale of Besov spaces defined on $\mathbb{R}^d$ by the seminorms

$$|f|_{B_p^{\lambda\theta}(\mathbb{R}^d)} := \left\{ \int_{\mathbb{R}_+} \left( \frac{\omega_k(f;t)_{Lp}}{t^\lambda} \right)^\theta \frac{dt}{t} \right\}^{\frac{1}{\theta}}, \tag{2.1}$$

where $\lambda > 0$, $0 < p, \theta \le \infty$, and $k$ is *the smallest integer* $> \lambda$. As usual, $\omega_k(f;\cdot)_{Lp}$ stands for the *k-modulus of continuity of* $f$ in $L_p(\mathbb{R}^d)$. For $p < 1$, there is the definition of this space due to Peetre [13] equivalent to that of (3), if $\frac{\lambda}{d} \ge \frac{1}{p} - 1$. In the remaining case Peetre's definition gives a proper subspace of $B_p^{\lambda\theta}(\mathbb{R}^d)$.

In order to avoid a complication irrelevant to the case, we define $B_p^{\lambda\theta}(\Omega)$ for a set $\Omega \subset \mathbb{R}^d$ to be *the trace space of* $B_p^{\lambda\theta}(\mathbb{R}^d)$ *to* $\Omega$. Hence

$$|f|_{B_p^{\lambda\theta}(\Omega)} := \inf \left\{ |g|_{B_p^{\lambda\theta}(\mathbb{R}^d)} : f = g|_\Omega \right\}, \tag{2.2}$$

where $g|_\Omega$ is the trace of $g$ to $\Omega$.

For a *Lipschitz domain* $\Omega$, this definition is equivalent to the "interior" definition based on $k$-modulus of continuity of $f$ in $L_p(\Omega)$.

Assume now that

(a) $f \in B_p^\lambda(\Omega)(:= B_p^{\lambda p}(\Omega))$ and $0 < p < q \le \infty$ satisfy

$$\frac{\lambda}{d} = \frac{1}{p} - \frac{1}{q}; \tag{2.3}$$

let, in addition, $p \le 1$, if $q = \infty$.

(b) $\mathcal{T}$ is a *quasidyadic tree* of partitions for a bounded $\Omega \subset \mathbb{R}^d$.

Under these assumptions the following is true.

**Theorem 2.1.** *Given* $N \in \mathbb{N}$, *there is a piecewise polynomial* $L_N(f) \in \mathcal{P}_k^N(\mathcal{T})$ *such that*

$$\|f - L_N(f)\|_{L_q(\Omega)} \le C N^{-\frac{\lambda}{d}} |f|_{B_p^\lambda(\Omega)} \tag{2.4}$$

*with constant $C$ independent of $f$ and $N$.*

Assume now that

(a) $f \in B_p^{\lambda\infty}(\Omega)$, and $0 < p < q \le \infty$ satisfy

$$\frac{\lambda}{d} > \frac{1}{p} - \frac{1}{q}. \tag{2.5}$$

(b) $\mathcal{T}$ is as in Theorem 2.1.

Then the following is true.

**Theorem 2.2.** *Given $N \in \mathbb{N}$, there is a piecewise polynomial $L_N(f) \in \mathcal{P}_k^N(\mathcal{T})$ such that (2.4) holds with $|f|_{B_p^{\lambda\infty}(\Omega)}$ substituted for $|f|_{B_p^{\lambda}(\Omega)}$, and $C$ independent of $f, N$ and $\varepsilon := \frac{\lambda}{d} - \frac{1}{p} + \frac{1}{q}$.*

Remark 2.3. It can be shown that in this case the approximating piecewise polynomial can be rewritten in a form $\sum_{\omega\in\tilde{B}} \tilde{p}_\omega \mathbf{1}_\omega$ with $\tilde{p}_\omega \in \mathcal{P}_k$ and $\tilde{\Delta} \subset \pi_{\mathcal{T}}$ consisting of at most $C(\varepsilon)N$ subsets. Here $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

The result of this type was firstly proved in her Ph.D. thesis by I. Irodova [11]. This author used an approximation algorithm that later was going to be named the *greedy algorithm.* In spite of Theorem 2.2, both constants $C$ and $C(\varepsilon)$ tend in this case to infinity, as $\varepsilon \to 0$.

To formulate a corollary of the latter result, define an *aproximation space* $\mathcal{N}_p^{\lambda\theta}(\mathcal{T})$ by the seminorm

$$|f|_{\mathcal{N}_p^{\lambda\theta}(\mathcal{T})} := \left\{ \sum_{N=1}^{\infty} \frac{1}{N} \left[N^{\lambda d} e_k^N(f; L_p)\right]^\theta \right\}^{\frac{1}{\theta}}. \tag{2.6}$$

Here $\lambda, k, p, \theta$ are as in (2.1) and

$$e_k^N(f; L_p) := \inf\left\{\|f - m\|_{L_q(\Omega)} : m \in \mathcal{P}_k^N(\mathcal{T})\right\}. \tag{2.7}$$

Replacing in this definition $L_p(\Omega)$ by $BMO(\Omega)$, the trace space of $BMO(\mathbb{R}^d)$, one defines the approximation space $\mathcal{E}_{BMO}^{\lambda\theta}(\mathcal{T})$.

**Corollary 2.4.** *Under the assumptions of Theorem 2.1, but with $p < 1$, if $q = \infty$, it is true that*

$$B_p^{\lambda}(\Omega) \subset \mathcal{N}_\infty^{\lambda p}(\mathcal{T}). \tag{2.8}$$

*In the remaining case $q = \infty$ and $1 < p < \infty$, the following embedding*

$$B_p^\lambda(\Omega) \subset \mathcal{N}_{BMO}^{\lambda p}(\Omega). \tag{2.9}$$

*holds.*

We now briefly discuss the similar results for anisotropic Besov spaces which can be carried out by the very same approach. The corresponding definition is now based on the seminorm

$$|f|_{B_p^{\bar\lambda,\theta}(\mathbb{R}^d)} := \sum_{i=1}^{d} \left\{ \int_{\mathbb{R}_+} \left( \frac{\omega_{k_i}^i(f;t)_{L_p}}{t^{\lambda_i}} \right)^\theta \frac{dt}{t} \right\}^{\frac{1}{\theta}}, \tag{2.10}$$

where $0 < p, \theta \le \infty$, $\bar\lambda = (\lambda_1, \dots, \lambda_d)$ is a vector with the strictly positive components, and $k_i$ is the smallest integer $> \lambda_i$, $1 \le i \le d$.

Recall that $\omega_k^i(f;\cdot)_{L_p}$ is *partiall $k$-modulus of continuity* of $f \in L_p(\mathbb{R}^d)$ with respect to variable $x_i$.

For a subset $\Omega \subset \mathbb{R}^d$ we then define $B_p^{\bar\lambda,\theta}(\Omega)$ in the very same fashion as in (2.2). This definition is equivalent to the corresponding "interior" one in the case of domains satisfying *the cube condition.* Recall that a domain $\Omega$ satisfies this condition, if every $x$ at the boundary $\partial\Omega$ is a vertex of an open cube containing in $\Omega$ and being congruent to a fixed cube $(0, \ell)^d$.

We also require the notion of a $\bar\lambda$-*quasidyadic tree* which is introduced by the conditions (a), (b) of Definition 1.1 and the condition (c), see (1.4) and (1.5), with the cube $Q_\omega$ replaced by the corresponding $\bar\lambda$-*cube.* In turn, a $\bar\lambda$-*cube* is a closed *$d$-interval*[3] whose sidelengths $\ell_i$ satisfy the condition

$$\lambda_i \log \ell_i = \text{const}, \quad 1 \le i \le d. \tag{2.11}$$

Together with the condition (1.5) this imply for the sidelengths $\ell_i(\omega)$ of the $\bar\lambda$-cube $Q_\omega$ the inequalities

$$\ell_i(\omega) \approx 2^{-j\langle\bar\lambda\rangle/\lambda_i}, \ \ 1 \le i \le d, \tag{2.12}$$

being fulfilled uniformly in $j$.

Here $j := h(\omega)$, see (1.3), and

$$\langle\bar\lambda\rangle := \left( \frac{1}{d} \sum_{i=1}^{d} \frac{1}{\lambda_i} \right)^{-1}. \tag{2.13}$$

Assume now that

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[3] *$d$-interval* is a parallelotope $\prod_{k=1}^{d} I_k$, where $I_k$ are intervals of $\mathbb{R}$.

(a) $f \in B_p^{\bar\lambda}(\Omega)$ and $0 < p < q \le \infty$ satisfy

$$\frac{\langle\bar\lambda\rangle}{d} = \frac{1}{p} - \frac{1}{q}; \tag{2.14}$$

let, in addition, $p \le 1$, if $q = \infty$.

(b) $\mathcal{T} := \{\pi_j\}$ is a $\bar\lambda$-*dyadic* tree of partitions for a *bounded* set $\Omega \subset \mathbb{R}^d$.

Under these assumptions the following is true.

**Theorem 2.5.** *Given $N \ge 1$, there is a piecewise polynomial $L_N(f) \in \mathcal{P}_{\bar k}^N(\mathcal{T})$, see Remark 1.3, such that*

$$\|f - L_N(f)\|_{L_q(\Omega)} \le CN^{-\frac{\langle\bar\lambda\rangle}{d}} |f|_{B_p^{\bar\lambda}(\Omega)}. \tag{2.15}$$

*Here $C$ is independent of $N$ and $f$.*

To formulate the second result, assume that:

(a) $f \in B_p^{\bar\lambda\infty}(\Omega)$ and $0 < p \le q \le \infty$ satisfy

$$\frac{\langle\bar\lambda\rangle}{d} > \frac{1}{p} - \frac{1}{q}; \tag{2.16}$$

(b) $\mathcal{T}$ and $\Omega$ are as in Theorem 2.7.

Then the following holds.

**Theorem 2.6.** *Given $N \ge 1$, there is a piecewise polynomial $L_N(f) \in \mathcal{P}_{\bar k}^N(\mathcal{T})$ such that (2.15) holds with $|f|_{B_p^{\bar\lambda\infty}(\Omega)}$ substituted for $B_{B_p^{\bar\lambda}(\Omega)}$ and $C$ independent of $f, N$ and $\varepsilon := \frac{\langle\bar\lambda\rangle}{d} - \frac{1}{p} + \frac{1}{q}$.*

Remark 2.7. It can be shown that in this case $L_N(f)$ can be rewritten in a form described in Remark 2.3 (with $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$). Such a representation can be used for the following estimation of the $\varepsilon$-entropy $H_\varepsilon(B_p^{\bar\lambda\infty}; L_q)$ of the unit ball of $B_p^{\bar\lambda\infty}(\Omega)$ in $L_q(\Omega)$.

$$\mathcal{H}_\varepsilon\left(B_p^{\bar\lambda,\infty}, L_q\right) \approx \varepsilon^{-\frac{d}{\langle\bar\lambda\rangle}}, \quad \varepsilon \to 0.$$

We shall present this result elsewhere.

## 3. The proofs of Theorem 2.1 and Corollary 2.4

P r o o f  of  T h e o r e m 2.1. We deduced the first two results of the present paper from the main approximation theorem of [7]. For the convenience of the reader, we formulate a special case of this theorem adapted to our setting.

Let $\mathcal{T} := \{\pi_j\}$ be a tree of partitions of a bounded set $\Omega \subset \mathbb{R}^d$ satisfying the following condition.

There is a constant $C = C(\mathcal{T})$ such that for every $j \in \mathbb{Z}$ and $\omega \in \pi_j$

$$2 \le \#\left\{\omega' \in \pi_{j+1} : \omega' \subset \omega\right\} \le C(\mathcal{T}). \tag{3.1}$$

Note that this condition is clearly true for quasidyadic or $\bar{\lambda}$-quasidyadic tree, since in these cases $|\omega| \approx 2^{-jd}$, see (1.5) and (2.14).

Let now $\mathcal{P}$ be a fixed finite dimensional space of polynomials in $x \in \mathbb{R}^d$. Given a positive weight $w : \pi_{\mathcal{T}} \to \mathbb{R}_+$ and $0 < p < \infty$, introduce a space of functions $B_p^w(\mathcal{T})$ by condition

$$|f|_{B_p^w(\mathcal{T})} := \inf\left\{\sum_{\omega\in\pi_{\mathcal{T}}}\left(w(\omega)\sup_{\omega}|f_\omega|\right)^p\right\}^{\frac{1}{p}} < \infty. \tag{3.2}$$

Here the infimum is taken over all decompositions,

$$f = \sum_{\omega\in\pi_{\mathcal{T}}} f_\omega \mathbf{1}_\omega \quad \text{(convergence in } L_p(\Omega)) \tag{3.3}$$

with $\{f_\omega\}_{\omega\in\pi_{\mathcal{T}}} \subset \mathcal{P}$.

Assume now that

(a) $\mathcal{T}$ is a tree of partitions for a bounded $\Omega \subset \mathbb{R}^d$ satisfying the condition (3.1).

(b) For some $q \in (p, +\infty]$ the following continuous embedding

$$B_p^w(\mathcal{T}) \subset L_q(\Omega) \tag{3.4}$$

holds.

Under these assumptions the following is true.

**Theorem 3.1.** (see [7]) *Given $f \in B_p^w(\mathcal{T})$ and $N \in \mathbb{N}$, there is a piecewise polynomial*

$$L_N(f) := \sum_{\omega\in\Delta} p_\omega \mathbf{1}_\omega \tag{3.5}$$

*with* $p_\omega \in \mathcal{P}$ *and* $\Delta \subset \pi_\mathcal{T}$ *containing, at most,* $4N$ *sets such that*

$$\|f - L_N(f)\|_{L_q(\Omega)} \le CN^{\frac{1}{q}-\frac{1}{p}} |f|_{B_p^w(\mathcal{T})}. \tag{3.6}$$

*The constant* $C$ *is independent of* $F$ *and* $N$.

We deduce Theorem 2.1 from this and the next result.

**Proposition 3.2.** *Let* $w_q : \pi_\mathcal{T} \to \mathbb{R}$ *be a weight defined by* $w_q(\omega) := |\omega|^{\frac{1}{q}}$, $0 < q \le \infty$, *and* $\mathcal{P}_k$ *is the space of polynomials of degree* $k-1$, $k \in \mathbb{N}$. *Then under the assumptions of Theorem 2.1,*

$$B_p^{w_q}(\mathcal{T}) \subset L_q(\Omega). \tag{3.7}$$

*If, in addition,* $\mathcal{P} = \mathcal{P}_k$, $k > \lambda$, *then*

$$B_p^\lambda(\Omega) \subset B_p^{w_q}(\mathcal{T}). \tag{3.8}$$

Proof. The embedding (3.7) follows from a more general result that will be proved in the last section, see Theorem 6.1. To prove the next embedding, introduce a family of subspaces of $L_p(\Omega)$ by

$$V_j := \left\{ \sum_{\omega \in \pi_j} p_\omega \mathbf{1}_\omega : p_\omega \in \mathcal{P} \right\}, \text{ if } j \ge 0 \tag{3.9}$$

and

$$V_j := \{0\}, \text{ if } j > 0.$$

Since $\pi_{j+1}$ is a refinement of $\pi_j$

$$V_j \subset V_{j+1}, \quad j \in \mathbb{Z}; \tag{3.10}$$

besides, $\bigcup_j V_j$ is dense in $L_p(\Omega)$, if $p < \infty$.

We then set

$$E_j(f) := \inf \left\{ \|f - g\|_{L_p(\Omega)} : g \in V_j \right\} \tag{3.11}$$

and introduce an approximation space $\mathcal{E}_p^\lambda(\mathcal{T})$ by the condition

$$\|f\|_{\mathcal{E}_p^\lambda(\mathcal{T})} := \left\{ \sum_{j \in \mathbb{Z}} \left( 2^{j\lambda} E_j(f) \right)^p \right\}^{\frac{1}{p}} < \infty. \tag{3.12}$$

It is easily seen that this space is quasi-Banach. ■

**Lemma 3.3.** *Up to the equivalence of the quasinorms, it is true that*

$$\mathcal{E}_p^\lambda(\mathcal{T}) = B_p^{w_q}(\mathcal{T}). \tag{3.13}$$

P r o o f. According to Theorem 1 of [8], see also [9], Lemma 4.3.21, for every $f \in \mathcal{E}_p^\lambda(\mathcal{T})$ the following equivalence

$$\|f\|_{\mathcal{E}_p^\lambda(\mathcal{T})} \approx \inf \left\{ \sum_{j\in\mathbb{Z}} \left( 2^{j\lambda} \|f_j - f_{j-1}\|_{L_p(\Omega)} \right)^p \right\}^{\frac{1}{p}} \tag{3.14}$$

holds with constants independent of $f$. Here the infimum is taken over all decompositions

$$f = \sum_{j\in\mathbb{Z}} (f_j - f_{j-1}) \tag{3.15}$$

with $f_j \in V_j$. Since for such $f_j$

$$f_j - f_{j-1} = \sum_{\omega\in\pi_j} p_\omega \mathbf{1}_\omega \tag{3.16}$$

with suitable polynomials $p_\omega \in \mathcal{P}$, see (3.10), the set of these decompositions for $f$ coincides with that involved in the definition of the space $B_p^{w_q}(\mathcal{T})$, see (3.3).

Show now that uniformly in $j$

$$2^{j\lambda} \|f_j - f_{j-1}\|_{L_p(\Omega)} \approx \left\{ \sum_{\omega\in\pi_j} \left( w_q(\omega) \sup_\omega |p_\omega| \right)^p \right\}^{\frac{1}{p}} \tag{3.17}$$

with $p_\omega$ from (3.16). Together with (3.14) this implies the equivalence of the quasinorms (3.12) and (3.2) with $w := w_q$.

To prove (3.16), note first that

$$|\omega| \approx |\mathrm{conv}(\omega)| \approx 2^{-jd}, \quad \omega \in \pi_j, \tag{3.18}$$

uniformly in $\omega$ and $j$, see (1.4) and (1.5). Hence the inequality of [5] implies for these $\omega$ the equivalence

$$\int_\omega |m|^p dx \approx |\omega| \left( \sup_\omega |m| \right)^p, \quad m \in \mathcal{P}_k, \tag{3.19}$$

which is uniform in $m$ and $\omega$.

Using this and (3.18), we then have

$$2^{j\lambda}\|f_j-f_{j-1}\|_{L_p(\Omega)} = \left\{\sum_{\omega\in\pi_j} 2^{j\lambda p}\int_\omega |p_\omega|^p dx\right\}^{\frac{1}{p}} \approx \left\{\sum_{\omega\in\pi_j} |\omega|^{-\frac{\lambda p}{d}+1}\left(\sup_\omega |p_\omega|\right)^p\right\}^{\frac{1}{p}}.$$

Since $-\frac{\lambda p}{d}+1=\frac{p}{q}$, see (2.3), the right-hand side coincides with that of (3.17). The lemma has been proved. ■

Remark 3.4. The equivalence (3.13) remains to be true for arbitrary space $\mathcal{P}$ of polynomials of finite degree, since (3.19) holds in this case. Virtually, one can replace $\mathcal{P}$ by a space of locally bounded functions satisfying (3.19).

Remark 3.5. The very same argument shows that for $f_j \in V_j$ the inequality

$$\|f_j - f_{j-1}\|_{L_\infty(\Omega)} \le C2^{\frac{jd}{p}}\|f_j - f_{j-1}\|_{L_p(\Omega)} \tag{3.20}$$

holds with $C$ independent of $j$ and $f_j - f_{j-1}$. This will be used in the last section.

We now prove the inequality

$$\|f\|_{\mathcal{E}_p^\lambda(\mathcal{T})} \le C\left\{|f|_{B_p^\lambda(\Omega)} + \|p_f\|_{L_p(\Omega)}\right\} \tag{3.21}$$

with $C$ independent of $f$, and $p_f$ being a polynomial of $\mathcal{P}_k$ such that

$$\|f-p_f\|_{L_p(\Omega)} = \inf\left\{\|f-m\|_{L_p(\Omega)} : m\in\mathcal{P}_k\right\}.$$

Together with Lemma 3.3, this implies the required embedding (3.8).

The key point in proving (3.21) is to estimate $E_j(f)$ by $k$-modulus of continuity of $f$. To do this, we first present $E_j(f)$ as follows. Let $f_j \in V_j$ be defined by

$$\|f-f_j\|_{L_p(\Omega)} := E_j(f).$$

If $f_j = \sum_{\omega\in\pi_j} p_\omega 1_\omega$, then

$$E_j(f) = \left\{\sum_{\omega\in\pi_j}\int_\omega |f-p_\omega|^p dx\right\}^{\frac{1}{p}}.$$

Because of optimality of $f_j$ the polynomial $p_\omega$ provides best approximation of $f$ in $L_p(\omega)$, i.e.,

$$E_k(f;\omega) := \inf_{g\in\mathcal{P}_k} \|f-g\|_{L_p(\omega)} = \|f-p_\omega\|_{L_p(\omega)}.$$

In particular, $p_\Omega$ is a polynomial of best approximation, i.e., it coincides with the polynomial $p_f$ in (3.21).

We will prove later that $\{Q_\omega;\omega\in\pi_j\}$ is $C$-*disjoint*[4] , with $C$ independent of $j$. Using this fact we have

$$E_j(f) \le C^{\frac{1}{p}} \sup_{\pi} \left\{ \sum_{Q\in\pi} E_k(\tilde{f};Q)^p \right\}^{\frac{1}{p}},$$

where the supremum is taken over all nonoverlapping subfamilies $\pi$ of $\pi_j$.

Finally, we apply for these $\pi$ the inequality

$$\left\{ \sum_{Q\in\pi} E_k(\tilde{f};Q)^p \right\}^{\frac{1}{p}} \le C \sup_{Q\in\pi} \omega_k\left(\tilde{f};|Q|^{\frac{1}{k}}\right)_{L_p(\mathbb{R}^d)}, \tag{3.22}$$

see [2] for $p\ge 1$ and [12] for $p<1$. Since $|Q_\omega| \le C2^{-jd}$, the last two inequalities imply

$$E_j(f) \le C\omega_k(\tilde{f};2^{-j})_{L_p(\mathbb{R}^d)} \tag{3.23}$$

with $C$ independent of $j$ and $f$. It follows from here that

$$\|f\|_{\mathcal{E}_p^\lambda(\mathcal{T})} \le C \left\{ \sum_{j\in\mathbb{Z}_+} \left(2^{j\lambda}\omega_k(\tilde{f};2^{-j})_{L_p(\mathbb{R}^d)}\right)^p + \|p_f\|^p_{L_p(\Omega)} \right\}^{\frac{1}{p}} \le C\left\{ |\tilde{f}|_{B_p^\lambda(\mathbb{R}^d)} + \|p_f\|_{L_p(\Omega)} \right\}.$$

Taking the infimum over all extensions $\tilde{f}$ of $f$ and using (2.2), we prove (2.21).

To complete this proof it remains to establish that $\{Q_\omega : \omega\in\pi_j\}$ is $C$-*disjoint* with $C$ independent of $j$. Let

$$M := \sup_{x\in\mathbb{R}^d} M(x) := \sup_{x\in\mathbb{R}^d} \#\left\{Q_\omega\in\pi_j : x\in Q_\omega\right\}.$$

By the covering theorem of [6], the family $\{Q_\omega : \omega\in\pi_j\}$ is $\tilde{C}$-disjoint with

$$\tilde{C} := 2^{d-1}(M-1)+1. \tag{3.24}$$

[4] A family of subsets is $C$-*disjoint*, if it is a union of, at most, $C$ subfamilies consisting of pairwise nonoverlapping subsets.

According to (1.5),

$$\operatorname{diam} Q_\omega \le C2^{-j}, \quad \omega \in \pi_j,$$

with $C$ independent of $\omega$ and $j$. Hence all $Q_\omega$ containing $x$ are subsets of the ball $B$ centered at $x$ and of radius $C2^{-j}$. Let $S(x) := \{\omega \in \pi_j : x \in Q_\omega\}$. Since $\pi_j$ is disjoint, $S(x)$ is disjoint as well. Therefore

$$M(x) \min\{|\omega| : \omega \in S(x)\} \le \sum_{\omega \in S(x)} |\omega| \le |B|.$$

By (1.5) this minimum is, at least, $C_1 2^{-jd}$, while $|B| = C_2 2^{-jd}$ where $C_1, C_2$ independent of $j$ and positive. Hence

$$M := \sup_x M(x) \le \frac{C_2}{C_1}$$

and the desired estimate follows from (3.24).

To within the proof of the embedding (3.7), Proposition 3.2 has been done.

We now prove Theorem 2.1. Using Theorem 3.1 with $w := w_q$ and $\mathcal{P} := \mathcal{P}_k$ together with (3.7), we find a piecewise polynomial $L_N(f) \in \mathcal{P}_k^N(\mathcal{T})$ satisfying

$$\|f - L_N\|_{L_q(\Omega)} \le C N^{\frac{1}{q} - \frac{1}{p}} |f|_{B_p^{w_q}(\mathcal{T})}.$$

Applying this to the function $f - p_f$ and taking into account (3.21) and (2.3), we have

$$\begin{aligned} &\|f - p_f - L_N(f - p_f)\|_{L_q(\Omega)} \le C N^{\frac{1}{q} - \frac{1}{p}} |f - p_f|_{B_p^w(\mathcal{T})} \\ &\le C_1 N^{\frac{1}{q} - \frac{1}{p}} |f - p_f|_{B_p^\lambda(\Omega)} = C_1 N^{-\frac{\lambda}{d}} |f|_{B_p^\lambda(\Omega)}. \end{aligned}$$

Hence $p_f + L_N(f - p_f)$ is the required piecewise polynomial.

Theorem 2.1 has been proved. ■

P r o o f o f C o r o l l a r y 2.4. Using the notion of the approximation space $\mathcal{N}_p^{\lambda\theta}(\mathcal{T})$, see(2.6), we reformulate Theorem 2.1 as follows.

Let $\lambda, p, q$ satisfy

$$\frac{\lambda}{d} = \frac{1}{p} - \frac{1}{q} \ (> 0) \tag{3.25}$$

and, in addition, $p \le 1$, if $q = \infty$. Then it is true that

$$B_p^\lambda(\Omega) \subset \mathcal{N}_q^{\lambda\infty}(\mathcal{T}). \tag{3.26}$$

Using this we establish, first, the desired result for $q < \infty$, and $q = \infty$ and $p < 1$. Choose $p_0, p_1$ such that

$$p_0 < p < p_1, \quad \text{if} \quad q < \infty,$$

and

$$p_0 < p < p_1 < 1, \quad \text{if} \quad q = \infty.$$

Then for suitable $\theta \in (0,1)$

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

By (3.25), there are $0 < \lambda_0 < \lambda < \lambda_1$ such that

$$\frac{\lambda_i}{d} = \frac{1}{p_i} - \frac{1}{q}, \qquad i = 0, 1.$$

Using the embedding (3.26) for these $\lambda_i, p_i$ and $q$ and applying the real interpolation, we then have

$$\left(B_{p_0}^{\lambda_0}, B_{p_1}^{\lambda_1}\right)_{\theta p}(\Omega) \subset \left(\mathcal{N}_q^{\lambda_0 \infty}, \mathcal{N}_q^{\lambda_1 \infty}\right)_{\theta p}(\mathcal{T}). \tag{3.27}$$

The Peetre–Sparr interpolation theorem for approximation spaces, see e.g., [1], Chapter 7, yields

$$\left(\mathcal{N}_q^{\lambda_0 \infty}, \mathcal{N}_q^{\lambda_1 \infty}\right)_{\theta p}(\mathcal{T}) = \mathcal{N}_q^{\lambda p}(\mathcal{T}). \tag{3.28}$$

Use now the interpolation theorem for Besov spaces, see e.g., [10]. We have

$$\left(B_{p_0}^{\lambda_0}, B_{p_1}^{\lambda_1}\right)_{\theta p}(\mathbb{R}^d) = B_p^{\lambda}(\mathbb{R}^d). \tag{3.29}$$

Since $B_p^\lambda(\Omega)$ is, by our definition, a trace space, the trace operator acts boundedly from $B_p^\lambda(\mathbb{R}^d)$ to $B_p^\lambda(\Omega)$. By the real interpolation theorem this operator acts continuously from the space (3.29) to the space $\left(B_{p_0}^{\lambda_0}, B_{p_1}^{\lambda_1}\right)_{\theta p}(\Omega)$, as well. In other words, we have

$$B_p^\lambda(\Omega) = B_p^\lambda(\mathbb{R}^d)\Big|_\Omega \subset \left(B_{p_0}^{\lambda_0}, B_{p_1}^{\lambda_1}\right)_{\theta p}(\Omega).$$

Together with (3.28) and (3.27), this implies in this case the required embedding

$$B_p^\lambda(\Omega) \subset \mathcal{N}_p^{\lambda p}(\mathcal{T}).$$

Let now $q = \infty$ and $1 < p < \infty$. Then (3.26) yields

$$\left(BMO, B_1^d\right)_{\theta p}(\Omega) \subset \left(\mathcal{N}_{BMO}^{0\infty}, \mathcal{N}_{\infty}^{d\infty}\right)_{\theta p}(\Omega)$$

with $\theta \in (0,1)$ determined by

$$\frac{1}{p} = \frac{1-\theta}{1}.$$

Then the Peetre–Sparr theorem gives

$$\left(\mathcal{N}_{BMO}^{0\infty}, \mathcal{N}_{\infty}^{d\infty}\right)_{\theta p}(\mathcal{T}) \subset \left(\mathcal{N}_{BMO}^{0\infty}, \mathcal{N}_{BMO}^{d\infty}\right)_{\theta p}(\mathcal{T}) = \mathcal{N}_{BMO}^{(1-\theta)d,p}(\mathcal{T}).$$

But $\frac{\lambda}{d} = \frac{1}{p} = 1 - \theta$. Hence $(1-\theta)d = \lambda$ and we have

$$\left(BMO, B_1^d\right)_{\theta p}(\Omega) \subset \mathcal{N}_{BMO}^{\lambda p}(\mathcal{T}). \tag{3.30}$$

On the other hand, the Peetre–Svensson theorem [14] states that

$$\left(BMO, B_1^d\right)_{\theta p}(\mathbb{R}^d) = B_p^\lambda(\mathbb{R}^d).$$

Since $BMO(\Omega)$ is also defined as a trace space, as before, we obtain the embedding

$$B_p^\lambda(\Omega) \subset \left(BMO, B_1^d\right)_{\theta p}(\Omega).$$

Together with (3.30) this yields the desired result in this case.

## 4. Proof of Theorem 2.2

Let $V_j$ and $E_j(f)$ be defined by (3.9) and (3.10) and (3.11), and $f_j \in V_j$ be determined by

$$E_j(f) = \|f - f_j\|_{L_p(\Omega)}. \tag{4.1}$$

Given $N$, one defines $J \in \mathbb{Z}_+$ by

$$2^{Jd} \le N < 2^{(J+1)d} \tag{4.2}$$

and introduce the function

$$g_J := f - f_J. \tag{4.3}$$

Applying Theorem 3.1 to this $g$ we have[5]

$$\|g_J - L_N(g_J)\|_{L_q(\Omega)} \le C N^{\frac{1}{q}-\frac{1}{p}} \, |f_J|_{B_p^{wq}(\mathcal{T})}. \tag{4.4}$$

Note that

$$g_J - L_N(g_J) = f - f_J - L_N(f - f_J) =: f - \tilde{L}_N(f)$$

and $\tilde{L}_N(f)$ is a sum of terms $\tilde{p}_\omega \mathbf{1}_\omega$, $\omega \in \tilde{\Delta}$, where $\tilde{p}_\omega \in \mathcal{P}_k$ and

$$\#\tilde{\Delta} \le \dim \pi_J + 4N.$$

By (1.5) and (4.2) $\dim \pi_J \le CN$ and therefore $\tilde{L}_N(f) \in \mathcal{P}_k^{CN}(\mathcal{T})$. Hence it remains to derive from (4.4) the desired upper bound.

**Lemma 4.1.** *Let $0 < \mu < \lambda$ be defined by*

$$\frac{\mu}{d} := \frac{1}{p} - \frac{1}{q}. \tag{4.5}$$

*Then uniformly in $J$ and $F$ the following is true:*

$$|f - f_J|_{B_p^{wq}(\mathcal{T})} \le C 2^{-J(\lambda-\mu)} |f|_{B^{\lambda\infty}(\Omega)}. \tag{4.6}$$

Proof. By (4.3),

$$g = \sum_{j>J} (f_j - f_{j+1}) = \sum_{j>J} \sum_{\omega \in \pi_j} f_\omega \mathbf{1}_\omega$$

with suitable $f_\omega \in \mathcal{P}_k$. This and (3.2) imply

$$|f - f_J|_{B_p^{wq}(\mathcal{T})} \le \left\{ \sum_{j>J} \sum_{\omega \in \pi_j} \left( |\omega|^{\frac{1}{q}} \sup_\omega |f_\omega| \right)^p \right\}^{\frac{1}{p}}. \tag{4.7}$$

Using (3.17) and the inequality

$$\|f_j - f_{j+1}\|_{L_p(\Omega)} \le 2^{\frac{1}{p^*}-1} \left( E_j(f) + E_{j+1}(f) \right),$$

[5] Hereafter $C$ stands for a constant independent of $f$ and $N$ which may vary from line to line.

where $p^* := \min(1,p)$, we then bound the right hand side of (4.7) by

$$C\left\{\sum_{j\geq J}\left(2^{j\mu}E_j(f)\right)^p\right\}^{\frac{1}{p}} \leq C2^{-J(\lambda-\mu)}\sup_{j\geq J} 2^{j\lambda}E_j(f).$$

Using now the inequality (3.23) we have for $j \geq J$

$$E_j(f) \leq C\omega_k\left(\tilde{f}; 2^{-j}\right)_{L_p(\Omega)},$$

whence, in turn, one gets

$$\sup_{j\geq J} 2^{j\lambda}E_j(f) \leq C\sup_{j\geq 0} 2^{j\lambda}\omega_k\left(\tilde{f}; 2^{-j}\right)_{L_p(\Omega)} \leq C|f|_{B_p^{\lambda\infty}(\Omega)}.$$

Combine all these inequalities to prove (4.6). ■

To complete the proof of Theorem 2.2, it remains to estimate the right hand side of (4.4) by (4.6). Then for $\tilde{L}_N(f) \in \mathcal{P}_k^{CN}(\mathcal{T})$ we get by (4.5) and (4.2)

$$\|f - \tilde{L}_N(f)\|_{L_q(\Omega)} \leq CN^{\frac{1}{p}-\frac{1}{q}}2^{-J(\lambda-\mu)}|f|_{B_p^{\lambda\infty}(\Omega)} \leq CN^{-\frac{\lambda}{d}}|f|_{B_p^{\lambda\infty}(\Omega)}.$$

## 5. The anisotropic case

The proofs of Theorem 2.5 and 2.6 are based on the very same arguments as ones of the corresponding isotropic results. We only single out the changes required in these cases, remaining the details for the reader. In Theorem 3.1 and Proposition 3.1, we replace the space $\mathcal{P}_k$ of polynomials of degree $k-1$ by that of vector degree $\bar{k} - \bar{e} = (k_1 - 1, \ldots, k_d - 1)$ (written $\mathcal{P}_{\bar{k}}$). Since Theorem 3.1 and the embedding (3.7) are true for arbitrary finite dimensional subspace $\mathcal{P}$ of polynomials in $x$, it remains only to check that the analog of (3.8) holds in this situation.

So we have to prove that for $\mathcal{P} := \mathcal{P}_{\bar{k}}$, $k_i > \lambda_i$, $1 \leq i \leq d$, the following embedding

$$B_p^{\bar{\lambda}}(\Omega) \subset B_p^{w_q}(\mathcal{T}) \tag{5.1}$$

holds. In turn, this embedding follows from the anisotropic analogue of the inequality (3.22) which asserts that

$$\left\{\sum_{Q\in\pi} E_{\bar{k}}(\tilde{f}; Q)^p\right\}^{\frac{1}{p}} \leq C\sup_{Q\in\pi}\left\{\sum_{i=1}^{d}\omega_{k_i}^i\left(\tilde{f}; \ell_i(Q)\right)_{L_p(\mathbb{R}^d)}\right\}. \tag{5.2}$$

Here $\pi$ is an nonoverlapping subfamily of the family $\{Q_\omega : \omega \in \pi_j\}$, and $\ell_i(Q)$ is the $i$th sidelength of the $\bar{\lambda}$-cube $Q_\omega$, see (2.11). The inequality (5.2) is, in fact, known and can be carried out by the argument presented, e.g. in [3], Appendix 1. Using now (2.12) and (5.2), we complete the proof of the embedding (5.1) repeating word by word the corresponding end of the proof for the embedding (3.7).

In the case of Theorem 2.6, we simply repeat the proof of Theorem 2.2 replacing in Lemma 4.1, $\mathcal{P}_k$ by $\mathcal{P}_{\bar{k}}$ and applying (5.2) instead of (3.22).

## 6. An embedding theorem

We now present a general embedding result which will immediately yield the required embedding (3.7).

Let $\mathcal{A} := \{A_j\}_{j\in\mathbb{Z}}$ be a family of subsets from $L_p(\Omega)$ satisfying

$$A_j \subset A_{j+1}, \quad j \in \mathbb{Z}, \tag{6.1}$$

and $A_j := \{0\}, \ \ j < 0$. Define the best approximation

$$E_j(f) := \inf_{g\in A_j} \|f - g\|_{L_p(\Omega)} \tag{6.2}$$

and introduce an *approximation space* $\mathcal{E}_p^\lambda(\mathcal{A})$, $\lambda > 0$, $0 < p < \infty$, by the quasinorm

$$\|f\|_* := \left\{ \sum_{j\in\mathbb{Z}} \left(2^{j\lambda} E_j(f)\right)^p \right\}^{\frac{1}{p}} \tag{6.3}$$

Assume that for $j \in \mathbb{Z}$ and $f \in A_j$

$$\|f\|_{L_\infty(\Omega)} \le C 2^{\frac{jd}{p}} \|f\|_{L_p(\Omega)} \tag{6.4}$$

with $C$ independent of $j$ and $f$. Assume also that $|\Omega| < \infty$. Without loss of generality we let

$$|\Omega| = 1. \tag{6.5}$$

Under these assumptions the following holds.

**Theorem 6.1.** *Let $\frac{\lambda}{d} \le \frac{1}{p}$, and $0 < p < q \le \infty$ be defined by*

$$\frac{\lambda}{d} = \frac{1}{p} - \frac{1}{q}. \tag{6.6}$$

*Let, in addition,* $p \le 1$, *if* $q = \infty$. *Then the following embedding holds:*

$$\mathcal{E}_p^\lambda(\mathcal{A}) \subset L_q(\Omega). \tag{6.7}$$

P r o o f. Let $f \in \mathcal{E}_p^\lambda(\mathcal{A})$, and $f_j \in A_j$ be defined by

$$E_j(f) = \|f - f_j\|_{L_p(\Omega)}. \tag{6.8}$$

Show that

$$f = \sum_{j\in\mathbb{Z}} (f_j - f_{j+1}) \ \ \text{(convergence in } L_p(\Omega)). \tag{6.9}$$

In fact, for $p^* := \min(1, p)$

$$\|f_j - f_{j+1}\|_{L_p(\Omega)} \le 2^{\frac{1}{p^*}-1} \left(E_j(f) + E_{j+1}(f)\right), \tag{6.10}$$

and (6.9) is majorated by the series $\sum E_j(f)^{p^*}$. This, in turn, is convergent because of finiteness of (6.3).

Now by (6.4), (6.10) and (6.2)

$$\|f_j - f_{j+1}\|_{L_p(\Omega)} \le C 2^{\frac{jd}{p}} E_j(f), \quad j \ge -1.$$

Let now $S \subset \Omega$ be a subset of measure $2^{-jd}$. Then for $C := 2^{\frac{1}{p^*}-1}$ we get

$$\begin{aligned} \|f\|_{L_p(S)} &\le C\left\{\|f - f_j\|_{L_p(\Omega)} + |S|^{\frac{1}{p}} \|f_j\|_{L_\infty(\Omega)}\right\} \\ &\le C\left\{\|f - f_j\|_{L_p(\Omega)} + 2^{\frac{-jd}{p}} \sum_{i\le j} \|f_i - f_{i+1}\|_{L_\infty \Omega)}\right\}. \end{aligned}$$

Together with previous inequality this gives

$$\|f\|_{L_p(S)} \le C 2^{\frac{-jd}{p}} \sum_{i\le j} E_i(f). \tag{6.11}$$

To specify $S$, one introduces the set

$$\left\{x \in \Omega : |f(x)| \ge f^*(2^{-jd})\right\}$$

defined by the rearrangement $f^*$ of $f$. By the definition of $f^*$, this set contains a subset $S$ of measure $2^{-jd}$. With this choice of $S$ one gets

$$\|f\|_{L_p(S)} \ge |S|^{\frac{1}{p}} f^*(|S|) = 2^{\frac{-jd}{p}} f^*(2^{-jd}).$$

Together with (6.11) this implies that

$$f^*(2^{-jd}) \le C \sum_{i \le j} 2^{\frac{id}{p}} E_i(f). \tag{6.12}$$

We now derive from this the desired embedding (6.7). Since

$$\|f\|_{L_q(\Omega)} = \left\{ \int_0^1 f^*(t)^q dt \right\}^{\frac{1}{q}} \le C \left\{ \sum_{j \in \mathbb{Z}_+} \left( 2^{-jd} f^*(2^{-jd}) \right)^q \right\}^{\frac{1}{q}},$$

the inequality (6.12) and the Hardy inequalities with $0 < p < q < \infty$ imply that

$$\|f\|_{L_q(\Omega)} \le C \left\{ \sum_{j \in \mathbb{Z}_+} \left( 2^{jd\left(\frac{1}{p} - \frac{1}{q}\right)} E_j(f) \right)^p \right\}^{\frac{1}{p}} = C\|f\|_*.$$

By (6.3), this completes the proof for $q < \infty$. In the case $q = \infty$ the inequality (6.12) implies that

$$\|f\|_{L_\infty(\Omega)} = \lim_{j \to \infty} f^*(2^{-jd}) \le C \sum_{i \in \mathbb{Z}_+} 2^{\frac{id}{p}} E_i(f).$$

Since in this case $p \le 1$ and $\frac{\lambda}{d} = \frac{1}{p}$, the right hand side is at most

$$C \left\{ \sum_{j \in \mathbb{Z}_+} \left( 2^{j\lambda} E_j(f) \right)^p \right\}^{\frac{1}{p}} \le C\|f\|_*.$$

Hence the result holds for $q = \infty$, as well. ∎

To derive the embedding (3.7) from Theorem 6.1, one sets $A_j = V_j$, see (3.9), and then apply Lemma 3.3. The lemma asserts that for this choice of $A_j$ the space $\mathcal{E}_p^\lambda(\mathcal{A})$ coincides with the space $B_p^{w_q}(\mathcal{T})$. Then (6.6) implies the desired result (3.7).

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