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On Grüss Type Multivariate Integral Inequalities

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Dedicated to the 70th birthday of Academician Bl. Sendov

Grüss type multidimentional integral inequalities are established involving two functions of any number of independent variables. These estimates open new directions in the study of multivariate integral inequalities in general.

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0. Introduction

One of the most famous integral inequalities was given by Grüss [6] in 1935 and it can be stated as follows (see [7, p. 296]),

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right| \le \frac{1}{4}(M-m)(N-n),$$

where f and g are integrable functions on $[a,b]$ and satisfy the conditions

$$m \le f(x) \le M, \qquad n \le g(x) \le N,$$

for all $x \in [a,b]$, where m, M, n, N are given real numbers.

A great deal of attention has been given to the above inequality and many articles related to various generalizations, extensions, and variants of it have appeared in the literature; see Chapter X of the book [7] by Mitrinović, Pečarić, and Fink, where more references are given.

Here we establish the multivariate analog of Grüss inequality for as many as possible independent variables, given in two variations. Grüss inequalities for functions of two variables were first given by B. Pachpatte in [8]. His paper is one of our motivations. Next we mention one of his results. We follow the notations of [8] exactly.

Here R denotes the set of real numbers and $\Delta = [a,b] \times [c,d]$, a, b, c, $d \in R$. We denote by $G(\Delta)$ the set of continuous functions $z\colon \Delta \to R$ for which $D_2 D_1 z(x,y) = \frac{\partial^2 z(x,y)}{\partial y \partial x}$ exists and is continuous on Δ and belong to $L_\infty(\Delta)$. For any function $z(x,y) \in L_\infty(\Delta)$, we define $\|z\|_\infty = \sup_{(x,y)\in\Delta} |z(x,y)|$.

We need the following notation

$$
\begin{aligned}
k &= (b-a)(d-c),\\
H_1(x) &= \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right],\\
H_2(y) &= \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2}\right)^2\right],\\
F(x,y) &= \left[(d-c)\int_a^b f(t,y)\,dt + (b-a)\int_c^d f(x,s)\,ds\right],\\
G(x,y) &= \left[(d-c)\int_a^b g(t,y)\,dt + (b-a)\int_c^d g(x,s)\,ds\right],\\
M(x,y) &= |g(x,y)|\,\|D_2D_1 f\|_\infty + |f(x,y)|\,\|D_2D_1 g\|_\infty,
\end{aligned}
$$

for $f, g \in G(\Delta)$.

Then we have the following

Theorem 2 ([8]). *Let $f, g \in G(\Delta)$. It holds*

$$
\begin{aligned}
&\Bigg|\frac{1}{k}\int_a^b\int_c^d f(x,y)g(x,y)\,dy\,dx\\
&\qquad + \left(\frac{1}{k}\int_a^b\int_c^d f(x,y)\,dy\,dx\right)\left(\frac{1}{k}\int_a^b\int_c^d g(x,y)\,dy\,dx\right)\\
&\qquad - \frac{1}{2k^2}\int_a^b\int_c^d \big(g(x,y)F(x,y) + f(x,y)G(x,y)\big)\,dy\,dx\Bigg|\\
&\quad \le \frac{1}{2k^2}\int_a^b\int_c^d M(x,y)H_1(x)H_2(y)\,dy\,dx.
\end{aligned}
$$

Another motivation for this work is the approach in [3].

1. Auxiliary Result

We need the following generalized Montgomery type identity.

Theorem 2 ([1]). *Let* $f\colon \times_{i=1}^{n}[a_i,b_i] \to R$ *be a continuous mapping on* $\times_{i=1}^{n}[a_i,b_i]$, *and* $\frac{\partial^n f(x_1,\ldots,x_n)}{\partial x_1\cdots\partial x_n}$ *exists on* $\times_{i=1}^{n}[a_i,b_i]$ *and is integrable. Let also* $(x_1,\ldots,x_n) \in \times_{i=1}^{n}[a_i,b_i]$ *be fixed. We define the kernels* $p_i\colon [a_i,b_i]^2 \to R$:

$$p_i(x_i,s_i) := \begin{cases} s_i - a_i, & s_i \in [a_i,x_i], \\ s_i - b_i, & s_i \in (x_i,b_i], \end{cases} \qquad \text{for all } i = 1,\ldots,n.$$

Then it holds

$$\begin{aligned}
(1)\quad \theta_{1,n} &:= \int_{\times_{i=1}^{n}[a_i,b_i]} \prod_{i=1}^{n} p_i(x_i,s_i) \frac{\partial^n f(s_1,\ldots,s_n)}{\partial s_1\cdots\partial s_n} ds_1\cdots ds_n \\
&= \left\{ \left(\prod_{i=1}^{n} (b_i - a_i) \right) \cdot f(x_1,\ldots,x_n) \right\} \\
&\quad - \left[\sum_{i=1}^{\binom{n}{1}} \left(\prod_{\substack{j=1\\ j\neq i}}^{n} (b_j - a_j) \int_{a_i}^{b_i} f(x_1,\ldots,s_i,\ldots,x_n) ds_i \right) \right] \\
&\quad + \left[\sum_{\ell=1}^{\binom{n}{2}} \left(\prod_{\substack{k=1\\ k\neq i,j}}^{n} (b_k - a_k) \int_{a_i}^{b_i}\int_{a_j}^{b_j} f(x_1,\ldots,s_i,\ldots,s_j,\ldots,x_n) ds_i ds_j \right)_{(\ell)} \right] \\
&\quad - + \cdots - + \cdots + (-1)^{n-1} \\
&\quad \times \left[\sum_{j=1}^{\binom{n}{n-1}} (b_j - a_j) \int_{\times_{\substack{i=1\\ i\neq j}}^{n}[a_i,b_i]} f(s_1,\ldots,x_j,\ldots,s_n) ds_1\cdots \widehat{ds_j} \cdots ds_n \right] \\
&\quad + (-1)^n \int_{\times_{i=1}^{n}[a_i,b_i]} f(s_1,\ldots,s_n) ds_1\cdots ds_n =: \theta_{2,n}.
\end{aligned}$$

The above ℓ counts all the (i,j)'s, $i<j$ and $i,j = 1,\ldots,n$. Also $\widehat{ds_j}$ means ds_j is missing.

2. Main Results

We present now the following

Theorem 1. *Let* $f, g \in C^n(B)$, $n \in N$, *where* $B := \times_{i=1}^{n} [a_i, b_i]$, $a_i, b_i \in R$, *with* $a_i < b_i$, $i = 1, \dots, n$. *Denote by* ∂B *the boundary of the box* B. *Assume that* $f(x) = g(x) = 0$, *for all* $x = (x_1, \dots, x_n) \in \partial B$ *(in other words we assume that*

$$\begin{aligned} f(\dots, a_i, \dots) = g(\dots, a_i, \dots) &= f(\dots, b_i, \dots) \\ &= g(\dots, b_i, \dots) = 0, \end{aligned}$$

for all $i = 1, \dots, n$*). Call* $V_n := \prod_{i=1}^{n} (b_i - a_i)$. *Then*

$$\begin{aligned} \text{(2)} \qquad & \frac{1}{V_n} \int_B |f(x_1, \dots, x_n)| \, |g(x_1, \dots, x_n)| \, dx_1 \cdots dx_n \\ & \leq \frac{1}{2^{n+1}} \left[\int_B |g(x_1, \dots, x_n)| \left\| \frac{\partial^n f}{\partial s_1 \cdots \partial s_n} \right\|_\infty \right. \\ & \left. + |f(x_1, \dots, x_n)| \left\| \frac{\partial^n g}{\partial s_1 \cdots \partial s_n} \right\|_\infty dx_1 \cdots dx_n \right]. \end{aligned}$$

Proof. Let $(x_1, \dots, x_n) \in B$, i.e., $a_i \leq x_i \leq b_i$, for all $i = 1, \dots, n$. By the assumptions we get

$$f(x_1, \dots, x_n) = \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} \frac{\partial^n f(s_1, \dots, s_n)}{\partial s_1 \cdots \partial s_n} ds_1 \cdots ds_n,$$

and

$$f(x_1, \dots, x_n) = (-1)^n \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} \frac{\partial^n f(s_1, \dots, s_n)}{\partial s_1 \cdots \partial s_n} ds_1 \cdots ds_n.$$

In general we introduce the subintervals

$$I_{i,0} = [a_i, x_i] \qquad \text{and} \qquad I_{i,1} = [x_i, b_i], \quad i = 1, \dots, n.$$

Then we find that

$$\text{(3)} \quad f(x_1, \dots, x_n) = (-1)^{\varepsilon_1 + \cdots + \varepsilon_n} \int_{I_{1,\varepsilon_1}} \cdots \int_{I_{n,\varepsilon_n}} \frac{\partial^n f(s_1, \dots, s_n)}{\partial s_1 \cdots \partial s_n} ds_1 \cdots ds_n,$$

where each ε_i can be either 0 or 1. Adding up (3) for all 2^n choices for $(\varepsilon_1, \dots, \varepsilon_n)$ we obtain

$$\begin{aligned} \text{(4)} \qquad 2^n f(x_1, \dots, x_n) = \sum_{\varepsilon_1, \dots, \varepsilon_n} (-1)^{\varepsilon_1 + \cdots + \varepsilon_n} \int_{I_{1,\varepsilon_1}} \\ \cdots \int_{I_{n,\varepsilon_n}} \frac{\partial^n f(s_1, \dots, s_n)}{\partial s_1 \cdots \partial s_n} ds_1 \cdots ds_n. \end{aligned}$$

Next by taking absolute values in (4) and using basic properties of integrals (noticing that the 2^n “sub-boxes” $I_{1,\varepsilon_1} \times \cdots \times I_{n,\varepsilon_n}$ form a partition of B) and the subadditivity property of the absolute value we find that

$$\begin{aligned} |f(x_1,\ldots,x_n)| &\le \frac{1}{2^n}\int_B \left|\frac{\partial^n f(x_1,\ldots,x_n)}{\partial x_1\cdots\partial x_n}\right| dx_1\cdots dx_n \\ &\le \frac{V_n}{2^n}\left\|\frac{\partial^n f}{\partial x_1\cdots\partial x_n}\right\|_\infty . \end{aligned}$$

I.e., we have that

$$|f(x_1,\ldots,x_n)| \le \frac{V}{2^n}\left\|\frac{\partial^n f}{\partial x_1\cdots\partial x_n}\right\|_\infty , \tag{5}$$

is true for all $(x_1,\ldots,x_n) \in B$. Similarly it holds

$$|g(x_1,\ldots,x_n)| \le \frac{V_n}{2^n}\left\|\frac{\partial^n g}{\partial x_1\cdots\partial x_n}\right\|_\infty , \tag{6}$$

true for all $(x_1,\ldots,x_n) \in B$. Hence

$$|f(x_1,\ldots,x_n)|\,|g(x_1,\ldots,x_n)| \le \frac{V_n}{2^n}|g(x_1,\ldots,x_n)|\left\|\frac{\partial^n f}{\partial x_1\cdots\partial x_n}\right\|_\infty , \tag{7}$$

and

$$|f(x_1,\ldots,x_n)|\,|g(x_1,\ldots,x_n)| \le \frac{V_n}{2^n}|f(x_1,\ldots,x_n)|\left\|\frac{\partial^n g}{\partial x_1\cdots\partial x_n}\right\|_\infty . \tag{8}$$

Therefore by adding (7) and (8) we see that

$$\begin{aligned} &|f(x_1,\ldots,x_n)|\,|g(x_1,\ldots,x_n| \\ &\quad \le \frac{V_n}{2^{n+1}}\left(|g(x_1,\ldots,x_n)|\left\|\frac{\partial^n f}{\partial x_1\cdots\partial x_n}\right\|_\infty \right. \\ &\qquad \left. + |f(x_1,\ldots,x_n)|\left\|\frac{\partial^n g}{\partial x_1\cdots\partial x_n}\right\|_\infty\right), \end{aligned} \tag{9}$$

is true for all $(x_1,\ldots,x_n) \in B$. Integrating (9) over B we obtain (2). ∎

Remark 1. Inequality (9) has by itself its own merits.

We also give

Theorem 2. *Consider the class of functions* $G := \{f\colon \times_{i=1}^{n} [a_i,b_i] \to R$ *continuous,* $n \in N$*: the partial* $\frac{\partial^n f}{\partial x_1\cdots\partial x_n}$ *exists on* $\times_{i=1}^{n}[a_i,b_i]$ *and belongs to*

$L_\infty(\times_{i=1}^{n}[a_i,b_i])$ with norm $\|\cdot\|_\infty\}$. Here $(x_1,\dots,x_n)\in \times_{i=1}^{n}[a_i,b_i]$. Let $f,g\in G$. Denote $V_n := \prod_{i=1}^{n}(b_i-a_i)$, and

$$H_j(x_j) := \frac{(x_j-a_j)^2+(b_j-x_j)^2}{2}, \quad j=1,\dots,n.$$

Call

$$F_1(x_1,\dots,x_n) := \sum_{i=1}^{\binom{n}{1}}\left(\prod_{\substack{j=1\\ j\neq i}}^{n}(b_j-a_j)\int_{a_i}^{b_i} f(x_1,\dots,s_i,\dots,x_n)ds_i\right),$$

$$G_1(x_1,\dots,x_n) := \sum_{i=1}^{\binom{n}{1}}\left(\prod_{\substack{j=1\\ j\neq i}}^{n}(b_j-a_j)\int_{a_i}^{b_i} g(x_1,\dots,s_i,\dots,x_n)ds_i\right);$$

$$F_2(x_1,\dots,x_n) := -\left[\sum_{\ell=1}^{\binom{n}{2}}\left(\prod_{\substack{k=1\\ k\neq i,j}}^{n}(b_k-a_k)\int_{a_i}^{b_i}\int_{a_j}^{b_j} \times f(x_1,\dots,s_i,\dots,s_j,\dots,x_n)ds_i ds_j\right)_{(\ell)}\right],$$

where ℓ counts all (i,j)'s, $i<j$, and $i,j=1,\dots,n$, also

$$G_2(x_1,\dots,x_n) := -\left[\sum_{\ell=1}^{\binom{n}{2}}\left(\prod_{\substack{k=1\\ k\neq i,j}}^{n}(b_k-a_k)\int_{a_i}^{b_i}\int_{a_j}^{b_j} \times g(x_1,\dots,s_i,\dots,s_j,\dots,x_n)ds_i ds_j\right)_{(\ell)}\right];$$

$$\cdots\cdots\cdots\cdots\cdots\cdots;$$

$$F_{n-1}(x_1,\dots,x_n) := (-1)^n\left[\sum_{j=1}^{\binom{n}{n-1}}(b_j-a_j)\int_{\times_{\substack{i=1\\ i\neq j}}^{n}[a_i,b_i]} \times f(s_1,\dots,x_j,\dots,s_n)ds_1\cdots\widehat{ds_j}\cdots ds_n\right],$$

where $\widehat{ds_j}$ means ds_j is missing, also

$$G_{n-1}(x_1,\ldots,x_n) := (-1)^n \Bigg[\sum_{j=1}^{\binom{n}{n-1}} (b_j - a_j) \int_{\substack{\times_{i=1}^{n} [a_i,b_i] \\ i\neq j}} \times g(s_1,\ldots,x_j,\ldots,s_n) ds_1 \cdots \widehat{ds_j} \cdots ds_n \Bigg].$$

Also define

$$M_n(x_1,\ldots,x_n) := |g(x_1,\ldots,x_n)| \left\| \frac{\partial^n f}{\partial s_1 \cdots \partial s_n} \right\|_\infty + |f(x_1,\ldots,x_n)| \left\| \frac{\partial^n g}{\partial s_1 \cdots \partial s_n} \right\|_\infty .$$

Then

$$\begin{aligned} \Gamma_n := \Bigg| & \frac{1}{V_n} \int_{\times_{i=1}^{n} [a_i,b_i]} f(x_1,\ldots,x_n) g(x_1,\ldots,x_n) dx_1 \cdots dx_n \\ & + (-1)^n \left(\frac{1}{V_n} \int_{\times_{i=1}^{n} [a_i,b_i]} f(x_1,\ldots,x_n) dx_1 \cdots dx_n \right) \\ & \cdot \left(\frac{1}{V_n} \int_{\times_{i=1}^{n} [a_i,b_i]} g(x_1,\ldots,x_n) dx_1 \cdots dx_n \right) \\ & - \frac{1}{2V_n^2} \Bigg(\int_{\times_{i=1}^{n} [a_i,b_i]} \Bigg[g(x_1,\ldots,x_n) \sum_{j=1}^{n-1} F_j(x_1,\ldots,x_n) \\ & + f(x_1,\ldots,x_n) \sum_{j=1}^{n-1} G_j(x_1,\ldots,x_n) \Bigg] dx_1 \cdots dx_n \Bigg) \Bigg| \\ \le\ & \frac{1}{2V_n^2} \int_{\times_{i=1}^{n} [a_i,b_i]} M_n(x_1,\ldots,x_n) \left(\prod_{j=1}^{n} H_j(x_j) \right) dx_1 \cdots dx_n. \end{aligned} \tag{10}$$

medskip

P r o o f. We define the kernels $p_i \colon [a_i,b_i]^2 \to R$:

$$p_i(x_i,s_i) := \begin{cases} s_i - a_i, & s_i \in [a_i,x_i] \\ s_i - b_i, & s_i \in (x_i,b_i], \end{cases}$$

for all $i = 1, \ldots, n$.

We also have that

$$\int_{[a_j,b_j]} |p_j(x_j, s_j)| ds_j = H_j(x_j), \quad j = 1, \ldots, n.$$

¿From Theorem 2 ([1]) we obtain

$$\begin{aligned}(11)\quad V_n f(x_1, \ldots, x_n) = &\sum_{j=1}^{n-1} F_j(x_1, \ldots, x_n) \\ &+ (-1)^{n+1} \int_{\times_{i=1}^{n} [a_i,b_i]} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \\ &+ \int_{\times_{i=1}^{n} [a_i,b_i]} \prod_{i=1}^{n} p_i(x_i, s_i) \frac{\partial^n f(s_1, \ldots, s_n)}{\partial s_1 \cdots \partial s_n} ds_1 \cdots ds_n.\end{aligned}$$

And also

$$\begin{aligned}(12)\quad V_n g(x_1, \ldots, x_n) = &\sum_{j=1}^{n-1} G_j(x_1, \ldots, x_n) \\ &+ (-1)^{n+1} \int_{\times_{i=1}^{n} [a_i,b_i]} g(s_1, \ldots, s_n) ds_1 \cdots ds_n \\ &+ \int_{\times_{i=1}^{n} [a_i,b_i]} \prod_{i=1}^{n} p_i(x_i, s_i) \frac{\partial^n g(s_1, \ldots, s_n)}{\partial s_1 \cdots \partial s_n} ds_1 \cdots ds_n.\end{aligned}$$

Next we multiply (11) by $g(x_1, \ldots, x_n)$ and (12) by $f(x_1, \ldots, x_n)$ and we add the resulting identities, to get

$$\begin{aligned}(13)\qquad &2V_n f(x_1, \ldots, x_n) g(x_1, \ldots, x_n) \\ &= \Bigg[g(x_1, \ldots, x_n) \sum_{j=1}^{n-1} F_j(x_1, \ldots, x_n) \\ &\quad + f(x_1, \ldots, x_n) \sum_{j=1}^{n-1} G_j(x_1, \ldots, x_n) \Bigg] \\ &\quad + A_n(x_1, \ldots, x_n) + B_n(x_1, \ldots, x_n).\end{aligned}$$

Here we have

$$(14)\quad A_n(x_1, \ldots, x_n) := (-1)^{n+1} \Bigg[g(x_1, \ldots, x_n)$$

$$\times \int_{\substack{n \\ \times \\ i=1}[a_i,b_i]} f(s_1,\dots,s_n)ds_1\cdots ds_n$$
$$\left. + f(x_1,\dots,x_n)\int_{\substack{n \\ \times \\ i=1}[a_i,b_i]} g(s_1,\dots,s_n)ds_1\cdots ds_n\right],$$

and

$$B_n(x_1,\dots,x_n) := \left[g(x_1,\dots,x_n)\int_{\substack{n \\ \times \\ i=1}[a_i,b_i]} \prod_{i=1}^{n} p_i(x_i,s_i)\right.$$
$$\times \frac{\partial^n f(s_1,\dots,s_n)}{\partial x_1\cdots\partial s_n}ds_1\cdots ds_n$$
$$+ f(x_1,\dots,x_n)\int_{\substack{n \\ \times \\ i=1}[a_i,b_i]} \prod_{i=1}^{n} p_i(x_i,s_i)$$
$$\left.\times \frac{\partial^n g(s_1,\dots,s_n)}{\partial s_1\cdots\partial s_n}ds_1\cdots ds_n\right]. \tag{15}$$

We observe that

$$|B_n(x_1,\dots,x_n)| \le M_n(x_1,\dots,x_n)\left(\prod_{j=1}^{n} H_j(x_j)\right). \tag{16}$$

Next we integrate (13) over $\overset{n}{\underset{i=1}{\times}}[a_i,b_i]$ and we find

$$\frac{1}{V_n}\int_{\substack{n \\ \times \\ i=1}[a_i,b_i]} f(x_1,\dots,x_n)g(x_1,\dots,x_n)dx_1\cdots dx_n$$
$$= \frac{1}{2V_n^2}\left(\int_{\substack{n \\ \times \\ i=1}[a_i,b_i]}\left[g(x_1,\dots,x_n)\sum_{j=1}^{n-1}F_j(x_1,\dots,x_n)\right.\right.$$
$$\left.\left. + f(x_1,\dots,x_n)\sum_{j=1}^{n-1}G_j(x_1,\dots,x_n)\right]dx_1\cdots dx_n\right)$$
$$+ (-1)^{n+1}\left[\left(\frac{1}{V_n}\int_{\substack{n \\ \times \\ i=1}[a_i,b_i]} f(x_1,\dots,x_n)dx_1\cdots dx_n\right)\right.$$
$$\left.\times\left(\frac{1}{V_n}\int_{\substack{n \\ \times \\ i=1}[a_i,b_i]} g(x_1,\dots,x_n)dx_1\cdots dx_n\right)\right]$$
$$+ \frac{1}{2V_n^2}\int_{\substack{n \\ \times \\ i=1}[a_i,b_i]} B_n(x_1,\dots,x_n)dx_1\cdots dx_n. \tag{17}$$

Consequently we obtain

$$\Gamma_n \leq \frac{1}{2V_n^2} \int_{\times_{i=1}^{n}[a_i,b_i]} |B_n(x_1,\ldots,x_n)| dx_1 \cdots dx_n. \tag{18}$$

At last using (18) along with (16) we have established (10). ■

Remark 2. From (13), (14), (16) we get

$$\begin{aligned} &\Bigg| 2V_n f(x_1,\ldots,x_n) g(x_1,\ldots,x_n) - g(x_1,\ldots,x_n) \\ &\times \sum_{j=1}^{n-1} F_j(x_1,\ldots,x_n) - f(x_1,\ldots,x_n) \sum_{j=1}^{n-1} G_j(x_1,\ldots,x_n) - A_n(x_1,\ldots,x_n) \Bigg| \\ &\quad \leq M_n(x_1,\ldots,x_n) \left(\prod_{j=1}^{n} H_j(x_j) \right), \text{ for } (x_1,\ldots,x_n) \in \times_{i=1}^{n} [a_i,b_i]. \end{aligned} \tag{19}$$

We also give

Corollary 1 (to Theorem 1). *Consider the class of functions* $\mathcal{F} := \{f \in C^n(B)$, *where* $n \in N$, $B := \times_{i=1}^{n} [a_i,b_i]$ *such that* $f(x) = 0$, *for all* $x = (x_1,\ldots,x_n) \in \partial B$ *(the boundary of* B*)*$\}$. *Let* $f \in \mathcal{F}$. *Let also* $g \in C^n(B)$. *Set* $V_n := \prod_{i=1}^{n} (b_i - a_i)$. *Then*

$$\begin{aligned} &\frac{1}{V_n} \int_B |f(x_1,\ldots,x_n)| \, |g(x_1,\ldots,x_n)| \, dx_1 \cdots dx_n \\ &\qquad \leq \frac{1}{2^n} \int_B \left| \frac{\partial^n (fg)}{\partial x_1 \cdots \partial x_n} (x_1,\ldots,x_n) \right| dx_1 \cdots dx_n. \end{aligned} \tag{20}$$

Proof. Totally the same way as in the proof of Theorem 1 we obtain

$$|f(x_1,\ldots,x_n)| \leq \frac{1}{2^n} \int_B \left| \frac{\partial^n f(x_1,\ldots,x_n)}{\partial x_1 \cdots \partial x_n} \right| dx_1 \cdots dx_n, \quad \forall f \in \mathcal{F}.$$

Integrating over B the last one we get

$$\int_B |f(x_1,\ldots,x_n)| \, dx_1 \cdots dx_n \leq \frac{V_n}{2^n} \int_B \left| \frac{\partial^n f(x_1,\ldots,x_n)}{\partial x_1 \cdots \partial x_n} \right| dx_1 \cdots dx_n, \tag{21}$$

true for any $f \in \mathcal{F}$.

The inequality (21) was also proved in [3]. Clearly here $f \cdot g \in \mathcal{F}$.
Finally applying (21) for $f \cdot g$ we establish (20). ■

Finally we have

Corollary 2 (to Theorem 2). *Case of* $n = 3$. *Here we consider the class of functions* $G^* := \{f\colon \times_{i=1}^{3} [a_i, b_i] \to R$ *continuous:* $\frac{\partial^3 f}{\partial x_1 \partial x_2 \partial x_3}$ *exists on* $\times_{i=1}^{3} [a_i, b_i]$ *and belongs to* $L_\infty(\times_{i=1}^{3} [a_i, b_i])$ *with norm* $\|\cdot\|_\infty\}$. *Let* $(x_1, x_2, x_3) \in \times_{i=1}^{3} [a_i, b_i]$ *and* $f, g \in G^*$. *Denote* $V_3 := \prod_{i=1}^{3} (b_i - a_i)$, *and*

$$H_j(x_j) := \frac{(x_j - a_j)^2 + (b_j - x_j)^2}{2}, \quad j = 1, \dots, n.$$

Call

$$F_1(x_1, x_2, x_3) := \sum_{i=1}^{3} \left(\prod_{\substack{j=1 \\ j \neq i}}^{3} (b_j - a_j) \int_{a_i}^{b_i} f(x_1, s_i, x_3) ds_i \right),$$

$$G_1(x_1, x_2, x_3) := \sum_{i=1}^{3} \left(\prod_{\substack{j=1 \\ j \neq i}}^{3} (b_j - a_j) \int_{a_i}^{b_i} g(x_1, s_i, x_3) ds_i \right);$$

$$\begin{aligned} F_2(x_1, x_2, x_3) := -\Bigg[&(b_3 - a_3) \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(s_1, s_2, x_3) ds_1 ds_2 \\ &+ (b_2 - a_2) \int_{a_1}^{b_1} \int_{a_3}^{b_3} f(s_1, x_2, s_3) ds_1 ds_3 \\ &+ (b_1 - a_1) \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x_1, s_2, s_3) ds_2 ds_3 \Bigg], \end{aligned}$$

$$\begin{aligned} G_2(x_1, x_2, x_3) := -\Bigg[&(b_3 - a_3) \int_{a_1}^{b_1} \int_{a_2}^{b_2} g(s_1, s_2, x_3) ds_1 ds_2 \\ &+ (b_2 - a_2) \int_{a_1}^{b_1} \int_{a_3}^{b_3} g(s_1, x_2, s_3) ds_1 ds_3 \\ &+ (b_1 - a_1) \int_{a_2}^{b_2} \int_{a_3}^{b_3} g(x_1, s_2, s_3) ds_2 ds_3 \Bigg]; \end{aligned}$$

$$\begin{aligned} M_3(x_1, x_2, x_3) := &|g(x_1, x_2, x_3)| \left\| \frac{\partial^3 f}{\partial s_1 \partial s_2 \partial s_3} \right\|_\infty \\ &+ |f(x_1, x_2, x_3)| \left\| \frac{\partial^3 g}{\partial s_1 \partial s_2 \partial s_3} \right\|_\infty . \end{aligned}$$

Then

$$\begin{aligned}
(22)\qquad & \left| \frac{1}{V_3} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x_1,x_2,x_3) g(x_1,x_2,x_3) dx_1 dx_2 dx_3 \right. \\
& - \left(\frac{1}{V_3} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} f(x_1,x_2,x_3) dx_1 dx_2 dx_3 \right) \\
& \left(\frac{1}{V_3} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} g(x_1,x_2,x_3) dx_1 dx_2 dx_3 \right) \\
& - \frac{1}{2V_3^2} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} [g(x_1,x_2,x_3)(F_1(x_1,x_2,x_3) \\
& + F_2(x_1,x_2,x_3)) + f(x_1,x_2,x_3)(G_1(x_1,x_2,x_3) \\
& \left. + G_2(x_1,x_2,x_3))] dx_1 dx_2 dx_3 \right| \\
& \le \frac{1}{2V_3^2} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} M_3(x_1,x_2,x_3) \left(\prod_{j=1}^{3} H_j(x_j) \right) dx_1 dx_2 dx_3 .
\end{aligned}$$

Comment. Our Theorem 2 clearly generalizes Theorem 2 of [8], i.e., for $n = 2$ the corresponding inequalities coincide.

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