**M a t h e m a t i c a**
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**New Series Vol. 17, 2003, Fasc. 1-2**

# Mathematical Model of the Distributed Van der Pol Self-oscillator

*N. Kh. Rozov, V. A. Sadovnichii*

*To dear Blagovest Khristovich Sendov*
*with sincere best regards, for remarkable jubilee*

We establish the bufferness phenomenon in the distributed analog of Van der Pol generator, whose mathematical model is a system of linear telegraph equations with nonlinearity in one of the boundary conditions.

*AMS Subj. Classification*: 35L60, 35B05

*Key Words*: linear telegraph equation, boundary value problems, Van der Pol self-oscillator

Theoretical and numerical research of mathematical models in mechanics, physics, technics, biology, economics etc has always been an important instrument for studying different real processes. At the same time the deep analysis of such models often lead to setting up new pure mathematical problems, and became the reason for developing new mathematical methods.

The brilliant example of mutually enriching interaction between theoretical study of mathematical model and deep penetration into the main point of nature process is the bufferness phenomenon. This phenomenon became the matter of overall theoretical research rather recently, but gave an opportunity to understand better the possible mechanism of complex relations between order and chaos in reality.

**0**. It is widely known that the stable limit cycles of the ordinary differential equations play an exclusive role in the analysis of mathematical models of various *oscillatory systems with lumped parameters*. There are models where the matter of principle is the existence of unique stable limit cycle. In other models, on the contrary, it is interesting to discover the multiplicity of such

cycles. Everything depends on the properties of concrete applied problem under consideration. Moreover, it is clear that, generally speaking, there exists different number of stable limit cycles, depending on the values of parameters in the system of differential equations. (These parameters describe actual properties of the object under investigation.)

It is easy to give examples of systems of ordinary differential equations, where, under appropriate choice of parameter values, one can guarantee the existence of arbitrary a priori defined finite number of stable limit cycles. Let us consider the following second order system (in polar coordinates)

$$\dot{r} = r\sin(1/\alpha r), \qquad \dot{\varphi} = 1,$$

where $\alpha, 0 < \alpha < 1/\pi$, is a parameter. For any natural number $N$ it is possible to indicate the value of parameter $\alpha$, positive and close to zero, providing exactly $N$ stable limit cycles, which are located in the area $r > 1$ of phase plane.

Nowadays in science, new technics and modern technologies the so-called *oscillatory systems with distributed parameters* or, more shortly, *the distributed oscillatory systems* are widely spread. (For example, the study of autogenerators with distributed parameters became actual as a result of the intensive development of microelectronics, after the opportunity appeared to construct self-oscillating systems on distributed structures.) These are the objects, the state of which depends on both time and space variables: the state of each point of space changes periodically on time. Thus, the periodic self-oscillations are generated, or *auto-wave processes*, following the term introduced by R. V. Khokhlov.

The dynamics of these objects is simulated, as a rule, by the partial differential equations (with boundary and initial conditions). Each actual self-oscillatory regime is corresponded by *the stable cycle* (it is a solution of the appropriate equation, periodic on time and satisfying the boundary conditions). The partial differential equation, being an adequate mathematical model of an oscillatory system with distributed parameters, can have one or several stable cycles. It is extremely important to find out the possible number of such cycles, because it will help to learn the number of really existing self-oscillating regimes. It is natural that the number of such cycles, generally speaking, may be different depending on the values of parameters in the equation.

Thus, the pure mathematical problem appears: to study how the number of stable cycles of partial differential equation (or the system) with boundary condition depends on involved parameters.

We will say that *the bufferness phenomenon* takes place in the distributed oscillatory system, if the system possesses the following property. The partial differential equation, which is the mathematical model of the system, possesses,

under the appropriate values of parameters, an arbitrary a priori fixed finite number of different stable cycles. Theoretically speaking, for any natural number $N$ one can choose physical characteristics of the system so that it will have $N$ auto-oscillatory regimes.

It appears that the bufferness phenomenon proves to be the property of many distributed oscillating systems, whose mathematical model is described by partial differential equation of hyperbolic or parabolic type. This phenomenon helps to understand better the features of such systems, to describe the set of possible self-oscillating regimes, and even to propose some hypotheses about the ways of transition from order to chaos in nature.

This paper contains our results in theoretical analysis for existing of bufferness phenomenon in a particular example of distributed oscillating system – in one autogenerator containing two long $LCR-$lines with a tunnel diode, posessing standart characteristic (so-called $LCR-$autogenerator). This system is a natural distributed analog of a classical one-lamp Van der Pol generator with oscillating circuit in anode chain.

**1**. Omitting the description of physical details, the deduction of equations (it is standard enough under a conventional assumptions — in particular, under cubic approximation of the diode characteristic), technical transforms and changes of variables, we consider the mathematical model being the non-linear boundary value problem for the system of telegraph equations

$$\begin{aligned} &\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial x}, \qquad \frac{\partial v}{\partial t} = -\frac{\partial u}{\partial x} - \varepsilon v, \\ &u|_{x=0} = 0, \qquad v|_{x=1} + \beta_0\sigma + \beta_1\sigma^2 - \beta_2\sigma^3 = 0; \\ &\sigma = u|_{x=1} + \varepsilon \int_0^1 v(t,x)\,dx. \end{aligned} \tag{1}$$

Here $t$ is time, $x$ is the coordinate along a line, $u(t,x)$ and $v(t,x)$ are the voltage and the current in a line; all the parameters are constant.

For the space of initial conditions $(u(x), v(x))$ of boundary problem (1) let us consider the non-linear manifold $\Sigma$ in Hilbert space $W_2^1([0,1]; R^2)$, consisting of vector-functions, the components of which satisfy the boundary conditions from (1). Let us point out that the unique solvability of the mixed problem, corresponding to (1), with initial conditions from $\Sigma$ can be proved easily: it is enough to integrate the linear system of equations from (1) along characteristics, and to substitute the result to the boundary conditions from (1).

We will study the $t-$periodic solutions of the boundary problem (1) under

three additional assumptions

$$\beta_0 = \frac{\varepsilon}{2} + \gamma\varepsilon^2, \qquad \beta_1 = 0, \qquad \beta_2 = 1, \tag{2}$$

where $0 < \varepsilon \ll 1$, and the parameter $\gamma > 0$ has the order of unit. The first condition from (2) follows from purely mathematical reasons (they will become clear later), the second assumption (meaning the symmetry of the diod characteristic) is not crusial and is posed just for simplicity, and the third equation from (2) can be achieved by suitable normalization of functions $u, v$.

Substituting the conditions (2) to the equations (1) and keeping in the second boundary condition only terms essential for future, we obtain the following boundary problem

$$\begin{gathered} \frac{\partial u}{\partial t} = -\frac{\partial v}{\partial x}, \qquad \frac{\partial v}{\partial t} = -\frac{\partial u}{\partial x} - \varepsilon v, \\ u|_{x=0} = 0, \\ v|_{x=1} + \left(\frac{\varepsilon}{2} + \gamma\varepsilon^2\right) u|_{x=1} + \frac{\varepsilon^2}{2}\int_0^1 v(t,x)\,dx - u^3|_{x=1} = 0. \end{gathered} \tag{3}$$

The last boundary problem is the very problem for which the bufferness phenomenon will be stated. To construct the $t-$periodic solutions of the problem (3), we will use the infinite-dimensional normalization method, which is the special version of the well-known Krylov — Bogolyubov — Mitropol'skii asymptotic method, and is associated with the technics of quasinormal forms developed by Yu.S. Kolesov.

**2**. First we substitute to the boundary problem (3) the formal series

$$u = \sum_{k=0}^{\infty} \varepsilon^{k+1} u_k(t,\tau,x), \qquad v = \sum_{k=0}^{\infty} \varepsilon^{k+1} v_k(t,\tau,x), \tag{4}$$

where $\tau = \varepsilon^2 t$, all the coefficients are 4-periodic in respect to $t$,

$$\begin{gathered} u_0 = \sum_{n=1}^{\infty} [z_n(\tau)\exp(i\omega_n t) + \overline{z}_n(\tau)\exp(-i\omega_n t)] \\ \times(-1)^{n-1}\sin(\omega_n x), \\ v_0 = i\sum_{n=1}^{\infty} [z_n(\tau)\exp(i\omega_n t) - \overline{z}_n(\tau)\exp(-i\omega_n t)] \\ \times(-1)^{n-1}\cos(\omega_n x), \end{gathered} \tag{5}$$

$\omega_n = \pi(2n-1)/2, n = 1, 2, \ldots$, and complex "amplitudes" $z_n, n = 1, 2, \ldots$ are such that the series with the general term $\omega_n^2 |z_n|^2$ converges (then $u_0, v_0 \in W_2^1$ in respect to variable $x$.) Let us point out, that searching of periodic solutions of the boundary problem (3) in the form (4) is quite natural, because formulas (5) (with fixed $\tau$) describe the arbitrary periodic solution of linear boundary problem, obtained from non-linear boundary problem (3) with $\varepsilon = 0$ and after neglecting of non-linearity.

Then, comparing the coefficients with the equal powers of $\varepsilon$, we can successively determine the functions $u_j, v_j, j = 1, 2$, and the unknown amplitudes $z_n, n = 1, 2, \ldots$.

During the first step of this algorithm we come to the boundary problem

$$\frac{\partial u_1}{\partial t} = -\frac{\partial v_1}{\partial x}, \qquad \frac{\partial v_1}{\partial t} = -\frac{\partial u_1}{\partial x} - v_0,$$

$$u_1|_{x=0} = 0, \qquad v_1|_{x=1} + \frac{1}{2} u_0|_{x=1} = 0;$$

we will seek its solution as

$$u_1 = \sum_{n=1}^{\infty} [z_n A_n(x) \exp(i\omega_n t) + \overline{z}_n \overline{A}_n(x) \exp(-i\omega_n t)],$$

$$v_1 = \sum_{n=1}^{\infty} [z_n B_n(x) \exp(i\omega_n t) + \overline{z}_n \overline{B}_n(x) \exp(-i\omega_n t)].$$

Then, to determine the functions $A_n, B_n, n = 1, 2, \ldots$, we obtain the linear non-homogeneous boundary problems; their solvability is provided by the term $(\varepsilon/2)u|_{x=1}$ in the second boundary condition from (3). (Thus the reason of the first condition in (2) becomes clear.) The simple analysis of this linear boundary problems leads to equations

$$\begin{aligned} A_n &= -\frac{i}{2}(-1)^{n-1} x \cos \omega_n x, \\ B_n &= \frac{i}{\omega_n}[A_n'(x) + i(-1)^{n-1} \cos \omega_n x], \qquad n = 1, 2, \ldots \end{aligned} \tag{6}$$

During the second step of the algorithm, after excluding $v_2$, we come to the boundary problem

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) u_2 = -2 \frac{\partial^2 u_0}{\partial t \partial \tau} - \frac{\partial u_1}{\partial t},$$

$$u_2|_{x=0} = 0, \qquad \left.\frac{\partial u_2}{\partial x}\right|_{x=1} = \frac{d}{dt}(\gamma u_0|_{x=1} - u_0^3|_{x=1});$$

we will seek its solution as

$$u_2 = \sum_{n=1}^{\infty}[C_n(x)\exp(i\omega_n t) + \overline{C}_n(x)\exp(-i\omega_n t)].$$

It is easy to verify that, taking (6) into account, we can obtain the following boundary problems for the functions $C_n, n = 1, 2, \ldots$ :

$$\begin{gathered} C_n'' + \omega_n^2 C_n = (-1)^{n-1}\left(2i\omega_n \frac{dz_n}{d\tau}\sin\omega_n x + \frac{\omega_n}{2} z_n x\cos\omega_n x\right), \\ C_n(0) = 0, \qquad C_n'(0) = i\omega_n f_n; \end{gathered} \tag{7}$$

here $f_n$ is a coefficient by the harmonic $\exp(i\omega_n y)$ of the Fourier series of function $\gamma w(\tau, y) - w^3(\tau, y)$, where

$$w(\tau, y) = \sum_{n=1}^{\infty}[z_n(\tau)\exp(i\omega_n y) + \overline{z}_n(\tau)\exp(-i\omega_n y)]. \tag{8}$$

The conditions of solvability of boundary problems (7) lead, in its turn, to the countable system of ordinary differential equations

$$i\omega_n \dot{z}_n = -\frac{1}{8}z_n + i\omega_n f_n, \qquad n = 1, 2, \ldots, \tag{9}$$

for unknown amplitudes $z_n, n = 1, 2, \ldots$. At last, with the help of function (8), we can make sure that the countable system of equations (9) can be "reduced" to the boundary problem

$$\frac{\partial^2 w}{\partial\tau\partial y} = -\frac{1}{8}w + (\gamma - 3w^2)\frac{\partial w}{\partial y}, \quad w(\tau, y+2) \equiv -w(\tau, y). \tag{10}$$

**3**. It is possible to attach strict reason to the above formal constructions in the way similar to the case of ordinary differential equations. In particular, using the constructed segments of series (4), we can construct the variable change transforming the original boundary-value problem (3), with precision of the terms of the order up to $\varepsilon$, to the form (9).

Thus the countable system (9), and hence the boundary problem (10) as well, present the "reduced" normal form of the problem (3). That is why the following standard assertion about the correspondence between their periodic solutions is valid.

**Theorem 1.** *Assume that the boundary-value problem (10) has a periodic solution* $w = w_0(\xi)$, $\xi = \sigma_0\tau + y$, *that is exponentially orbitally stable or dichotomous (in metrics* $W_2^1$*). Then there exists* $\varepsilon_0 > 0$, *such that for all* $0 < \varepsilon \ll \varepsilon_0$ *the original boundary problem (3) possesses a corresponding cycle (t− periodic solution) with the same stability and asymptotics (4). At the same time, in the formulas (5) we should take* $z_n(\tau) = w_n^0 \exp(i\omega_n\sigma_0\tau)$, *where* $w_n^0, \overline{w}_n^0, n = 1, 2, \ldots,$ *are the Fourier coefficients of the function* $w_0(\xi)$ *in the system* $\exp(\pm i\omega_n\xi), n = 1, 2, \ldots.$

Let us note that the method used here is the infinite-dimensional analog of the method of slowly altering amplitudes, that is widely used in oscillations theory.

**4**. After appropriate normalizing of variables $w, \tau, y$ in the boundary-value problem (10) and changing $\tau$ to $t$ and $y$ to $x$, we can transform it to the more suitable form

$$\frac{\partial^2 w}{\partial t \partial x} = w + \lambda(1 - w^2)\frac{\partial w}{\partial x}, \qquad w(t, x + 1) = -w(t, x), \tag{11}$$

where $\lambda = 4\gamma > 0$. It follows from the theorem 1 that the matter of interest is the dynamics in respect to $\lambda$ of its periodic solutions of the type of travelling wave

$$w = \theta(y), \qquad y = \sigma t - x, \qquad \sigma > 0. \tag{12}$$

Substituting the representation (12) to the boundary-value problem (11), we obtain the ordinary differential equation for the function $\theta(y)$, which after change of variable $y/\sqrt{\sigma} \to y$ can be rewritten in the form

$$\theta'' + r(\theta^2 - 1)\theta' + \theta = 0, \tag{13}$$

where $r = \lambda/\sqrt{\sigma}$. Note that (13) is a classical Van der Pol equation, which for all $r > 0$ has a unique periodic solution

$$\theta = H(y, r), \qquad H(y, 0) = 2\cos y,$$

with a period $T = T(r)$, $T(0) = 2\pi$, also satisfying the condition

$$H(y + (T/2), r) \equiv -H(y, r).$$

However, the function $\theta(y)$ involved in (12) is periodic with the period 2, and that is why we are interested in the solution of the equation (13) which has the period $2/\sqrt{\sigma}$. Hence, we obtain the following equation for definition of the unknown parameter $r$

$$\lambda T(r) - 2r = 0. \tag{14}$$

With $\lambda \ll 1$, applying the implicit function theorem to the equation (14) in the point $\lambda = 0, r = 0$, we can uniquely define its solution $r = r_0(\lambda)$, $r_0(0) = 0$. Then, using the correspondence between the equation (13) and the original problem (11), we can conclude that the problem (11) has a periodic solution

$$w = \theta_0(y, \lambda), \qquad y = \sigma_0(\lambda)t - x;$$

$$\sigma_0(\lambda) = \left(\frac{\lambda}{r_0(\lambda)}\right)^2, \quad \theta_0(y, \lambda) = H\left(\frac{y}{\sqrt{\sigma_0(\lambda)}}, r_0(\lambda)\right). \tag{15}$$

Note that along with the solution (15) the boundary-value problem (11) with $\lambda \ll 1$ has other periodic solutions, that can be obtained from the constructed solution using the similarity principle:

$$w = \theta_n(y, \lambda), \qquad y = \sigma_n(\lambda)t - x, \qquad n = 0, 1, 2, \ldots;$$

$$\theta_n = \theta_0((2n+1)y, (2n+1)\lambda), \quad \sigma_n(\lambda) = \frac{\sigma_0((2n+1)\lambda)}{(2n+1)^2}. \tag{16}$$

**Theorem 2.** *For any positive integer $N$ one can find a small enough $\lambda_N > 0$, such that for all $0 < \lambda \le \lambda_N$ the boundary-value problem (11) has exponentially orbitally stable periodic solutions (16) with subscripts $n = 0, 1, 2, \ldots, N$.*

Note that the defined solutions of the boundary-value problem (11) in the form of the waves with stationary profile, which are described by the Van der Pol equation, from the physical point of view are related to autosolitons, that can be observed for example in the ring lazer in the case of self-synchronization of modes. Under such interpretation the equation (14) is analogous to the nonlinear dispersion equation. Note also that in our case the "main" period of oscillations, i.e. the period of the function $\theta_0(y, \lambda)$, is equal to 2, and all the functions $\theta_n(y, \lambda), n \ge 1$, according to the similarity principle, have the periods $2/(2n+1)$. Then, using also the theorem 1, we can conclude that the $t-$periodic solutions $(u_n(t, x, \varepsilon), v_n(t, x, \varepsilon)), n = 0, 1, \ldots,$ of the original boundary-value problem (3), corresponding to the solutions (16), have the periods

$$T_n(\varepsilon) = \frac{4}{2n+1} + O(\varepsilon^2).$$

Let us complete Theorem 2: each periodic solution (16) can be continuously prolonged in respect to parameter $\lambda$ upon the interval

$$0 < \lambda < \lambda_n, \qquad \lambda_n = \frac{\lambda_0}{2n+1}, \qquad \lambda_0 = \frac{2}{3 - 2\ln 2},$$

and becomes discontinuous when $\lambda \geq \lambda_n$. It is quite clear from the physical point of view: increasing of the energetic parameter $\lambda$, along with the effect of self-synchronization of modes, first leads to complication of the shape of each cycle (16), and then (when $\lambda \geq \lambda_n$) to its complete destruction.

In conclusion we will state the following consideration. It follows from Theorems 1 and 2 that the original boundary-value problem (3) under the appropriate decrease of parameters $\varepsilon$ and $\gamma$ exhibits any a priori fixed number of stable cycles. If the system with several stable cycles can be considered as a determined system, then it is natural to consider the behaviour of the system with big enough number of stable cycles as chaotic. Hence, we can state that the bufferness phenomenon in a particular real system indicates that in this system the transition from the determined to the chaotic behaviour is possible. The considered distributed oscillating system, analogous to the Van der Pol self-oscillator, is a particular example of such system.

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*Lomonosov Moscow State University*
*119 899 Moscow, RUSSIA*

*Received: 07.03.2003*