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On C-Mappings Images of Metric Spaces [1]

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In this paper, we prove that a space X is a C-mapping image of a metric space if and only if X has a compact-star network. Using this result, we establish the characterizations of images of metric spaces under some C-mappings. As some applications of the above results, we obtain that some C-mappings preserve metrizability.

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To find the internal characterizations of certain images of metric spaces is one of the central questions in general topology. Since Arhangel'skii published the famous paper "Mappings and spaces"([2]) in 1966, the behaviour of certain P-mappings images and C-mappings images on metric spaces has attracted considerable attention, and some noticeable results have been obtained. In [5], [6], [1] and [3] Heath, Kofner, Alexander and Burke gave internal characterizations of open P, quotient P, pseudo-open P and countably bi-quotient P-mappings images of metric spaces respectively. In [12], Tanaka introduced g-developable spaces and characterized quotient P-mappings images of metric spaces. Recently, Lin introduced the concept of point-star network to discuss sequence-covering (1-sequence-covering, sequentially quotient) P-mappings on metric spaces and established "The point-star network characterizations" for some P-mappings images of metric spaces ([10],[11]). Then, what are the internal characterizations of certain C-mappings images of metric spaces? we know that open C-mappings images of metric spaces are metric spaces ([5]). However,

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except for this result, we know nothing about C-mappings. This arouses our interest in the C-mappings images of metric spaces.

In this paper, we introduced the concept of compact-star network, and prove that a space X is a C-mapping image of a metric space if and only if X has a compact-star network. Using this result, we establish the characterizations of images of metric spaces under some C-mappings. As some applications of the above results, we obtain that some C-mappings preserve metrizability.

Throughout this paper, all spaces are regular and T_1, all mappings are continuous and onto. N denotes the set of all natural numbers. $\{x_n\}$ or $\{x_n : n \in N\}$ denotes a sequence, the n-th term is x_n. (α_n) denotes a point in a Tychonoff-product space, the n-th coordinate is α_n. A sequence $\{P_n : n \in N\}$ of subsets of a space abbreviates to $\{P_n\}$. Similarity, a sequence $\{\mathcal{P}_n : n \in N\}$ of families of subsets of a space abbreviates to $\{\mathcal{P}_n\}$. Let X be a space, $x \in X$, A be a subset of X and $\mathcal{U}$ be a family of subsets of X. Then A° denotes the interior of A, $st(x,\mathcal{U}) = \cup\{\mathcal{U} \in \mathcal{U} : x \in U\}$, $st(A,\mathcal{U}) = \cup\{\mathcal{U} \in \mathcal{U} : U \cap A \neq \phi\}$. For terms which are not defined here, refer to [4].

Definition 1. Let $f : X \longrightarrow Y$ be a mapping.

(1) Let X be a metric space with a metric d. f is a C-mapping ([5]), if for each compact subset of Y and each open subset $U \supset K$, there is an $\varepsilon > 0$ such that $f(B(f^{-1}(K),\varepsilon)) \subset U$, here $B(f^{-1}(K),\varepsilon) = \{x \in X : d(f^{-1}(K),x) < \varepsilon\}$;

(2) f is an almost open mapping ([9]), if for each $y \in Y$ there is $x \in f^{-1}(y)$ such that if U is a neighborhood of x, then $y \in (f(U))^\circ$;

(3) f is a pseudo-open mapping ([1]), if $y \in Y$ and open subset $U \supset f^{-1}(y)$, then $y \in (f(U))^\circ$;

(4) f is a quotient mapping ([6]), if $f^{-1}(U)$ is open in X if and only if U is open in Y;

(5) f is a sequence-covering mapping (sequentially quotient mapping) ([8]), if for each convergent sequence S of Y, there is a convergent sequence L of X such that $f(L) = S$ ($f(L) \subset S$);

(6) f is a 1-sequence-covering mapping ([8]), if for each $y \in Y$, there is $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$; f is a 2-sequence-covering mapping ([9]), if $y \in Y$ and $x \in f^{-1}(y)$ then whenever $\{y_n\}$ is a sequence converging to y in Y there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

Remark 1. The following are known for mappings ([9],[10]):

(1) Almost open $\Longrightarrow$ pseudo-open $\Longrightarrow$ quotient $\Longrightarrow$ sequentially quotient;

(2) 2-sequence-covering $\Longrightarrow$ 1-sequence-sequence $\Longrightarrow$ sequence-covering $\Longrightarrow$ sequentially quotient;

(3) If images are sequential spaces, then sequentially quotient $\Longrightarrow$ quotient;

(4) If images are *Fréchet*-spaces, then quotient $\Longrightarrow$ pseudo-open.

Definition 2. ([8]) Let X be a space X, $x \in P \subset X$. P is a sequential neighborhood of x in X if whenever $S = \{x_n\}$ is a sequence converging to x, then S is eventually in P, that is $\{x_n : n \geq m\} \subset P$ for some $m \in N$.

Remark 2. It is well known that P is a sequential neighborhood of x in X if and only if whenever $S = \{x_n\}$ is a sequence converging to x, then S is frequently in P, that is $\{x_n : n \geq m\} \cap P \neq \phi$ for each $m \in N$.

Definition 3. ([9]) A family $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ of subsets of a space X is a network of X, if whenever $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}_x$, each $\mathcal{P}_x$ is a network for x; A family $\mathcal{P} = \cup\{\mathcal{P}_K : K$ is compact in $X\}$ of subsets of a space X is a pseudo-base of X, if whenever compact subset $K \subset U$ with U open in X, then $K \subset P \subset U$ for some $P \in \mathcal{P}_K$, each $\mathcal{P}_K$ is a pseudo-base for K. A sequence $\{\mathcal{P}_n\}$ of covers of X is a point-star network (compact-star network), if for each $x \in X$ (compact subset K of X), $\{st(x, \mathcal{P}_n) : n \in N\}$ ($\{st(K, \mathcal{P}_n) : n \in N\}$) is a network for x (pseudo-base for K).

Definition 4. ([8]) Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a network of a space X and if $P_1, P_2 \in \mathcal{P}_x$ then $P \subset P_1 \cap P_2$ for some $P \in \mathcal{P}_x$ whenever $x \in U$.

(1) $\mathcal{P}$ is a weak base for X if whenever $U \subset X$ satisfying for each $x \in U$ there is $P \in \mathcal{P}_x$ with $P \subset U$, then U is open in X, here $\mathcal{P}_x$ is a wn-netword (weak neighborhood network) for x, the element of $\mathcal{P}_x$ is weak neighborhood of x. X is g-first countable if $\mathcal{P}_x$ is countable for each $x \in X$.

(2) $\mathcal{P}$ is an sn-network for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X for each $x \in X$, here $\mathcal{P}_x$ is an sn-network for x. X is sn-first countable if $\mathcal{P}_x$ is countable for each $x \in X$.

Remark 3. The following are known ([7],[10]):

(1) weak neighborhood $\Longrightarrow$ sequential neighborhood, and in a sequential space, sequential neighborhood $\Longrightarrow$ weak neighborhood.

(2) In a *Fréchet*-space X, if $x \in X$ and P is a weak (sequential) neighborhood of x, then $x \in P^\circ$.

Definition 5. Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space X. $\{\mathcal{P}_n\}$ is a compact-star wn-network (sn-network), if $\{\mathcal{P}_n\}$ is a compact-star network of X and $st(x, \mathcal{P}_n)$ is a weak (sequential) neighborhood of x in X for each $n \in N$ and each $x \in X$.

Definition 6. ([9]) Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is a *cs*-cover (*cs**-cover), if each convergent sequence is eventually (frequently) in P for some $P \in \mathcal{P}$;

(2) $\mathcal{P}$ is a *g*-cover (an *sn*-cover), if each element of $\mathcal{P}$ is a weak (sequential) neighborhood of some point in X, and for each $x \in X$, some $P \in \mathcal{P}$ is a weak (sequential) neighborhood of x.

Theorem 1. *A space X is a C-mapping image of a metric space if and only if X has a compact-star network.*

Proof. *Necessity*: Let $f : M \longrightarrow X$ be a C-mapping, (M, d) be a metric space. Put $\mathcal{B}_n = \{B(a, 1/n) : a \in M\}$ and $\mathcal{P}_n = f(\mathcal{B}_n)$. Then $\{\mathcal{P}_n\}$ is a sequence of covers of X. Let K be compact in X, U be open in X and $K \subset U$. f is C-mapping, so there is $n \in N$ such that $f(B(f^{-1}(K), 2/n)) \subset U$, thus $st(K, \mathcal{P}_n) \subset U$. In fact, if $x \in st(K, \mathcal{P}_n)$, then there is $P \in \mathcal{P}_n$ such that $x \in P$ and $K \cap P \neq \phi$. Pick $x' \in K \cap P$. Let $P = f(B(a, 1/n)$ for some $a \in M$ and $b, b' \in B(a, 1/n)$ such that $f(b) = x$ and $f(b') = x'$. Then $d(b, f^{-1}(K)) \leq d(b, b') < 2/n$, that is $b \in B(f^{-1}(K), 2/n)$, thus $x \in f(B(f^{-1}(K), 2/n)) \subset U$.

Sufficiency: Suppose $\mathcal{P}_n$ is a compact-star network of X. For each $n \in N$, let $\mathcal{P}_n = \{P\alpha : \alpha \in A_n\}$, the topology on A_n is the discrete topology. Put $M = \{a = (\alpha_n) \in \Pi_{n\in N} A_n : \{P_{\alpha_n}\}$ is a network for some $x_a \in X\}$. Then M is a subspace of the product space $\Pi_{n\in N} A_n$, and X is a metric space with metric d define as follows. Let $a = (\alpha_n), b = (\beta_n) \in M$, if $a = b$, then $d(a, b) = 0$; if $a \neq b$, then $d(a, b) = 1/min\{n \in N : \alpha_n \neq \beta_n\}$. Define $f : M \longrightarrow X$ by $f(a) = x_a$ for each $a \in M$. It is easy to see that x_a is unique for each $a \in M$ by T_1-property of X, so f is a function.

(1) f is onto: Let $x \in X$. For each $n \in N$, there is $\alpha \in A_n$, such that $x \in P_{\alpha_n}$. As $\mathcal{P}_n$ is a compact-star network of X, $\{P_{\alpha_n}\}$ is a network for x. Put $a = (\alpha_n)$, then $f(a) = x$.

(2) f is continuous: Let $a = (\alpha_n) \in M$, U be a neighborhood of $x = f(a)$. Then there is $k \in N$ such that $P_{\alpha_k} \subset U$. Put $V = \{b = (\beta_n) \in M : \beta_k = \alpha_k\}$. Then V is open in M containing a and $f(V) \subset P_{\alpha_n} \subset U$, thus f is continuous.

(3) f is a C-mapping: Let K be compact in X, U be open in X and $K \subset U$. As $\mathcal{P}_n$ is a compact-star network of X, there is $n \in N$ such that $st(K, \mathcal{P}_n) \subset U$. It is easy to prove that $f(B(f^{-1}(K), 1/2n)) \subset U$. In fact, let $a = (\alpha_n) \in B(f^{-1}(K), 1/2n)$. Then $d(f^{-1}(K), a) < 1/2n$, there is $b = (\beta_n) \in f^{-1}(K)$ such that $d(a, b) < 1/n$, so $\alpha_k = \beta_k$ if $k \leq n$, thus $P_{\alpha_n} = P_{\beta_n}$, hence $f(a) \in P_{\alpha_n} = P_{\beta_n} \subset st(K, \mathcal{P}_n) \subset U$.

By the above, X is a C-mapping image of a metric space. ∎

Lemma 1. Let $\{\mathcal{P}_n\}$ be a sequence of *cs**-covers of a space X, and S be a sequence in X converging to a point $x \in X$. Then there is a subsequence S' of S such that for each $n \in N$, there is $P_n \in \mathcal{P}_n$ such that S' is eventually in P_n.

P r o o f. For each $n \in N$, $\mathcal{P}_n$ is a cs^*-cover of X, so there is $P_n \in \mathcal{P}_n$ such that S is frequently in P_n. As S is frequently in P_1, there is a subsequence S_1 of S such that $S_1 \subset P_1$. Put x_{n_1} is the first term of S_1. Similarly, S_1 is frequently in P_2, there is a subsequence S_2 of S_1 such that $S_2 \subset P_2$. Put x_{n_2} is the second term of S_2. By the inductive method, for each $k \in N$, As S_{k-1} is frequently in P_k, there is a subsequence S_k of S_{k-1} such that $S_k \subset P_k$. Put x_{n_k} is the k-th term of S_k. Let $S' = \{x_{n_k} : k \in N\} \cup \{x\}$. Then S' is a subsequence of S such that for each $n \in N$ and for each $n \in N$ $x_{n_k} \in S_k \subset S_n \subset P_n$ if $k > n$, so S' is eventually in P_n. ■

Theorem 2. *The following are equivalent for a space X:*
(1) X is a sequentially quotient and C-mapping image of a metric space;
(2) X has a compact-star sn-network consisting of cs-covers;*
(3) X has a compact-star network consisting of cs-covers.*

P r o o f. (1) $\Longrightarrow$ (2). Let $f : M \longrightarrow X$ be a sequentially quotient and C-mapping, (M, d) be a metric space. By the method in proof of necessity in Theorem 1, we can obtain a compact-star network $\{\mathcal{P}_n\}$ of X and

(i) For each $n \in N$, $\mathcal{P}_n$ is a cs^*-cover of X: Let $x \in X$ and S be a sequence in X converging to the point x. f is sequentially quotient, so there is a sequence L in M converging to a point $a \in f^{-1}(x)$ such that $f(L) = S'$ is a subsequence of S. As $a \in B(a, 1/n)$, L is eventually in $B(a, 1/n)$, so $S' = f(L)$ is eventually in $P = f(B(a, 1/n)) \in \mathcal{P}_n$, Thus S is frequently in P.

(ii) For each $x \in X$ and $n \in N$, $st(x, \mathcal{P}_n)$ is a sequential neighborhood of x: Let S be a sequence in X converging to the point x. By the proof in the above (i), S is frequently in some $P \in \mathcal{P}_n$. Notice that $x \in P$, S is frequently in $st(x, \mathcal{P}_n)$.

By the above (i),(ii), X has a compact-star sn-network consisting of cs^*-covers;

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1). Let $\{\mathcal{P}_n\}$ be a compact-star network consisting of cs^*-covers of X. By the method in proof of sufficiency in Theorem 1, we can obtain a metric space M and a C-mapping $f : M \longrightarrow X$. Now we only need to show f is sequentially quotient. Let S be a sequence in X converging to a point $x \in X$. By Lemma 2, there is a subsequence $S' = \{x_k : k \in N\} \cup \{x\}$ of S such that for each $n \in N$, there is $\alpha_n \in A_n$ such that S' is eventually in P_{α_n}. Put $a = (\alpha_n)$. Obviously, $a \in M$ and $f(a) = x$. We pick $b_k \in f^{-1}(x_k)$ for each $x_k \in S'$ as follows. For each $n \in N$, if $x_k \in P_{\alpha_n}$, put $\beta_{k_n} = \alpha_n$; if $x_k \notin P_{\alpha_n}$, pick $\alpha_{k_n} \in A_n$ such that $x_k \in P_{\alpha_{k_n}}$, and put $\beta_{k_n} = \alpha_{k_n}$. Put $b_k = (\beta_{k_n}) \in \Pi_{n \in N} A_n$. Obviously, $b_k \in M$ and $f(b_k) = x_k$. It is easy to prove that $L = \{b_k : k \in N\} \cup \{a\}$ is a sequence in M converging to the point a. In fact, let U is open in M containing

a. By the definition of Tychonoff-product spaces, we can assume there is $m \in N$ such that $U = ((\Pi\{\{\alpha_n\} : n \leq m\}) \times (\Pi\{A_n : n > m\})) \cap M$. For each $n \leq m$, S' eventually P_{α_n}, so there is $k(n) \in N$ such that $y_k \in P_{\alpha_n}$ if $k > k(n)$, thus $\beta_{k_n} = \alpha_n$. Put $k_0 = max\{k(1), k(2), ..., k(m), m\}$. It is easy to see that $\beta_k \in U$ if $k > k_0$, so L converge to a. Thus there is converging sequence L in M such that $f(L) = S'$ is a subsequence of S, so f is sequentially quotient. ∎

Similarly to proof of Theorem 2, we can obtain the following theorem.

Theorem 3. *(1) A space X is a sequence-covering C-mapping image of a metric space if and only if X has a compact-star network consisting of cs-covers;*

(2) A space X is a 1-sequence-covering C-mapping image of a metric space if and only if X has a compact-star network consisting of sn-covers.

Corollary 1. *The following are equivalent for a space X:*

(1) X is a quotient and C-mapping image of a metric space;

(2) X is a sequential space with a compact-star sn-network consisting of cs-covers;*

(3) X is a sequential space with a compact-star network consisting of cs-covers;*

(4) X has a compact-star wn-network consisting of cs-covers.*

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) from Theorem 2 and Remark 1. (2) $\Longrightarrow$ (4) from Remark 3. Notice that a space with a compact-star wn-network is g-first countable, hence sequential, thus (4) $\Longrightarrow$ (3). ∎

Corollary 2. *The following are equivalent for a space X:*

(1) X is a 1-sequence-covering, quotient and C-mapping image of a metric space;

(2) X has a compact-star network consisting of g-covers;

(3) X is a sequential space with a compact-star network consisting of sn-covers.

Proof. (1) $\Longleftrightarrow$ (3) from Theorem 3 and Remark 1. Notice that a space with a compact-star network consisting of g-covers is g-first countable, (2) $\Longleftrightarrow$ (3) from Remark 3. ∎

We can obtain the following corollary from Theorem 2 and Remark 1.

Corollary 3. *The following are equivalent for a space X:*

(1) X is a pesudo-open and C-mapping image of a metric space;

(2) X is a Fréchet-space with a compact-star network consisting of cs-covers.*

Lemma 2. ([11]) *Let $f : X \longrightarrow Y$ be a mapping and X be first countable. Then:*

(1) f is an almost open mapping if and only if f is a 1-sequence-covering and pseudo-open mapping.

(2) f is an open mapping if and only if f is a 2-sequence-covering and quotient mapping.

Theorem 4. *The following are equivalent for a space X:*

(1) X is a metric spaces;

(2) X is an open and C-mapping image of a metric space;

(3) X is an almost open and C-mapping image of a metric space;

(4) X is a 2-sequence-covering, quotient and C-mapping image of a metric space;

(5) X is a 1-sequence-covering, pseudo-open and C-mapping image of a metric space;

(6) X is a Fréchet-space with a compact-star network consisting of open covers;

(7) X is a Fréchet-space with a compact-star network consisting of g-covers;

(8) X is a Fréchet-space with a compact-star network consisting of sn-covers.

Proof. (1) $\Longleftrightarrow$ (2) from [5, Theorem 2]. (1) $\Longleftrightarrow$ (6) from [4, 5.4.E]. (2) $\Longrightarrow$ (3) is obvious. (2) $\Longleftrightarrow$ (4) and (3) $\Longleftrightarrow$ (5) from Lemma 2. (5) $\Longrightarrow$ (8) from Corollary 2. (6) $\Longleftrightarrow$ (7) $\Longleftrightarrow$ (8) from Remark 3. ■

In [11], Lin proves that a space with a point-star network consisting of *cs*-covers has a point-star network consisting of *sn*-covers, thus a space is a sequence-covering P-mapping image of a metric space if and only if it is a 1-sequence-covering P-mapping image of a metric space. But we do not know whether (1) $\Longleftrightarrow$ (2) in Theorem 3, so the following question is open.

Question. *Do sequence-covering, pseudo-open and C-mappings preserve metric spaces?*

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