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Cauchy Problem for Nonlinear Degenerate Parabolic Equation in One Dimension

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Existence of solution and uniqueness for Cauchy problem for a class of degenerate nonlinear parabolic equations in one dimension are studied. A regularizing effect, pointwise estimates, gradient estimates and qualitative properties of solution are obtained.

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1. Introduction

We study the existence and properties of the solutions of Cauchy problem for the equation:

$$w_t = \partial_x \left(w a' \left(\frac{\varphi(w)}{w} w_x \right) \right) + \frac{w}{\varphi(w)} b \left(\frac{\varphi(w)}{w} w_x \right) \tag{1.1}$$

in the strip $S_T = (0, T) \times R^1$. All properties of the smooth functions a, b, φ and initial conditions we specify later on in Section 2. A typical example for the equation (1.1) is the double-powered nonlinear degenerate parabolic equation with $w \geq 0$

$$w_t = \partial_x \left(| (w^m)_x |^{p-2} (w^m)_x \right), \text{ for } m > 0, p > 1. \tag{1.2}$$

In this case $b = 0$, $\varphi(s) = m s^{m - \frac{1}{p-1}}$, $a(q) = \dfrac{1}{p} |q|^p$, $\lambda = m - \dfrac{1}{p-1}$.

The equation (1.1) has the following "Barenblatt type solutions" if $\varphi(s) = |s|^\lambda$ and $b(q) = (k+\lambda)a(q) - (\lambda+1)a'(q)$

$$(1.3)\qquad \begin{aligned} w^1_{\lambda,\tau}(t,x) &= \left[C(t+\tau)^{-\frac{\lambda}{k}} - \lambda k(t+\tau)\alpha\left(\tfrac{x}{k(t+\tau)}\right)\right]_+^{\frac{1}{\lambda}}, \\ w^2_{\lambda,T}(t,x) &= \left[C(T-t)^{-\frac{\lambda}{k}} + \lambda k(T-t)\alpha\left(\tfrac{x-x_0}{k(T-t)}\right)\right]_+^{\frac{1}{\lambda}}, \end{aligned}$$

where $[s]_+ = \max[s,0]$. The properties of the solutions of (1.1) are well illustrated by means of the solutions (1.3). Functions $w^1_{\lambda,\tau}$ and $w^2_{\lambda,T}$ are weak solutions of (1.1) with corresponding initial data.

The most complete studies for the equation (1.2) were made in the cases $m = 1$ or $p = 2$, see Aronson, Benilan [1], Crandall, Pierre [3] and Di Benedetto, Herrero [6]. When the equation (1.1) is double powered, i.e. $m \neq 1$ and $p \neq 2$ and $b = 0$, the existence, uniqueness and properties of solutions were proved in Esteban, Vazquez [4, 5] and for unbounded initial data in Kalashnikov [7]. Existence results and propagation properties for weak solutions of equation (1.1) with non-powered non-linearity were obtained in Kalashnikov [8] for bounded initial data with Lipshitz continuous velocities. For more detailed information see the survey of Kalashnikov [9]. The properties of the smooth solutions for the multidimensional equation (1.1) and for non-power like non-linearity were studied in Fabricant, Marinov, Rangelov [11].

The aim of the work is to improve the results in Esteban, Vazquez [5] for non-powered case and for an equation with lower order terms. The main questions that we study for the Cauchy problem for the equation (1.1) are: existence , uniqueness, regularizing effect, pointwise estimates, gradient estimates, qualitative properties etc. The plan is as follows: the main results are formulated in Section 2; a regularizing effect and a pointwise estimate for solutions of the regularized Cauchy problem for (1.1) are studied in Section 3. In Section 4 estimates for smooth solutions of (1.1) are shown. In Section 5 we get the main theorems through limiting process.

2. The main results

We deal with the Cauchy problem :

$$(2.1)\qquad \left| \begin{array}{l} w_t = \partial_x(wa'(v_x)) + \frac{w}{\varphi(w)}b(v_x), \quad (t,x) \in S_T \\ w(0,x) = w_0, \quad x \in \mathbf{R}^1 \end{array} \right.$$

where $v_x = \dfrac{\varphi(w)}{w}w_x$ with $v = \Gamma(w)$, and the following conditions are satisfied:

$$(2.2)\qquad \left| \begin{array}{l} \varphi(s) \in C^3(\mathbf{R}^1 \setminus 0), \varphi(s) > 0 \text{ for } s \neq 0 \text{ and the function } \lambda(s) \equiv \frac{s\varphi'(s)}{\varphi(s)} \\ \text{is nonincreasing for } s \neq 0, \lambda(0) < \infty; \end{array} \right.$$

$$(2.3)\quad \left| \begin{array}{l} a(q) \in C^2(\mathbf{R}^1 \setminus 0) \cap C^1(\mathbf{R}^1), a(q) \geq 0, a(0) = 0, a'(0) = 0, \\ a''(q) \geq 0 \text{ for } q \neq 0, b(q) \in C(\mathbf{R}^1); \\ (r_0 - 1)qa'(q) \leq b(q) \leq rqa'(q), \text{ for some } r_0 > 0, r \geq [r_0 - 1]_+; \end{array} \right.$$

$$(2.4)\quad \left| \begin{array}{l} \text{there exists } k > 0, \text{ such that } \tilde{b}(q) + \lambda(w)\alpha(a'(q)) - ka(q) \\ \text{is convex in } q \text{ for all } w \geq 0, \text{ where } \tilde{b}(q) \equiv b(q) + qa'(q); \\ \text{with } \alpha - \text{ the Joung conjugate to the function } a, \\ a_1 = \alpha(a'(q)) = qa' - a, \text{ there exist constants } 0 < C_1, C \\ \text{such that } C_1\alpha(a'(q)) + C \geq |q|. \end{array} \right.$$

Let us separate the four principle cases with respect to λ. Since $\lambda(\sigma)$ is nonincreasing we have the inequality $\dfrac{\varphi(s)}{s} \leq \dfrac{\varphi(\sigma)}{\sigma}\left(\dfrac{s}{\sigma}\right)^{\lambda(\sigma)-1}$ for all $s, \sigma \in \mathbf{R}^1$.

- Case $(+)$: $\lambda(s) \geq 0$, $\lambda \neq 0$ then φ nondecreasis, $\varphi(0) = 0$, $\Gamma(w) = = \displaystyle\int_0^w \frac{\varphi(s)}{s}\, ds$, $\Gamma(+0) = 0$, $\Gamma(\infty) = \infty$.
- Case $(\pm)$: $\lambda(s) \begin{cases} > 0, & s < s_0 \\ = 0, & s_0 < s < s_1, \quad 0 < s_0 \leq s_1 < \infty, \\ < 0, & s_1 < s < \infty \end{cases}$
  then $\varphi(0) = \varphi(\infty) = 0$, φ is bounded, $\Gamma(w) = \displaystyle\int_0^w \frac{\varphi(s)}{s}\, ds$, $\Gamma(+0) = 0$, $\Gamma(\infty) < \infty$.
- Case (-) : $\lambda(s) \leq 0$, $\lambda \neq 0$, then φ nonincreasis, $\varphi(\infty) = 0$, $\Gamma(w) = = -\displaystyle\int_0^\infty \frac{\varphi(s)}{s}\, ds$, $\Gamma(+0) = -\infty$, $\Gamma(\infty) = 0$.
- Case (0) : $\lambda(s) \equiv 0$, then $\varphi(s) \equiv 1$, $\Gamma(w) = \ln w$, $\Gamma(+0) = -\infty$, $\Gamma(\infty) = +\infty$.

Note that $\Gamma(s)$ increases in all cases.

We define for a non-negative and nondecreasing in $\mathbf{R}^1_+$ function $H(s)$, $\tau > 0$, $A > 0$ the function $f_H(\tau, A) = \max\limits_{\alpha(\sigma) \leq A[\Gamma(\tau) - \Gamma(s)], 0 < s < \tau} H(s)|\sigma|$. Let the following inequality holds

$$(2.5)\quad \left| \text{ for } H_{r_0}(s) = s^{r_0}, \quad \int_0^T f_{H_{r_0}}\left(\tau, \frac{1}{kt}\right) dt < \infty \right. .$$

Note that in the case when $\Gamma(0)$ is finite, (2.5) is a condition only on the function α of the type $\displaystyle\int_0^T \alpha^{-1}\left(\frac{C}{kt}\right) dt < \infty$. If $sH'(s) \geq \overline{r}H(s)$, $\overline{r} \geq r_0$, then $f_H\left(\tau, \dfrac{1}{kt}\right)$ is also integrable over $(0, T)$. In the double powered case for equation (1.2) we have $f_{H_{r_0}}\left(\tau, \dfrac{1}{kt}\right) \sim t^{-\frac{1}{p'}}\tau^{\frac{k_0}{p}}$ if $k_0 = r_0p + \lambda(p-1) > 0$.

Let $H(s)$ be a non-negative, increasing on $\mathbf{R}^1_+$ function, $H(0) = 0$ and $sH''(s) \geq rH'(s)$ with the same $r \geq 0$ as in the condition (2.3). Denote $S_{\tau,T} = (\tau, T) \times \mathbf{R}^1$.

Theorem 2.1. (existence) *For every $w_0 \geq 0$, $H(w_0) \in L^1(\mathbf{R}^1) \cap L^\infty(\mathbf{R}^1)$ there exists a week solution $w(t,x)$ of (2.1) which is a nonnegative continuous function such that*
(a) $w(t,x)$ is Hölder continuous and $w(t,x) \leq M_0$ for $t > 0$, where $M_0 = \max w_0(x)$, $w(t,x) \to 0$ for $|x| \to \infty$.
(b) there exists a.e. $v_x(t,x)$ and for a.e. $t > 0$, $wa'(v_x(t,x))$ is continuous in x ; $v_x(t,x) \in L^\infty(S_{\tau,T})$ in cases (+) and (±) and $w_x(t,x) \in L^\infty(S_{\tau,t})$ in cases (-), (0) for $\tau > 0$; w_t, $\partial_x wa'(v_x)$, $\dfrac{w}{\varphi(w)} b(v_x) \in L^{1+\beta}_{loc}(S_T)$ for some $\beta > 0$ and w satisfies equation (2.1) a.e. in S_T.
(c) $\|H(w)\|_1 \leq \|H(w_0)\|_1$, $\left\| H'(w) \dfrac{w}{\varphi(w)} v_x a'(v_x) \right\|_1 \leq A\|H(w_0)\|_1$ and $\left\| \dfrac{H(w)}{w} w_t \right\|_1 \leq 2AC_0 \|H(w_0)\|_1$, for every $t > 0$ where $A = \dfrac{1}{kt}$, $C_0 = const$ and $\|\cdot\|_1 = \|\cdot\|_{L^1(\mathbf{R}^1)}$.
(d) $w(0,x) = w_0(x)$ a.e. in $\mathbf{R}^1$.

Remark 2.1. ($L^1 - L^\infty$ estimate) If for every $A > 0$ and H as in theorem 2.1 $\lim\limits_{\tau \to \infty} f_H(\tau, A) = \infty$, then theorem 2.1 is true when $H(w_0) \in L^1(\mathbf{R}^1)$. Moreover $w(t,x)$ is bounded for $t > 0$ which follows by means of the inequality $f_H \left(\max\limits_x w(t,x), \dfrac{1}{kt} \right) \leq \dfrac{1}{kt} \|H(w_0)\|_1$. Note that in the cases (+) and (0) this condition is satisfied since $\Gamma(+\infty) = +\infty$. The same result is true in all of the cases of double powered equation (1.2) since then $f_H(\tau, A) \sim A^{\frac{1}{p'}} \tau^{\frac{k}{p}}$ and $k > 0$.

Remark 2.2. (uniqueness) For $t > 0$ there exist finite $L_\pm = \lim\limits_{x \to \pm\infty} H(w(t,x)) a'(v_x(t,x))$ such that $L_+(t) \leq 0 \leq L_-(t)$, $L_-(t) - L_+(t) \leq \dfrac{1+r}{kt} \displaystyle\int_{\mathbf{R}^1} H(w_0)\, dx$. If for every $A > 0$ and H as in theorem 2.1 $\lim\limits_{\tau \to +0} f_H(\tau, A) = 0$ then $L_+(t) = L_-(t) = 0$. If in addition $b(q) = rqa'(q)$ then the solution, given from theorem 2.1 is unique. Moreover if w_j, $j = 1, 2$ are two solutions in the same sense with initial data w_{j0} then

$$\int_{\mathbf{R}^1} \left[w_1^{1+r}(t,x) - w_2^{1+r}(t,x) \right]_+ dx \leq \int_{\mathbf{R}^1} \left[w_{10}^{1+r}(x) - w_{20}^{1+r}(x) \right]_+ dx.$$

Note that theorem 2.1 for the double powered equation (1.2) was proved in Esteban, Vazquez [4] for $m > 1$.

Denote $W(t,x) = \partial_x a'(v_x(t,x))$ and let k is from (2.3).

Theorem 2.2. (regularizing effect) *The following estimates are hold:*

$$W \geq -\frac{1}{kt}, \;\; t > 0, \;\; \textit{in } D'(\mathbf{R}^1). \tag{2.6}$$

$$w_t \geq -\frac{1}{kt} w \;\; \textit{a.e. in } \mathbf{R}^1, \;\; t > 0. \tag{2.7}$$

The estimates (2.6), (2.7) are obtained for equation (1.2) with $p = 2$ in Aronson, Benilan [1] for $m > 1$; for more general φ in Crandall, Pierre [3] and in Fabricant, Marinov, Rangelov [12] and for $d = 1$, $m > 0$ in Esteban, Vazquez [4].

We define the function

$I_\lambda(t_0, t, x_0, x) = \inf\limits_{z(s)} \left\{ \int_{t_0}^{t} \left(\frac{s}{t_0} \right)^{\frac{\lambda}{k}} \beta(\dot{z}(s))\, ds : z(t_0) = x_0, \;\; z(t) = x \right\}$, with convex function β, $0 < t_0 < t < T$, $x_0, x \in \mathbf{R}^1$. Denote $\Gamma_\sigma(s) = \Gamma(s) - \Gamma(\sigma) + \dfrac{\varphi(\sigma)}{\lambda(\sigma)}$ for the fixed parameter $\sigma \in (0, \infty)$. Note that $\lambda(\sigma)\Gamma_\sigma(s) \geq \varphi(s)$.

Theorem 2.3. (pointwise estimate) *If $\tilde{b}(q)$ is greater than the Joung conjugate of β then the solution $w(t,x)$ satisfies*

$$\left| \begin{array}{l} \Gamma_\sigma(w(t_0, x_0)) \leq \left(\frac{t}{t_0}\right)^{\frac{\lambda(\sigma)}{k}} \Gamma_\sigma(w(t,x)) + I_{\lambda(\sigma)}(t_0, t; x_0, x) \\ \textit{with } \Gamma_\sigma(s) \textit{ for } \sigma \in (\min w(t,x), \max w(t,x)), \lambda(\sigma) \neq 0 \end{array} \right. \tag{2.8}$$

$$\left| \begin{array}{l} \Gamma(w(t_0, x_0)) \leq \Gamma\left(\left(\frac{t}{t_0}\right)^{\frac{\lambda(m)}{k}} w(t,x) \right) + I_{\lambda(m)}(t_0, t; x_0, x) \\ \textit{if } \lambda(m) \textit{ is finite }, m = \min w(t,x). \end{array} \right. \tag{2.9}$$

The estimates (2.8),(2.9) are obtained in Fabricant, Marinov, Rangelov [11, 12] for smooth solutions of multidimensional analogue of (1.1). They corresponds to the classical Mozer pointwise estimate, see Mozer [13] in the linear case. Note that on the examples of solutions (1.3), estimates (2.6)-(2.9) are sharp.

3. Regularized problem

The Cauchy problem (2.1) can be regularized with $\varepsilon > 0$, $\delta > 0$, in such a way that

(3.1)
$$\left|\begin{array}{l} a_\varepsilon, b_\varepsilon \in C^\infty, a'_\varepsilon(0) = a''_\varepsilon(0) = 0, a_\varepsilon \text{ is strictly convex and} \\ \varepsilon \le a''_\varepsilon(q) \le \frac{1}{\varepsilon}; \\ a_\varepsilon, a'_\varepsilon, b_\varepsilon \text{ satisfy (2.3), (2.4) with appropriate } r_\varepsilon, r_{0\varepsilon}, k_\varepsilon \\ \text{ and tend uniformly on compact subsets of } \mathbf{R}^1 \text{ to } a, a', b \\ \text{ respectively for } \varepsilon \to 0. \end{array}\right.$$

(3.2)
$$\left|\begin{array}{l} \text{There exists a constant } M_0, \text{ such that } 0 < \delta \le w_{0,\delta} \le M_0 \text{ and } w_{0,\delta} \\ \text{ have Hölder continuous and bounded second order derivatives;} \\ H_\delta(s) \text{ is non-negative, increasing in } [\delta, M_0], H_\delta(\delta) = 0 \text{ and} \\ sH''_\delta(s) \ge rH'_\delta(s) \text{ with the same } r \ge 0 \text{ as in the condition (2.3);} \\ H_\delta(s) \to H(s) \text{ uniformly on compact subsets} \\ \text{ and } H_\delta(w_{0,\delta}) \to H(w_0) \text{ in } L^1(\mathbf{R}^1) \text{ for } \delta \to 0; \\ \text{ there exists a constant } C > 0 \text{ and } \|H_\delta(w_{0,\delta})\|_1 \le C\|H(w_0)\|_1. \end{array}\right.$$

Let $H_{\delta,\varepsilon}$ have the same properties as H_δ but with $r = r_\varepsilon$, and approximate H_δ for $\varepsilon \to 0$.

The regularization is described in details for the double powered equation in [12]. It follows by Ladyzenskaja, Solonnikov, Ural'tzeva [10] (Sect-s 1, 4 and 5) that under the conditions (3.1),(3.2) the Cauchy problem (2.1) has a unique solution $w_{\delta,\varepsilon}(t,x) \in H^{2+l,1+l/2}(S_T), l \in (0,1)$ more over $\delta \le w_{\delta,\varepsilon}(t,x) \le M_0$. Indeed, under (2.2) Γ is a smooth function increasing on $(0,\infty)$, which may have singularities only at 0 or ∞. From (2.1) the change $v = \Gamma(w)$ leads to Cauchy problem for v (we omit indices ε, δ for simplicity):

(3.3)
$$\left|\begin{array}{l} v_t = \chi(v)a''(v_x)v_{xx} + v_x a'(v_x) + b(v_x), \ (t,x) \in S_T = (0,T) \times \mathbf{R}^1 \\ v(0,x) = v_{0,\delta}, \quad x \in \mathbf{R}^1, \end{array}\right.$$

where $\chi(v) = \varphi(\Gamma^{-1}(v))$.

In order to apply the results in [10] we first regularize $\chi(v)$ such that

$$\chi_n(v) = \left\{ \begin{array}{cccc} \Gamma\left(\frac{1}{n}\right), & v & < & \Gamma\left(\frac{1}{n}\right) \\ \chi(v), & \Gamma\left(\frac{1}{n}\right) & \le & v \le \Gamma(n). \\ \Gamma(n), & v & > & \Gamma(n) \end{array}\right.$$

Then all conditions of Theorem 8.1, Sect.5 in [10] are satisfied and the Cauchy problem (3.3) with $\chi_n(v)$ in the place of $\chi(v)$ has unique solution $v(t,x) \in$

$H^{2+l,1+l/2}(S_T)$. Moreover $v(t,x) \in H^{4+l,2+l/2}(S_T)$ if $v_{0,\delta} \in H^{2+l/2}(\mathbf{R}^1)$ and its derivatives up to order 2 are bounded. Now by Corollary 2.1 and Theorem 2.5, Sect.1 in [10] $\Gamma(\delta) \leq v \leq \Gamma(M)$ and for large n , $\chi_n(v) = \chi(v)$ on these intervals. So $v(t,x) \in C^{4,2}(S_T)$ with bounded derivatives, which is enough for our aims. The same is as well true for the regularized Cauchy problem (2.1) for w.

Denote $W = \partial_x a'(v_x)$. In the theorem bellow we get an estimate known as "regularizing effect" and we proof the Theorem 2.2 for smooth solutions of (2.1).

Proposition 3.1. *Let* $w(0,x) \geq -\dfrac{1}{C_0}$, *with some* $C_0 > 0$ *then*

$$W \geq -\frac{1}{kt + C_0}, \ t > 0. \tag{3.4}$$

P r o o f. The proof is as in [12] but we repeat it briefly here in a more simple one dimensional situation. With the notation $\tilde{b}(q)$ the equation in (2.1) becomes $w_t = wW + \dfrac{w}{\varphi(w)}\tilde{b}(v_x)$. Differentiating W in t

$$W_t = \partial_x a''(v_x)\partial_x\left[\varphi(w)W + \tilde{b}(v_x)\right]$$

$$= \varphi(w)a''(v_x)\partial_x^2 W + \varphi(w)\frac{a'''(v_x)}{a''(v_x)}W\partial_x W + \left[2\lambda(w)v_x a''(v_x) + \tilde{b}'(v_x)\right]\partial_x W$$

$$+\lambda'(w)\frac{w}{\varphi(w)}(v_x)^2 a''(v_x)W + \frac{\tilde{b}''(v_x) + \lambda(w)(v_x a'''(v_x) + a''(v_x))}{a''(v_x)}W^2.$$

Under the condition (2.4) we get that the last term on the right is not less then kW^2. So W satisfies the parabolic differential inequality $W_t - L(W) \geq kW^2$ with some quasilinear elliptic operator L. With C_0 as in the theorem, the function $z = -\dfrac{1}{kt + C_0}$ satisfies the inverse differential inequality due to (2.2), i.e. $\lambda'(s) \leq 0$. Now using the comparison principle - Lemma 3.2 in [12], we obtain (3.4) in the whole S_T. ∎

By using the regularizing effect estimate the next pointwise estimate holds. This proves Theorem 2.3 for smooth solutions.

Proposition 3.2. *Let in addition to (3.4)* $\tilde{b}(q)$ *majorate some convex function conjugate to* $\beta(s)$ *.*
(i) With $\Gamma_\sigma(s)$ *for* $\sigma \in (\min w(t,x), \max w(t,x))$, $\lambda(\sigma) \neq 0$

$$\Gamma_\sigma(w(t_0,x_0)) \leq \left(\frac{t}{t_0}\right)^{\frac{\lambda(\sigma)}{k}} \Gamma_\sigma(w(t,x)) + I_{\lambda(\sigma)}(t_0,t;x_0,x). \tag{3.5}$$

(ii) If $\lambda(m)$ *is finite,* $m = \min w(t,x)$*, (3.5) may be written as*

$$\Gamma(w(t_0,x_0)) \le \Gamma\left(\left(\frac{t}{t_0}\right)^{\frac{\lambda(m)}{k}} w(t,x)\right) + I_{\lambda(m)}(t_0,t;x_0,x). \tag{3.6}$$

P r o o f. From the regularizing effect (3.4) $w_t \ge -\dfrac{1}{kt}w + \dfrac{w}{\varphi(w)}\tilde{b}\left(\dfrac{\varphi(w)}{w}w_x\right)$ for $0 < t_0 < t$.

(i) Note that $\lambda(\sigma)\Gamma_\sigma(s) \ge \varphi(s)$ since $\lambda(\sigma)$ nonincreasis and $\lambda(\sigma)\Gamma_\sigma(s) - \varphi(s) = \displaystyle\int_\sigma^s \frac{\varphi(\tau)}{\tau}[\lambda(\sigma) - \lambda(\tau)]\,d\tau \ge 0$. Assume that $\tilde{b}(q)$ is convex together with its conjugate $\beta(s)$ and define $E(\tau) = \left(\dfrac{\tau}{\tau_0}\right)^{\frac{\lambda(\sigma)}{k}}\Gamma_\sigma(w(\tau,z(\tau))) + \displaystyle\int_{t_0}^\tau \left(\frac{s}{t_0}\right)^{\frac{\lambda(\sigma)}{k}}\beta(\dot{z}(s))\,ds$, where $z(s)$ is a smooth curve with $z(t_0) = x_0$, $z(t) = x$. Then

$$\frac{d}{dt}E(\tau) = \left(\frac{\tau}{t_0}\right)^{\frac{\lambda(\sigma)}{k}}\left[\frac{1}{k\tau}\lambda(\sigma)\Gamma_\sigma(w) + \frac{\varphi(w)}{w}w_\tau + \frac{\varphi(w)}{w}w_x z + \beta(\dot{z})\right]$$
$$\ge \left(\frac{\tau}{t_0}\right)^{\frac{\lambda(\sigma)}{k}}\left[\tilde{b}\left(\frac{\varphi(w)}{w}w_x\right) + \frac{\varphi(w)}{w}w_x\dot{z} + \beta(\dot{z})\right] \ge 0,$$

through the Young inequality for $\tilde{b}$ and β. So $E(t_0) \le E(t)$ for $0 < t_0 < t$ and we get (3.6), minimizing in z.

(ii) By the properties of φ in (2.2) and corresponding inequality for φ: $\varphi(\mu w) \le$ $\le \varphi(w)\mu^{\lambda(w)} \le \varphi(w)\mu^{\lambda(m)}$ if $\mu > 1$. If $\tilde{b}(q)$ is convex then $\dfrac{1}{s}\tilde{b}(sq)$ is nondecreasing in s, so

$$\frac{1}{\varphi(w)}\tilde{b}\left(\varphi(w)\frac{w_x}{w}\right) \ge \frac{\mu^{\lambda(m)}}{\varphi(\mu w)}\tilde{b}\left(\mu^{-\lambda(m)}\varphi(\mu w)\frac{w_x}{w}\right).$$

We use this inequality with $\mu = \mu(\tau) = \left(\dfrac{\tau}{t_0}\right)^{\frac{1}{k}} > 1$ for $\tau > t_0 > 0$.

Let again $E(\tau) = \Gamma(\mu(\tau)w(\tau,z(\tau))) + \displaystyle\int_{t_0}^\tau \mu(s)^{\lambda(m)}\beta(\dot{z}(s))\,ds$, then

$$\frac{d}{d\tau}E(\tau) = \frac{\varphi(\mu w)}{\mu w}\left[\frac{1}{k\tau}\mu(\tau)w + \mu(\tau)w_\tau + \mu(\tau)w_x\dot{z}\right] + \mu^{\lambda(m)}\beta(\dot{z})$$
$$\ge \frac{\varphi(\mu w)}{\varphi(w)}\tilde{b}\left(\varphi(w)\frac{w_x}{w}\right) + \frac{\varphi(\mu w)}{w}w_x\dot{z} + \mu^{\lambda(m)}\beta(\dot{z}) \ge 0,$$

by the inequality on $\tilde{b}$ above and Joung inequality. So $E(t_0) \le E(t)$ for $0 < t_0 < t$ and we get (3.6). ∎

4. Estimates for smooth solutions

Below we omit for simplicity the parameters ε, δ and variable t, and write v' in the place of $v_x(t,x)$. For $A_1 > A = -\dfrac{1}{kt+C_0}$, $\mu = const$, $x, z \in \mathbf{R}^1$ and $h(z) = v(z) + \dfrac{1}{A_1}\alpha(\mu + A_1(x-z))$, with α - Young conjugate of a, the regularizing effect estimate (3.4) leads to:

Lemma 4.1. $\max\limits_{z\in I} h(z) = \max\limits_{z\in\partial I} h(z)$ *for every compact interval* $I \subset \mathbf{R}^1$, *and*

(i) $v(x) \le v(x+\rho) + \dfrac{1}{A}\alpha(\mu - A\rho) - \dfrac{1}{A}\alpha(\mu)$ *if* $\rho(v'(x) - \alpha'(\mu)) \ge 0$*;*

(ii) $v(x) + \dfrac{1}{A}\alpha(a'(v'(x))) \le v(z) + \dfrac{1}{A}\alpha(a'(v'(x)) + A(x-z))$*;*

(iii) $a'(v'(x)) \le (\ge) A\rho + a'\left(\dfrac{v(x+\rho)-v(x)}{\rho}\right)$ *for* $\rho > (<)\ 0$*.*

P r o o f. Assume that $h(z_0) = \max\limits_{z\in I} h(z)$, $z_0 \in \operatorname{Int} I$. Then $h'(z_0) = 0$, i.e. $v'(z_0) - a'(\mu + A(x-z_0)) = 0$, and $h''(z_0) \le 0$, i.e.
$v''(z_0) + A_1\alpha''(\mu + A(x-z_0)) \le 0$, but

$$a''(v'(z_0)) = a''(\alpha'(\mu + A_1(x-z_0))) = \frac{1}{\alpha''(\mu + A_1(x-z_0))}.$$

So

$$0 \ge a''(v'(z_0))[v''(z_0) + A_1\alpha''(\mu + A(x-z_0))] = \partial_x a'(v'(z))\big|_{z=z_0} + A_1 > 0$$

which is a contradiction, so the first part of the lemma is proved.

Since $h'(x) = v'(x) - \alpha'(\mu)$ then:

- If $h(z)$ increases in x then $h(z)$ nondecreases for every $z > x$ so $h(x) \le h(x+\rho)$, $\rho > 0$ and $\rho h'(x) > 0$.
- If $h(z)$ decreases in x then $h(z)$ nonincreases for every $z < x$ so $h(x) \le h(x+\rho)$, $\rho > 0$ and $\rho h'(x) > 0$.
- If $h' = 0$ since $h'' \ge 0$, $h(x) = \inf\limits_{z\in\mathbf{R}^1} h \le h(x+\rho)$ for every ρ.

So with every $A_1 > A$ we get (i) and (ii) is its consequence with $\mu = a'(v'(x))$ and $x + \rho = z$.

To prove (iii) we take (ii) again with $z = x + \rho$ and the inequality for the convex function α :

$$\frac{1}{A}\alpha(a'(v'(x))) \ge \frac{1}{A}\alpha(a'(v'(x)) - A\rho) + \rho\alpha'(a'(v'(x)) - A\rho)$$

so $\rho\alpha'(a'(v'(x)) - A\rho) \le v(x+\rho) - v(x)$, which leads to (iii). ■

Let H_δ be nonnegative, strictly monotone function on some interval containing (δ, M). We shall omit δ in the notation of H_δ for simplicity. Then the functional $J_R(f) = H^{-1}\left(\frac{1}{R}\int_0^R H(f(y))\,dy\right)$ is defined and monotone over continuous functions $f : \mathbf{R}^1 \to [\delta, M]$ and $J_R(C) = C$ for $C = const$, $R \in \mathbf{R}^1 \setminus \{0\}$.

Property (i) from Lemma 4.1 with $\mu = 0$ can be written as $\Gamma(w(x)) - \frac{1}{A}\alpha(-AR) \le \Gamma(w(x+\rho))$ for all $\rho \in (0, R)$, $R > 0$ or $\rho \in (R, 0)$, $R < 0$ and $Rv'(x) \ge 0$ since for such ρ: $\frac{1}{A}\alpha(-A\rho) \le \frac{1}{A}\alpha(-AR)$. Define as $[P]_Q = \max(P, Q)$. Since $\Gamma(w) \in (\Gamma(\delta), \Gamma(M))$, $M = \max_x w(x)$ we can apply to the inequality $[\Gamma(w(x)) - \frac{1}{A}\alpha(-AR)]_{\Gamma(\delta)} \le \Gamma(w(x+\rho))$ subsequently the inverse increasing function Γ^{-1} and the monotone functional $J_R(f)$. So we get

Lemma 4.2. *For $R \ne 0$, $Rv'(x) \ge 0$ and H as above:*
(i) $[\Gamma(w(x)) - \frac{1}{A}\alpha(-AR)]_{\Gamma(\delta)} \le \Gamma\left[H^{-1}\left(\frac{1}{R}\int_x^{x+R} H(w(y))\,dy\right)\right]$
If $H(\delta) = 0$, $H(w) \in L^1(\mathbf{R}^1)$ then
(ii) $w(x) \to_{x\to\infty} \delta$ $a'(v'(x)) \to_{x\to\infty} 0$ *and* $f_H\left(\max_x w(x), A\right) \le A\|H(w)\|_1$, *where $f_H(\tau, A)$ is defined in item 2.*

P r o o f. We have to prove only (ii). Indeed, fix $R > 0$ (< 0) for x's such that $v'(x) \ge 0$ (< 0). $\frac{1}{R}\int_x^{x+R} H(w(y))\,dy \to_{x\to\infty} 0$ for every such R under the condition $H(w) \in L^1(\mathbf{R}^1)$. Since $H^{-1}(0) = \delta$, then

$$\overline{\lim}_{\substack{x\to\infty,\\ v'\ge(<)0}} [\Gamma(w(x)) - \frac{1}{A}\alpha(-AR)]_{\Gamma(\delta)} \le \Gamma(\delta).$$

Letting $R \to +(-)0$ we get $\lim_{x\to\infty} \Gamma(w(x)) = \Gamma(\delta)$.

Since $v(x+\rho) - v(x) \to_{x\to\infty} 0$ for all ρ then from lemma 4.1 (iii) we get $\overline{\lim}_{x\to\infty} a'(v'(x)) \le A\rho$, $\rho > 0$ and $\underline{\lim}_{x\to\infty} a'(v'(x)) \ge A\rho$, $\rho < 0$ so we obtain the second assertion of (ii) letting $\rho \to 0$. ■

Since $w(x)$ achieves its maximum $M(t)$ at some finite x then $w'(x) = v'(x) = 0$ and from (i) $|\sigma| H\left(\Gamma^{-1}\left([\Gamma(M(t)) - \frac{1}{A}\alpha(\sigma)]_{\Gamma(\delta)}\right)\right) \le A\|H(w)\|_1$ for all $\sigma = -AR$. Let $[\Gamma(M(t)) - \frac{1}{A}\alpha(\sigma)]_{\Gamma(\delta)} = \Gamma(s)$, then $\delta \le s \le M(t)$ and for all such s the inequality above can be written as $H(s)|\sigma| \le A\|H(w)\|_1$ with σ such that $\alpha(\sigma) \le A(\Gamma(M(t)) - \Gamma(s))$. So $f_H(\tau, A) \le A\|H(w)\|_1$ for $\tau = M(t)$.

Let $\alpha^{-1}_{+(-)}(\alpha(s)) = s$ for $s > (<)\ 0$. Then $\alpha^{-1}_{+(-)}(\tau)$ is nondecreasing and nonnegative (nonincreasing and nonpositive), concave (convex) function of $\tau > 0$, $\alpha^{-1}_{+(-)}(0) = 0$. For even $a(q)$: $\alpha^{-1}_{-}(\tau) = -\alpha^{-1}_{+}(\tau)$. If $a''(q) \le \frac{1}{\varepsilon}$ for $q \in \mathbf{R}^1$, then $\alpha(s) \ge \frac{\varepsilon s^2}{2}$ and $\alpha^{-1}_{+}(\tau) \le \sqrt{\frac{2\tau}{\varepsilon}}$, $\alpha^{-1}_{-}(\tau) \ge -\sqrt{\frac{2\tau}{\varepsilon}}$. We can also take as definition of $\alpha^{-1}_{+(-)}$: $\alpha^{-1}_{+}(\tau) = \inf_{\rho>0} \frac{\tau + a(\rho)}{\rho}$, $\alpha^{-1}_{-}(\tau) = \sup_{\rho<0} \frac{\tau + a(\rho)}{\rho}$. So we have the next estimate for x derivative.

Lemma 4.3. $\alpha(a'(v'(x))) \le A \int_{w(x)}^{w(z)} \frac{\varphi(s)}{s}\, ds \equiv \tau$, $z = x + \frac{1}{A}\alpha'(v'(x))$, *so* $a'(v'(x)) \in \left(\alpha^{-1}_{-}(\tau), \alpha^{-1}_{+}(\tau)\right)$.

P r o o f. We apply Lemma 4.1 (ii) with $z = x + \frac{1}{A} a'(v'(x))$ and use the fact that $v(z) - v(x) = \int_{w(x)}^{w(z)} \frac{\varphi(s)}{s}\, ds$ and $\alpha(0) = -\inf a(s) = 0$. ∎

Let us write down some properties of functions $H = H_{\delta,\varepsilon}$ defined in (3.2), $r = r_\varepsilon$.

Lemma 4.4. *For* $s, \sigma \in [\delta, M_0]$*:*
(i) $H([s-\sigma]_+ + \delta) \le [H(s) - H(\sigma)]_+$ *and the analogue of this with* $|.|$ *instead of* $[.]_+$.
(ii) The function $\tilde{H}(s) = \int_\delta^s \frac{H(\sigma)}{\sigma}\, d\sigma$ *is of the same class and* $\tilde{H}(s) \le \frac{1}{1+r} H(s)$.

P r o o f. (i) If $\delta < s < \sigma$ the both sides are 0, now if $\delta < \sigma < s$: $[H(s-\sigma+\delta) - H(s)]'_s = H'(s-\sigma+\delta) - H'(s) \le 0$, so $H(s-\sigma+\delta) - H(s) \le H(\delta) - H(\sigma) = -H(\sigma)$. Further

$$H(|s-\sigma| + \delta) = H([s-\sigma]_+ + \delta) + H([\sigma - s]_+ + \delta)$$

$$\le [H(s) - H(\sigma)]_+ + [H(\sigma) - H(s)]_+ = |H(s) - H(\sigma)|.$$

(ii) Note that $sH' - (1+r)H(s)$ is nonnegative and increasing so $s\tilde{H}''(s) - r\tilde{H}'(s) = \left(s\tilde{H}'(s)\right)' - (1+r)\tilde{H}'(s) = \frac{sH'(s) - (1+r)H}{s} \ge 0$ and $\tilde{H}(s) \le \frac{1}{1+r} \int_\delta^s \frac{\sigma H'(\sigma)}{\sigma}\, d\sigma = \frac{1}{1+r} H(s)$. ∎

Lemma 4.5. *Let* w *is a smooth solution of the regularized Cauchy problem (2.1) and let* $H(w_0) \in L^1(\mathbf{R}^1)$ *, then for* $t > 0$ *and* $A = \frac{1}{kt}$*:*
(i) $\|H(w)\|_1 \le \|H(w_0)\|_1$.

(ii) $\left\| H'(w)\dfrac{w}{\varphi(w)}v'a'(v')\right\|_1 \le A\|H(w_0)\|_1$.

(iii) $\displaystyle\int_0^t\int_{\mathbf{R}^1}\frac{w}{\varphi(w)}\left[wH''(w)v'a'(v') - H'(w)b(v')\right]\,dxd\tau \le \|H(w_0)\|_1$.

(iv) $\left\|\dfrac{H(w)}{w}w_t\right\|_1 \le 2AC_{r_0}\|H(w_0\|_1$ *and* $\displaystyle\int_{\mathbf{R}^1}\frac{H(w)}{w}w_t \le 0$.

Proof. Since $wH''(w)v'a'(v')-H'(w)b(v') \ge (wH''(w) - rH'(w))\,v'a'(v') \ge 0$ then in every bounded interval K we get

$$\begin{aligned}(4.1)\qquad &\tfrac{d}{dt}\textstyle\int_K H(w)\,dx = \int_K H'(w)w_t\,dx = \int_K H'(w)\left[\partial w a'(v') + \frac{w}{\varphi(w)}b(v')\right]\\ &= wH'a'(v')|_{\partial K} - \textstyle\int_K \frac{w}{\varphi(w)}\left[wH''v'a'(v') - H'b(v')\right]\\ &\le wH'(w)a'(v')|_{\partial K}\,.\end{aligned}$$

But $wH'(w) \le M_0H'(M_0)$, where $M_0 = \max w_0$ and $|a'(v')| \le \dfrac{C}{\sqrt{t}}$ with some $C(\delta,\varepsilon,M_0)$ as we noted preparing Lemma 4.3. So we integrate the inequality (4.1) over $(0,t)$ and get

$$(4.2)\qquad \int_K H(w)\,dx \le \int_K H(w_0) + \int_0^t wH'(w)a'(v')\,d\tau\Big|_{\partial K}$$

then $\displaystyle\int_K H(w)\,dx \le \int_K H(w_0)+C_1\sqrt{t}$, that leads to $H(w)\in L^1(\mathbf{R}^1)$. According to Lemma 4.2 (ii) $a'(v'(x)) \to_{x\to\infty} 0$ by Lebegue theorem we get from (4.2) the first inequality (i).

It follows from the regularizing effect (3.4) that the function

$$H(w)a'(v')(x) + A\int_{x_0}^x H(w)\,dy - \int_{x_0}^x H'(w)\frac{w}{\varphi(w)}v'a'(v')$$

is nondecreasing. So for every interval K : $0 \le \displaystyle\int_K H'(w)\frac{w}{\varphi(w)}v'a'(v')\,dx \le H(w)a'(v')|_{\partial K} + A\int_K H(w)\,dx$ and letting $K\to R^1$ we get (ii).

Inequality (iii) is obvious from (4.1) since $wH''v'a'(v') - H'b(v') \ge 0$. From the regularizing effect (3.4) it follows that $w_t \ge -Aw + \dfrac{w}{\varphi(w)}\tilde b(v')$. So $|w_t| = \left|w_t + Aw - \dfrac{w}{\varphi(w)}\tilde b(v') + \dfrac{w}{\varphi(w)}\tilde b(v') - Aw\right| \le w_t + 2Aw + 2\dfrac{w}{\varphi(w)}\cdot$ $\cdot\left[-\tilde b(v')\right]_+$ where $\left[-\tilde b\right]_+ = [-b-qa'(q)]_+ \le [r_0-1]_+\,qa'(q)$. Multiplying by $\tilde H'(w) = \dfrac{H(w)}{w}$ and integrating over K as in (4.2),

$$\int_K \frac{H(w)}{w}|w_t|\,dx \le \int_K \tilde H'(w)w_t\,dx + 2A\int_K H(w)\,dx$$

$$+2[r_0-1]_+\int_K \tilde{H}'\frac{w}{\varphi(w)}v'a'(v')\,dx$$

$$\leq H(w)a'(v')|_{\partial K}+2A\int_K H(w)\,dx+\frac{2[r_0-1]_+}{1+r}A\int_K H(w)\,dx$$

and with $K\to R^1$, we get (iv). ■

In order to be in a position to pass to limit with $\delta\to 0$, $\varepsilon\to 0$, we need some additional and more precise estimates for the solutions $w_{\delta,\varepsilon}(t,x)$. The limiting process depends on the properties of $\Gamma(w)$. We summarize the estimates for the smooth solutions of (2.1) in two theorems.

Theorem 4.1. (Uniform bound for x derivatives)
Cases (+), (±):

> *There exists a function $G(\tau)$ concave and nondecreasing in $\tau\geq 0$ such that $|a'(v'(x))|\leq G(A\Gamma(M))$ and in the case (±) the argument of $G(\tau)$ above can be replaced by $A\Gamma(\infty)=A\int_0^\infty\frac{\varphi(s)}{s}\,ds$.*

Cases(0), (-) :

> *There exists independent in indices constant $C(M,A)$ such that $|w'|\leq C(M,A)$.*

P r o o f. In the cases (+), (±) the proof is a direct consequence of lemma 4.3, since $\displaystyle\int_{w(x)}^{w(z)}\frac{\varphi(s)}{s}\,ds\leq\int_0^M\frac{\varphi(s)}{s}\,ds=\Gamma(M)$. So $\alpha(a'(v'))\leq A(\Gamma(M))$, and $|a'(v'(x))|\leq G(A(\Gamma(M)))$, with $G(\tau)=\max(\alpha_+^{-1}(\tau),-\alpha_-^{-1}(\tau))$.

In the cases(0), (-) through lemma 4.3 $\displaystyle\frac{w}{\varphi(w)}\alpha(a'(v'))\leq A\frac{w}{\varphi(w)}\int_w^M\frac{\varphi(s)}{s}\,ds.$ So by the last inequality in (2.4)

$$|w'(x)|=\frac{w}{\varphi(w)}|v'|\leq\frac{w}{\varphi(w)}(C+C_1\alpha(a'(v')))\leq\frac{w}{\varphi(w)}C+AC_1\frac{w}{\varphi(w)}\int_w^M\frac{\varphi(s)}{s}\,ds.$$

But $\displaystyle\left(\frac{\sigma}{\varphi(\sigma)}\right)'=\frac{1-\lambda(\sigma)}{\varphi(\sigma)}>0$ and by l'Hospitale rule

$$\lim_{\sigma\to 0}\frac{\int_\sigma^M\frac{\varphi(s)}{s}\,ds}{\frac{\varphi(\sigma)}{\sigma}}=\lim_{\sigma\to 0}\frac{\sigma}{1-\lambda(\sigma)}=0.$$

So $\displaystyle\frac{w}{\varphi(w)}\leq\frac{M}{\varphi(M)}$, and $\displaystyle\sup_{w\leq M}\frac{w}{\varphi(w)}|v'|\leq C(M,A)$. ■

Theorem 4.2. *There exists an increasing, concave function $l(s)$, $l(0) = 0$, such that*

$$|w(t_1, x_1) - w(t_0, x_0)| \leq l(\tilde{C}\rho) \tag{4.3}$$

for t_0, $t_1 \in [\tau, \infty]$, $\tau > 0$, $|t_1 - t_0| \leq \rho^2$, $|x_1 - x_0| \leq \rho$, and $\tilde{C}$ depends on τ, M, $\|H(w_0)\|_1$.

Proof. Let us define the function $h(s)$ for different cases as:

Case (+): $\left| h(s) = \int_0^s \frac{\varphi(\sigma)}{\sigma^\nu} \, d\sigma \right.$, with $\nu = \min(1, \lambda(\infty))$.

Case (±):

$$\left| \begin{array}{l} h(s) = \begin{cases} \int_0^s \varphi(\sigma)\, d\sigma, & s < s_0 \\ \varphi(s_0)(s - s_0) + \int_0^{s_0} \varphi(\sigma)\, d\sigma, & s > s_0, \end{cases} \\ \text{if } \lambda(s) \begin{cases} > 0, & s < s_0 \\ \leq 0, & s > s_0 \end{cases} \text{ and } s_0 \in (0, \infty). \end{array} \right.$$

Cases(0),(-): $|h(s) = s$.

Then $h(0) = 0$, $h'(s)$ is nondecreasing and positive. So $h(s)$ has the property (i) of $H(s)$ with $\delta = 0$ from Lemma 4.4 i.e. $h(|s - t|) \leq |h(s) - h(t)|$ for $s, t \in (0, \infty)$ then $h\left(|w(x,t) - w(t,\hat{x})|\right) \leq |h(w(t,x)) - h(w(t,\hat{x}))| = \left|\frac{h'(w)w}{\varphi(w)}(\xi)v'(\xi)\right| |x - \hat{x}|$. From the previous Theorem 4.1 in the cases (+), (±) $v'(\xi)$ is uniformly bounded in $S_{\tau,T}$, $\frac{h'(w)w}{\varphi(w)} = w^{1-v}$ for the case (+) and $\begin{cases} w, & w < s_0 \\ \frac{\varphi(s_0)w}{\varphi(w)}, & w \geq s_0 \end{cases}$ for the case (±), which are increasing functions of w and so bounded by $\frac{h'(M)M}{\varphi(M)}$.

In the other two cases $\frac{h'(w)w}{\varphi(w)} v_x = w_x$ which is also uniformly bounded by the same theorem. So we get $|w(t,x) - w(t,\hat{x})| \leq h^{-1}\left(C_{M,\tau}|x - \hat{x}|\right)$ for all $t \geq \tau$. Let now $|x_1 - x_0| < \frac{\rho}{2}$, $0 < \tau < t_0 < t_1 < t_0 + \rho^2$, then for every x such that $|x - x_0| < \frac{\rho}{2}$:

$$|w(t_1,x_1) - w(t_0,x_0)| \leq |w(t_1,x_1) - w(t_1,x)| + |w(t_1,x) - w(t_0,x)|$$
$$+|w(t_0,x) - w(t_0,x_0)| \leq 2h^{-1}\left(C_{M,\tau}\rho\right) + |w(t_1,x) - w(t_0,x)|.$$

With $\tilde{H}(s) = \int_\delta^s \frac{H(\sigma)}{\sigma} \, d\sigma$ from Lemma 4.4 we have

$$J \equiv \tilde{H}\left(\left[|w(t_1,x_1) - w(t_0,x_0)| - 2h^{-1}\left(C_{M,\tau}\rho\right)\right]_+ + \delta\right)$$

$$\leq \tilde{H}\left(|w(t_1,x)-w(t_0,x)|+\delta\right) \leq |\tilde{H}(w(t_1,x))-\tilde{H}(w(t_0,x))|.$$

Integrating J over $|x-x_0|<\dfrac{\rho}{2}$ by the estimate (iv) from Lemma 4.5 we get

$$\rho J \leq \int_{|x-x_0|<\frac{\rho}{2}} \left|\tilde{H}(w(t_1,x))-\tilde{H}(w(t_0,x))\right| dx$$

$$\leq \int_{|x-x_0|<\frac{\rho}{2}} \int_{t_0}^{t_1} \tilde{H}'(w(t,x)|w_t|\, dtdx \leq \int_{t_0}^{t_1}\int_{\mathbf{R}^1} \frac{H(w(t,x))}{w(t,x)}|w_t(t,x)|\, dxdt$$

$$\leq 2C\int_{t_0}^{t_1} A\|H(w_0)\|_1\, dt = \frac{2C}{k}\|H(w_0)\|_1 \text{ Ln } \frac{t_1}{t_0} \leq \frac{2C\rho^2}{k\tau}\|H(w_0)\|_1.$$

So $J \leq \dfrac{2C}{k\tau}\rho\|H(w_0)\|_1$ and

$$|w(t_1,x_1)-w(t_0,x_0)| \leq 2h^{-1}(C_{M,\tau}\rho)+\tilde{H}^{-1}\left(\frac{2C}{k\tau}\rho\|H(w_0)\|_1\right)-\delta$$

for $|x_1-x_0|\leq\rho$, $|t_1-t_0|\leq\rho^2$ and we get the result of the theorem with

$$l(s)=2h^{-1}(s)+\max_{\delta>0}\left[\tilde{H}^{-1}(s)-\delta\right] \text{ and } \tilde{C}=\max\left(C_{M,\tau},\frac{2C}{k\tau}\|H(w_0)\|_1\right).$$

■

5. Limiting process, proof of the main theorems

Recall that we omit ε, δ in the notation of the solution $w(t,x)$ and other functions in the regularizing Cauchy problem (2.1). We shall use the notation with a bar: $\overline{w}$, $\overline{v}$ etc. for the corresponding limiting functions as $\varepsilon\to 0$, $\delta\to 0$ in suitable subsequences.

The proof of Theorems 2.1 - 2.3 is derived in three theorems:

Theorem 5.1. *(i) The solutions of the regularized problem $w(t,x)$ and the corresponding functions $v(t,x)$ are precompacts in $C_{loc}(S_{\tau,T})$ and for every compact $K\subset\mathbf{R}^1$ and $\tau>0$, $v_x(t,x)$ are uniformly bounded in $[\tau,T]\times K$. The limit $\overline{w}(t,x)$ is nonnegative in the cases (+), (±) and positive in cases (-), (0), Hölder continuous in $S_{\tau,T}$ for $\tau>0$ and $\|\overline{w}(t,.)\|_\infty\leq\|\overline{w}_0\|_\infty$, $\|\overline{H}(\overline{w}(t,.))\|_1\leq\|\overline{H}(\overline{w}_0)\|_1$, $\lim\limits_{|x|\to\infty}\overline{w}(t,x)=0$.*

(ii) There exist for $t>0$ finite left and right derivatives $\overline{v}_x(t,x-0)\leq\overline{v}_x(t,x+0)$ and a.e. continuous $\overline{v}_x(t,x)$ to which the approximate derivatives v_x converge in every $Q\subset S_{\tau,T}$.

(iii) $\partial_x \overline{a}'(\overline{v}_x) \geq -A$ *in* $D'(S_T)$ *and* $\overline{w}$ *satisfies the pointwise estimates (2.8) or (2.9).*

P r o o f. (i) is obvious in the cases (+), (±) since the function $\Gamma(x) = = \int_0^s \frac{\varphi(\sigma)}{\sigma} d\sigma$ is uniformly continuous in $[0, M]$ and the global boundness of $v_x(t,x)$ is done by Theorem 4.1. Thus precompactness of $w(t,x)$ by theorem 4.2 leads to precompactness of $v(t,x)$ and their uniform convergence may be applied to the previous lemmas and theorems. For instance the Hölder continuity of $\overline{w}(t,x)$ in $S_{\tau,T}$ and imbedding $H(\overline{w}) \in L^\infty(S_T) \cap L^\infty((0,T); L^1(\mathbf{R}^1))$ is inherited from Theorem 4.2 and Lemma 4.5 (i). We also have to use here the postulated in (3.2) convergence $H_\delta(w_{0,\delta}) \to \overline{H}(\overline{w}_0)$ in $L^1(\mathbf{R}^1)$. As a consequence of (i) we have $\overline{v}(t,x) \in W^{1,\infty}(S_{\tau,T})$.

The problem in the cases (-), (0) is that $\Gamma(+0) = -\infty$. Let assume for the moment that $\overline{w}(t,x)$ is positive for $t > 0$. Then in every compact $Q \subset S_{\tau T}$ the approximate sequence $w(t,x)$ is bounded away from 0 by a constant depending on $\inf_Q \overline{w}(t,x)$ and the uniform convergence of $v(t,x)$ in $Q' \subset Q$ takes place. Moreover $v_x(t,x)$ are uniformly bounded in Q by lemma 4.3 and we can get (i) as above. Here however $\overline{v}(t,x) \in W^{1,\infty}_{loc}(S_{\tau,T})$.

So it remains to prove the positivity of $\overline{w}(t,x)$ in the cases (-), (0). Let $\overline{w}(t,\overline{x}) > 0$, $\overline{w}(t,x_0) = 0$ for some $x_0 < \overline{x}$ and x_1 is such that $\overline{x} \in I = (x_0, x_1)$. Note that $v(t,\overline{x}) \to \overline{v}(t,\overline{x})$ and $v(t,x_0) \to -\infty$. By Lemma 4.1 the function $h(t,z) = v(t,z) + \frac{1}{A_1}\alpha(A_1(x_1 - z))$ achieves its maximum on ∂I. So

$$v(t,\overline{x}) + \frac{1}{A_1}\alpha(A_1(x_1 - \overline{x})) = h(t,\overline{x}) \leq \max\left(h(t,x_0), h(t,x_1)\right)$$

$$= \max\left(v(t,x_0) + \frac{1}{A_1}\alpha(A_1(x_1 - x_0)), v(t,x_1)\right)$$

Letting $\varepsilon \to 0$, $\delta \to 0$ the limit of $v(t,x_1)$ must be finite and thus equal to $\overline{v}(t,x_1)$. Moreover $\overline{v}(t,x_1) \geq \overline{v}(t,\overline{x}) + \frac{1}{A_1}\overline{\alpha}(A_1(x_1 - \overline{x}))$ and this is obviously true for all $x_1 > \overline{x}$. But for $x_1 \to +\infty$ the inequality contradicts to $\overline{v}(t,x) \leq \Gamma(M) < \infty$. So $\overline{w}$ is positive or identically 0 . Meanwhile we proved that the function $\overline{h}(t,z) = \overline{v}(t,z) + \frac{1}{A_1}\overline{\alpha}(A_1(x_1 - z))$ for fixed x and $t > 0$ has the extremum property of Lemma 4.1. So $\overline{v}(t,x)$ is of bounded variation in x on finite sets.

(ii) Denote

$$D^{\pm}(\overline{v}(t,x)) = \overline{\mathrm{Lim}}_{\rho \to \pm 0} \frac{\overline{v}(t,x+\rho) - \overline{v}(t,x)}{\rho},$$

$$D_{\pm}(\overline{v}(t,x)) = \underline{\mathrm{Lim}}_{\rho\to\pm 0}\frac{\overline{v}(t,x+\rho)-\overline{v}(t,x)}{\rho}.$$

We shall prove that $D_+\overline{v}(t,x) = D^+\overline{v}(t,x) \geq D^-\overline{v}(t,x) = D_-\overline{v}(t,x)$. Indeed from (iii), Lemma 4.1 we have $g(x,-\rho) \leq a'(v_x(t,x)) \leq g(x,\rho)$ for $\rho > 0$ with $g(x,\rho) = A\rho + a'\dfrac{v(t,x+\rho)-v(t,x)}{\rho}$ and its integrated variant

$$\frac{1}{s}\int_y^{y+s}\alpha'(g(x,-\rho))\,dx \leq \frac{v(t,y+s)-v(t,y)}{s} \leq \frac{1}{s}\int_y^{y+s}\alpha'(g(x,\rho))\,dx$$

with arbitrary $s \neq 0$. For fixed ρ the function $g(x,\rho)$ converge uniformly on compact sets to $\overline{g}(x,\rho) = A\rho + \overline{a}'\left(\dfrac{\overline{v}(t,x+\rho)-\overline{v}(t,x)}{\rho}\right)$. Subsequently the limits $\varepsilon\to 0$, $\delta\to 0$, $s\to 0^+$ or $s\to 0^-$ in the last inequality lead to $D^+\overline{v}(t,x) \leq \overline{\alpha}'(\overline{g}(y,\rho))$ and $\overline{\alpha}'(\overline{g}(y,-\rho)) \leq D_-\overline{v}(t,y)$ and from $\rho\to 0^+$ we get $D^+\overline{v} \leq D_+\overline{v}$, $D^-\overline{v} \leq D_-\overline{v}$. So there exist finite $\overline{v}_x(t,x+0)$, $\overline{v}_x(t,x-0)$. Subsequently the limits $\varepsilon\to 0$, $\delta\to 0$, $\rho\to 0^+$ in the first inequality lead to $\overline{a}'(\overline{v}_x(t,x-0)) \leq \underline{\lim}\,a'(v_x(t,x)) \leq \overline{\lim}\,a'(v_x(t,x)) \leq \overline{a}'(\overline{v}_x(t,x+0))$.

So $\overline{v}_x(t,x-0) \geq \overline{v}_x(t,x+0)$ and we have $v_x(t,x)\to\overline{v}_x(t,x)$ at the points of existence of $\overline{v}_x(t,x)$. Note that at the points of maximum of $\overline{v}$, the derivative $\overline{v}_x$ exists since there $\overline{v}_x(t,x-0) \geq \overline{v}_x(t,x+0)$ along with the inverse inequality we got above.

(iii) At last since $\overline{a}'(\overline{v}_x(t,x)) + A_1x$ is monotone increasing then $\overline{a}'(\overline{v}_x(t,x))$ is a.e. continuous and its derivative in $D'(Q)$ satisfies $\partial\overline{a}'(\overline{v}_x(t,x)) \geq -A$.

Let us write (2.1) in a weak form. For $\xi \in C_0^\infty(S_T)$

$$\int\!\!\int_{S_T}\left(w\xi_t - wa'(v_x)\xi_x + \frac{w}{\varphi(w)}b(v_x)\xi\right)dxdt = 0 \tag{5.1}$$

Note that in the cases (-) and (0): since $\dfrac{w}{\varphi(w)} \leq \dfrac{M_0}{\varphi(M_0)}$, $b(v_x)$ is uniformly bounded in supp ξ and $b(v_x)\to\overline{b}(\overline{v}_x)$ a.e., then $\dfrac{w}{\varphi(w)}b(v_x)\to\dfrac{\overline{w}}{\varphi(\overline{w})}\overline{b}(\overline{v}_x)$ a.e.; since $w(t,x)\to\overline{w}(t,x)$ uniformly and $wa'(v_x)\to\overline{wa}'(\overline{v}_x)$ a.e. $|wa'(v_x)| \leq C(K,M_0)$, $K\subset$ supp ξ, then we can pass to the limit in the integral in (5.1) by the Lebegues theorem and obtain that $\overline{w}$ is a weak solution of (2.1).

Note that in the cases (-) and (0) for every fixed compact K and $\tau > 0$, v_x is uniformly bounded in $[\tau,T]\times K$. Indeed $v_x = \dfrac{\varphi(w)}{w}w_x$ and w_x is globally bounded by theorem 4.1 and by the positivity of $\overline{w}$ proved in Theorem 5.1. ∎

Further we shall prove the possibility of this limit for cases $(\pm)$, $(+)$ along with regularity of derivatives $\overline{w}_t$, $\partial_x\overline{wa}'(\overline{v}_x)$ that gives Theorem 2.1.

Theorem 5.2. *$\overline{w}_t$, $\partial_x \overline{w}\overline{a}'(\overline{v}')$ and $\dfrac{\overline{w}}{\varphi(\overline{w})}\overline{b}(\overline{v}') \in L^1_{loc}(S_T)$ so the equation (2.1) is satisfied a.e. by $\overline{w}$ and its corresponding derivatives.*

Proof. It is sufficient to prove uniform $L^{1+\beta}_{loc}(S_T)$, $\beta > 0$ estimates for the derivatives of approximate solutions. For $\xi \in C_0^\infty(\mathbf{R}^1)$

$$\int_{t_0}^T \int_{\mathbf{R}^1} \xi^2 \frac{\varphi(w)}{w} w_t^2 \, dxdt = \int_{t_0}^T \int_{\mathbf{R}^1} \xi^2 v_t \left[\partial_x w a'(v_x) + \frac{w}{\varphi(w)} b(v_x)\right] dxdt$$

$$= -2\int_{t_0}^T \int_{\mathbf{R}^1} \xi\xi' v_t w a'(v_x)\, dxdt - \int_{t_0}^T \int_{\mathbf{R}^1} \xi^2 w \frac{d}{dt} a(v_x)\, dxdt$$

$$+ \int_{t_0}^T \int_{\mathbf{R}^1} \xi^2 w_t b(v_x)\, dxdt \le 2\int_{t_0}^T \int_{\mathbf{R}^1} |\xi\xi'|\varphi(w)|w_t||a'(v_x)|\, dxdt$$

$$+ \int_{\mathbf{R}^1} \xi^2 w a(v_x)\Big|_{t=t_0} + \int_{t_0}^T \int_{\mathbf{R}^1} \xi^2 |w_t||a(v_x) + b(v_x)|\, dxdt.$$

But $0 \le a(q) \le qa'(q)$ so $(r_0 - 1)qa'(q) \le a(q) + b(q) \le (r+1)qa'(q)$ and all expressions with v_x are bounded by some $C(t_0, M_0, K)$, where K is a fixed compact supp $\xi \subset K$. The same is true for w and $\varphi(w)$. Since $\tilde{b}(q) = qa'(q) + b(q) \ge r_0 q a_1 \ge 0$ and $w_t \ge -\dfrac{1}{kt} w + \dfrac{w}{\varphi(w)}\tilde{b}(v_x) \ge -\dfrac{1}{kt} w$, then $|w_t| \le w_t + \dfrac{2}{kt_0} w$ for $t \ge t_0$. So $\displaystyle\int_{t_0}^T \int_{\mathbf{R}^1} \xi^2 \frac{\varphi(w)}{w} w_t^2 \, dtdx \le C_1 \left[\int_{t_0}^T \int_K |w_t|\, dtdx + \int_K w(t_0)\, dx\right] \le$ C_2, with constants $C_1(t_0, M_0, K)$, $C_2(t_0, M_0, K)$. Also $\left|w_t - \dfrac{w}{\varphi(w)}\tilde{b}(v_x)\right| \le w_t + \dfrac{2}{kt_0} w$. So

$$\frac{\varphi(w)}{w}\left|w_t - \frac{w}{\varphi(w)}\tilde{b}(v_x)\right|^2 \le 2\left[\frac{\varphi(w)}{w} w_t^2 + \frac{4}{(kt_0)^2} w\varphi(w)\right] \in L^1_{loc}(S_{t_0,T}).$$

If we may drop the multiplier $\dfrac{\varphi(w)}{w}$ in these imbeddings, we will conclude that the sets $\{w_t\}$, $\left\{\dfrac{w}{\varphi(w)}\tilde{b}(v_x)\right\}$ are precompacts in $L^1_{loc}(S_T)$ and the same remains true for $\left\{\dfrac{w}{\varphi(w)} v_x a'(v_x)\right\}$, $\{\partial_x w a'(v_x)\}$ since $\dfrac{w}{\varphi(w)} v_x a'(v_x) \le \dfrac{1}{r_0}\dfrac{w}{\varphi(w)}\tilde{b}(v_x)$, $\partial_x w a'(v_x) = w_t - \dfrac{w}{\varphi(w)} b(v_x)$. Passing to limit in a subsequence we get the existence result of Theorem 2.1, by using the fact that $\dfrac{w}{\varphi(w)} b(v_x) \to \dfrac{\overline{w}}{\varphi(\overline{w})} b(\overline{v}_x)$

a.e. in $S_T^+ = \{(t,x) : \overline{w}(t,x) > 0\}$ and in the intervals, where

$$\overline{w} \equiv 0, \quad \frac{w}{\varphi(w)} b(v_x) \in ((r_0-1)w_x a'(v_x), (r+1)w_x a'(v_x)) \to 0.$$

Of course the interface $\overline{S}_T^+ \setminus S_T^+$ has measure 0.

Now we exploit an idea of Benilan [2]. Let $0 < \beta < \min\left(1, \frac{1}{\lambda(0)}\right)$ in the cases ($\pm$), (+) where by (2.2), $0 \leq \lambda(0) < \infty$. Then $\left[\dfrac{s}{\varphi(s)}\right]^{\frac{\beta}{1-\beta}}$ is integrable over $(0,w)$ and $g(w) \equiv \displaystyle\int_0^w \left[\frac{s}{\varphi(s)}\right]^{\frac{\beta}{1-\beta}} ds \leq \frac{1-\beta}{1-\beta\lambda(0)} wg'(w)$, $wg'(w)$ nondecreases. Indeed for every $0 < \beta \leq \dfrac{1}{\lambda(0)}$, for $s > 0$ $\left[s^{-\frac{1}{\beta}}\varphi(s)\right]' = s^{-\frac{1}{\beta}-1}\varphi(s)\left[\lambda(s) - \dfrac{1}{\beta}\right] \leq$ $\leq s^{-\frac{1}{\beta}-1}\varphi(s)[\lambda(s) - \lambda(0)] \leq 0$. So

$$\int_0^w \left[\frac{s}{\varphi(s)}\right]^{\frac{\beta}{1-\beta}} ds = \int_0^w \left[s^{-\lambda(0)}\varphi(s)\right]^{-\frac{\beta}{1-\beta}} s^{\frac{1-\lambda(0)}{1-\beta}\beta} ds \leq \left[w^{-\lambda(0)}\varphi(w)\right]^{-\frac{\beta}{1-\beta}}$$

$$\times \frac{1-\beta}{1-\lambda(0)\beta} w^{\frac{1-\lambda(0)}{1-\beta}\beta} = \frac{1-\beta}{1-\lambda(0)\beta}\left[w^{-\frac{1}{\beta}}\varphi(w)\right]^{-\frac{\beta}{1-\beta}} = \frac{1-\beta}{1-\lambda(0)\beta} wg'(w).$$

So $g(w)$ and $wg'(w)$ are bounded for $w \in [0, M_0]$. Then with $g'(w) = \left[\dfrac{w}{\varphi(w)}\right]^{\frac{\beta}{1-\beta}}$ and by the Hölder inequality,

$$\int_{t_0}^T \int_K \xi^2 w_t^{1+\beta}\, dxdt = \int_{t_0}^T \int_K \xi^2 \left[\frac{\varphi(w)}{w} w_t^2\right]^\beta \left[g'(w)|w_t|\right]^{1-\beta} dxdt$$

$$\leq \left(\int_{t_0}^T \int_K \xi^2 \frac{\varphi(w)}{w} w_t^2\, dxdt\right)^\beta \left(\int_{t_0}^T \int_K \xi^2 g'(w)|w_t|\, dxdt\right)^{1-\beta} \leq C_2^\beta J^{1-\beta}$$

and

$$J \leq \int_{t_0}^T \int_K \xi^2 \left[g'(w)w_t + \frac{2}{kt_0} g'(w)w\right] dxdt$$

$$\leq \int_K \xi^2 g(w) dx\Big|_{t=T} + \frac{2}{kt_0} g'(M_0)M_0(T-t_0)\int_K \xi^2\, dx$$

which is majorized by some constant $C_3(t_0, T, M_0, K)$, supp $\xi \subset K$.

The same can be done for $\left|w_t - \dfrac{w}{\varphi(w)}\tilde{b}(v_x)\right|$ instead of w_t. So $L_{loc}^{1+\beta}$ boundness of these quantities leads to corresponding precompactness in L_{loc}^1

and the above analysis takes place. Moreover we can pass to limit in the global integral inequalities of Lemma 4.5. Since in fact $\partial_x \overline{wa}'(\overline{v}_x)$ belongs to some $L^{1+\beta}_{loc}$ with $\beta > 0$, $\overline{wa}'(\overline{v}_x) \in L^{1+\beta}_{loc}\left((0,T); C^{\frac{\beta}{1+\beta}}_{loc}(\mathbf{R}^1)\right)$, so $\overline{wa}'(\overline{v}_x)$ is Hölder continuous in x for a.e. $t > 0$ and $\overline{v}_x(t,x)$ with $\overline{w}_x(t,x)$ are continuous on S^+_T for such $t > 0$. ■

Theorem 5.3. *The function $\overline{w}(t,x)$ satisfies the initial conditions $\overline{w}(0,x) = w_0(x)$ a.e. in $\mathbf{R}^1$.*

Proof. Let $w_\delta = \lim\limits_{\varepsilon\to 0} w_{\delta,\varepsilon}$, then $w_\delta \in C(\overline{S}_T)$. Moreover w_δ, v_δ satisfy all lemmas and theorems up to $t = 0$ since in this case $A = \dfrac{1}{kt + C_\delta}$ with some $C_\delta > 0$. Let also $H(s)$ is uniform limit in $[0, M]$ of $H_\delta(s)$ as $\delta \to \infty$, where $H_\delta(s)$ is the function defined in chapter 4. With $\tilde{H}(s) = \displaystyle\int_0^s \frac{H(\tau)}{\tau}\, d\tau$ and for fixed K - a compact set in $\mathbf{R}^1$

$$\int_K \tilde{H}(w_\delta)\, dx = \int_K \tilde{H}(w_{0\delta}) + \int_0^t \left[H'(w_\delta) wa'(v_{\delta x})\right]\Big|_{\partial K} - \int_0^t \int_K \overline{g}(w_\delta, v_{\delta x})\, dxdt,$$

where $\overline{g}(w, v_x) = \dfrac{w}{\varphi(w)}\left[w\tilde{H}'' v_x \overline{a}'(v_x) - \tilde{H}'\overline{b}(v_x)\right] \geq 0$. Here we used a part of the existence result, Theorem 5.2, applied with $\varepsilon \to 0$ only.

Define further an approximation of $[s]_+$,

$$f_\sigma(s) = \begin{cases} 0, & s \leq 0 \\ \frac{s^2}{2\sigma}, & 0 < s < \sigma \\ s - \frac{\sigma}{2}, & \sigma \leq s \end{cases}, \quad \text{then } 0 \leq f'_\sigma \leq 1 \text{ and}$$

$$f''_\sigma(s) = \begin{cases} 0, & s \leq 0 \\ \frac{1}{\sigma}, & 0 < s < \sigma. \\ 0, & \sigma \leq s \end{cases}$$

For given function $F(s,q)$ and $\delta_1, \delta_2 > 0$ denote $\{F(w, v_x)\} = F(w_{\delta_1}, v_{\delta_1 x}) - F(w_{\delta_2}, v_{\delta_2 x})$. Then in K

$$\begin{aligned} &\frac{d}{dt}\int_K f\left(\left\{\tilde{H}(w)\right\}\right) dx \\ &= \int_K f'\left(\left\{\tilde{H}(w)\right\}\right)\left\{\tilde{H}'(w)\partial w\overline{a}'(v_x) + \tilde{H}'(w)\frac{w}{\varphi(w)}\overline{b}(v_x)\right\} \\ &= f'\left(\left\{\tilde{H}(w)\right\}\right)\left\{\tilde{H}'(w) w\overline{a}'(v_x)\right\}\Big|_{\partial K} - \int_K f'\left(\left\{\tilde{H}(w)\right\}\right)\left\{\overline{g}(w, v_x)\right\} \\ &\quad - \int_K f''\left(\left\{\tilde{H}(w)\right\}\right)\left\{\tilde{H}'(w)\frac{w}{\varphi(w)} v_x\right\}\left\{\tilde{H}'(w) w\overline{a}'(v_x)\right\} \end{aligned}$$

and integrate it over $(0,t)$. Since

$$-\int_0^t\int_K f'\left(\left\{\tilde H(w)\right\}\right)\{\overline{g}(w,v_x)\}\,dxdt \le \int_0^t\int_K \overline{g}(w_{\delta_2},v_{\delta_2 x})\,dxdt$$
$$=\int_K \tilde H(w_{\delta_2 0})\,dx-\int_K \tilde H(w_{\delta_2}(t))\,dx+\int_0^t H(w_{\delta_2})w_{\delta_2}\overline{a}'(v_{\delta_2 x})\,ds\Big|_{\partial K},$$

we get

$$\begin{array}{ll}(5.2) & \int_K f\left(\left\{\tilde H(w)\right\}\right)(t)\,dx \le \int_K f\left(\left\{\tilde H(w)\right\}\right)(0)\,dx \\ & +\int_0^t\left|\tilde H'(w_{\delta_1})w_{\delta_1}\overline{a}'(v_{\delta_1 x})\right|\,ds\Big|_{\partial K}+2\int_0^t\left|\tilde H'(w_{\delta_2})w_{\delta_2}\overline{a}'(v_{\delta_2 x})\right|\,dt\Big|_{\partial K} \\ & +\int_K \tilde H(w_{\delta_2 0})\,dx-\int_K H(w_{\delta_2}(t))\,dx-J_\sigma,\end{array}$$

where $J_\sigma=\displaystyle\int_0^t\int_K f''\left(\left\{\tilde H(w)\right\}\right)\left\{\tilde H(w)\frac{w}{\varphi(w)}v_x\right\}\left\{\tilde H'(w)w\overline{a}'(v_x)\right\}\,dxdt$. To estimate J_σ we shall use the inequality:

$$\{s\}\left\{\mu\overline{a}'\left(\frac{s}{\eta}\right)\right\}\ge\{\eta\}\left\{\mu\overline{a}_1'\left(\frac{s}{\eta}\right)\right\}+\{\mu\}\left\{\eta\overline{a}\left(\frac{s}{\eta}\right)\right\}, \tag{5.3}$$

for $\mu\ge 0$, $\eta>0$, $s\in\mathbf{R}^1$ and $\overline{a}_1(q)=qa'-a$. To prove the estimate (5.3) note that the function $C(\eta,s)=\eta\overline{a}\left(\dfrac{s}{\eta}\right)$ is convex for $\eta>0$, so the convexity property $c(p)\ge c(q)+(p-q)\nabla c(q)$ gives

$$\eta_2\overline{a}\left(\frac{s_2}{\eta_2}\right)\ge\eta_1\overline{a}\left(\frac{s_1}{\eta_1}\right)-(\eta_2-\eta_1)\overline{a}_1\left(\frac{s_1}{\eta_1}\right)+(s_2-s_1)\overline{a}'\left(\frac{s_1}{\eta_1}\right)$$
$$\eta_1\overline{a}\left(\frac{s_1}{\eta_1}\right)\ge\eta_2\overline{a}\left(\frac{s_2}{\eta_2}\right)-(\eta_1-\eta_2)\overline{a}_1\left(\frac{s_2}{\eta_2}\right)+(s_1-s_2)\overline{a}'\left(\frac{s_2}{\eta_2}\right).$$

Multiplying the first inequality by $\mu_1>0$ and the second by $\mu_2>0$ and adding them together we get (5.3). Let $s=\tilde H'(w)w_x$, $\eta=\tilde H'(w)\dfrac{w}{\varphi(w)}$, $\mu=w\tilde H'(w)$. Apply (5.3) to the integrand of J_σ and get

$$-J_\sigma\le\frac{1}{\sigma}\int\int_{(0,t)\times K\cap\left\{\left\{\tilde H\right\}\le\sigma\right\}}\left(\left|\left\{\tilde H'(w)\frac{w}{\varphi(w)}\right\}\left\{w\tilde H'\overline{a}_1'(v')\right\}\right|\right.$$
$$\left.+\left|\left\{w\tilde H'(w)\right\}\left\{\tilde H'(w)\frac{w}{\varphi(w)}\overline{a}(v_x)\right\}\right|\right)\,dxds.$$

But in the first integral, for instance,

$$\frac{1}{\sigma}\left|\left\{\tilde H'(w)\frac{w}{\varphi(w)}\right\}\right|=\frac{\{H(w)\}}{\sigma}\frac{\left(\frac{\tilde H'(\tilde w)\tilde w}{\varphi(\tilde w)}\right)_{\tilde w}}{\tilde H'(\tilde w)}\le C_{\delta,M},\qquad \delta=\min(\delta_1,\delta_2)$$

at some $\tilde{w} \in (w_{\delta_1}, w_{\delta_2})$ by the Cauchy theorem. It is analogously in the second integral, so

$$\overline{\mathrm{Lim}}_{\sigma\to+0}(-J_\sigma) \le C_{\delta,M} \int\int_{(0,t)\times K\cap\{w_{\delta_1}=w_{\delta_2}=\tilde{w}\}} \left(\left|\tilde{w}\tilde{H}'(\tilde{w})\right| |\{\overline{a}_1'(v_x)\}|\right.$$

$$\left. + \left|\tilde{H}'(\tilde{w})\frac{\tilde{w}}{\varphi(\tilde{w})}\right| |\{\overline{a}(v_x)\}|\right) dxds = 0,$$

since on the set of integrating $v_{\delta_1 x} = v_{\delta_2 x}$ a.e.. Thus letting $\sigma_1 \to 0$ in (5.2) we omit J_σ.

The next limiting process $\delta_1 \to 0$ is available due to conditions (2.5), (3.2) and Lemma 4.3. That gives an integrable in $(0,t)$ majorant of $\max_{x\in\partial K} \left|H(w_{\delta_1}(s,x))\overline{a}'(v_{\delta_1 x}(s,x))\right|$. Recall that $\tilde{H}'(w)w = H(w)$. So

$$\lim_{\delta_1\to 0} \int_0^t |H(w_{\delta_1})\overline{a}'(v_{\delta_1 x})|\Big|_{\partial K} ds = \int_0^t |H(\overline{w})\overline{a}'(\overline{v}_x)|\ ds\Big|_{\partial K}.$$

Since $\left\{\tilde{H}(\overline{w}(t,.)\right\}_{t>0}$ is bounded in $L^1 \cap L^\infty$, then there exists for some $t_k \to +0$, the L^1_{loc} limit $\tilde{H}(\overline{w}(0,.)) \in L^1(\mathbf{R}^1)$. In fact the inequality $\int_K \tilde{H}(\overline{w}(t))\,dx \le \int_K \tilde{H}(\overline{w}(t'))\,dx + \int_{t'}^t \tilde{H}'(\overline{w})\overline{w}\overline{a}'(\overline{v}_x)\,ds\Big|_{\partial K}$ for $0 \le t' < t$ shows that the limit is the same for all $t \to +0$. So letting it we get from (5.2)

$$\int_K \left[\tilde{H}(\overline{w}(0)) - \tilde{H}(w_{\delta_2 0})\right]_+ dx \le \int_K \left[\tilde{H}(\overline{w}_0) - \tilde{H}(w_{\delta_2 0})\right]_+ dx.$$

Obviously it leads to $\tilde{H}(\overline{w}(0,x)) \le \tilde{H}(w_0(x))$ a.e. and so $\overline{w}(0,x) \le w_0(x)$ a.e. We choose $\tilde{H}_1(w) = w^{r_0}$ to obtain the opposite inequality. The corresponding $\overline{g}_1(w,v_x) = r_0\dfrac{w^{r_0}}{\varphi(w)}((r_0-1)v_x\overline{a}'(v_x) - \overline{b}(v_x)) \le 0$ in the equation for $\tilde{H}_1(w_\delta)$. So $\int_K w_\delta^{r_0}\,dx \ge \int_K w_{\delta 0}^{r_0}(t)\,dx + r_0\int_0^t (w_\delta^{r_0}\overline{a}'(v_\delta))\Big|_{\partial K} ds$. Basing once more on the condition (2.5) for $H_1(s) = s^{r_0}$, for $\delta \to 0$, $t \to 0$ we obtain $\int_K \overline{w}^{r_0}(0)\,dx \ge \int_K w_0^{k_0}\,dx$. So $\int_K [w_0^{r_0} - \overline{w}^{r_0}(0)]_+\,dx = \int_K w_0^{r_0}\,dx - \int_K \overline{w}^{r_0}\,dx \le 0.$ ∎

P r o o f. of Remark 2.1.
The proof of Remark 2.1 was given in the lines of the proof of Lemma 4.2. We need only the limit $\varepsilon \to 0$, $\delta \to 0$ in its assertion (i) and analogous arguments further more. Note that the condition on $f_H(\tau, A)$ is necessary in the cases $(\pm)$,

(-) when $\Gamma(\infty)$ is finite. The property $\overline{v}_x(t,x) \to_{|x|\to\infty} 0$ remains true only in the cases (+), (±). ■

P r o o f. of Remark 2.2.
The function $H(\overline{w}(t,x))\overline{a}'(\overline{v}'(t,x)) + A\int_z^x H(\overline{w}(t,y))\,dy$ is nondecreasing in x due to the regularizing effect and bounded by $f_H(M,A)+A\|H(w_0)\|_1$. So there exists finite $L_\pm(t) = \lim_{x\to\pm\infty} H(\overline{w}(t,x))\overline{a}'(\overline{v}'(t,x))$ and the monotonicity

$H(w)\overline{a}(v_x)(t,x_1) \le A\int_{x_1}^{x_2} H(w(t,y))\,dy + H(w)\overline{a}'(v_x)(t,x_2)$ for $x_1 < x_2$ leads to

$L_- - L_+ \le A\int_{\mathbf{R}^1} H(\overline{w}(t,x))\,dx \le A\int_{\mathbf{R}^1} H(w_0(x))\,dx$. Since $\overline{w}(t,x) \to_{|x|\to\infty} 0$, $v(t,x)$ approaches to its infinimum with $|x|\to\infty$. Then $\underline{\lim}_{x\to\infty} a'(v_x(t,x)) \le 0$ and $\overline{\lim}_{x\to-\infty} a'(v_x(t,x)) \ge 0$. So $L_+(t) \le 0 \le L_-(t)$. Under the condition on $f_H(\tau,A)$ we have for all sufficiently large x for which $w(t,x) \le \varepsilon$ that $H(\overline{w}(t,x))|\overline{a}'(\overline{v}_x(t,x))| \le f_H(\varepsilon,A)$. So $L_-(t) = L_+(t) = 0$. If $b(q) = rqa'(q)$ we take $H(s) = s^{1+r}$, then $g(w,v_x) = 0$ and as in the proof of (5.2), Theorem 5.3 we write

$$\int_K \left[w_1^{1+r}(t,x) - w_2^{1+r}(t,x)\right]_+ dx \le \int_K \left[w_1^{1+r}(0,x) - w_2^{1+r}(0,x)\right]_+ dx$$

$$+(1+r)\int_0^t \operatorname{sgn}\,(w_1(s,x) - w_2(s,x))\left\{w^{1+r}(s,x)\overline{a}'(v_x(s,x))\right\}\Big|_{\partial K}\,ds.$$

Since the last integrand is bounded by the integrable function $2f_H(M,\frac{1}{kt})$ and a.e. in s tends to 0 with $K \to \mathbf{R}^1$, the result is obtained from the Lebegues theorem. ■

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